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Department of Electronics

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Robust Feedback Linearization of a Quadrotor

Presented by:

- **SAIDI Lakhdar**

Supervisor:

DR. Kessal

Registration Number:........../2017

Abstract

Classical feedback linearization which transforms the original nonlinear system into a Brunovsky form has poor robustness properties and cannot be easily combined with $H\infty$ type control law. We propose here to transform by feedback the original nonlinear system into its tangent linearized system around an operating point, and prove that this allows to preserve the good robustness properties obtained by a linear control law which it is associated with. This method constitutes a way of robustly controlling an uncertain nonlinear system around an operating point.

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Introduction

Almost all of the controller design techniques used for various processes are based on well-established results in linear control theory. For nonlinear systems (NLS), in particular, the predominant approach is linearization around an operating point followed by one of the controller design techniques developed for linear systems. For a certain class of nonlinear systems, the particular nature of nonlinearity can create difficult stability and performance problems and therefore renders the linear controllers unacceptable (Ray, 1981).

Several authors proposed the method of feedback linearization (Chou & Wu, 1995), to design a nonlinear controller. The main idea with feedback linearization is based on the fact that the system is not entirely nonlinear, which allows to transform a nonlinear system into an equivalent linear system by effectively canceling out the nonlinear terms (Seo *et al.*, 2007). It provides a way of addressing the nonlinearities in the system while allowing one to use the power of linear control design techniques to address nonlinear closed loop performance specifications.

Nevertheless, the classical feedback linearization technique has certain disadvantages regarding robustness. A robust linear controller designed for the linearized system may not guarantee robustness when applied to the initial nonlinear system, mainly because the linearized system obtained by feedback linearization is in the Brunovsky form, a non robust form whose dynamics is completely different from that of the original system and which is highly vulnerable to uncertainties (Franco, *et al.*, 2006). To eliminate the drawbacks of classical feedback linearization, a robust feedback linearization method has been developed for uncertain nonlinear systems (Franco, et al., 2006; Guillard & Bourles, 2000; Franco *et al.*, 2005) and its efficiency proved theoretically by W-stability (Guillard & Bourles, 2000). The method proposed ensures that a robust linear controller, designed for the linearized system obtained using robust feedback linearization, will maintain the robustness properties when applied to the initial nonlinear system.

In this project, the robust feedback linearization method is presented after a brief discussion about the classical feedback linearization approach. The mathematical steps are given in both approaches. It is shown how the classical approach can be altered in order to obtain a linearized system that coincides with the tangent linearized system around the chosen operating point, rather than the classical chain of integrators. Further, a robust linear controller is designed for the feedback linearized system using loop-shaping techniques and then applied to the original nonlinear system. To test the robustness of the method, a flight dynamic model is given, concerning the control of an Unmanned Aerial Vehicle (UAV), a quad rotor.

The project is organized as follows. In chapter 1, the mathematical concepts of feedback linearization are presented both in the classical and robust approach. The authors propose a technique for disturbance rejection in the case of robust feed- back linearization, based on a feed-forward controller. Section 3 presents the $H\infty$ robust stabilization problem. To exemplify the robustness of the method described, the nonlinear robust control of a quadcopter is given in Section 4. Simulations results for reference tracking, as well as disturbance rejection are given, considering uncertainties in the process parameters.

Classical versus robust approach

Feedback linearization implies the exact cancelling of nonlinearities in a nonlinear system, being a widely used technique in various domains such as robot control (Robenack, 2005), power system control (Dabo et al., 2009), and also in chemical process control (Barkhordari Yazdi & Jahed-Motlagh, 2009; Pop & Dulf, 2010; Pop et al, 2010), etc. The majority of nonlinear control techniques using feedback linearization also use a strategy to enhance robustness. This section describes the mathematical steps required to obtain the final closed-loop control structure, to be later used with robust linear control.

I.1 Classical feedback linearization

I.1.1 Feedback linearization for SISO systems

In the classical approach of feedback linearization as introduced by Isidori (Isidori, 1995), the Lie derivative and relative degree of the nonlinear system plays an important role. For a single input single output system, given by:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1.1}$$

with $x \in R^n$ is the state, u is the control input, y is the output, f and g are smooth vector fields on R^n and h is a smooth nonlinear function. Differentiating y with respect to time, we obtain:

$$\begin{aligned} \dot{y} &= \frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x) \\ \dot{y} &= L_f h(x) + L_g h(x) u \end{aligned} \tag{1.2}$$

with $L_f h(x): R^n \to R$ and $L_g h(x): R^n \to R$, defined as the Lie derivatives of h with respect to f and g, respectively. Let U be an open set containing the equilibrium point x_0 , that is a point where $f(x)$ becomes null – $f(x_0 = 0$. Thus, if in equation (1.2), the

Lie derivative of h with respect to g - $L_g h(x)$ - is bounded away from zero for all $x \in U$ (Sastry, 1999), then the state feedback law

$$u = \frac{1}{L_g h(x)}(-L_f h(x) + v) \tag{1.3}$$

yields a linear first order system from the supplementary input v to the initial output of the system, y. Thus, there exists a state feedback law, similar to (1.3), that makes the nonlinear system in (1.2) linear. The relative degree of system (1.2) is defined as the number of times the output has to be differentiated before the input appears in its expression. This is equivalent to the denominator in (1.3) being bounded away from zero, for all $x \in U$. In general, the relative degree of a nonlinear system at $x_0 \in U$ is defined as an integer r satisfying:

$$\begin{aligned} &L_g L_f^i h(x) = 0, \forall x \in U, i = 0, \dots \dots, r-2 \\ &L_g L_f^{r-1} h(x_0) \neq 0 \end{aligned} \tag{1.4}$$

Thus, if the nonlinear system in (1.1) has relative degree equal to r, then the differentiation of y in (1.2) is continued until:

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) u \tag{1.5}$$

with the control input equal to:

$$u = \frac{1}{L_g L_f^{r-1} h(x)}(-L_f^r h(x) + v) \tag{1.6}$$

The final (new) input – output relation becomes

$$y^{(r)} = v \tag{1.7}$$

which is linear and can be written as a chain of integrators (Brunovsky form). The control law in (6) yields (n-r) states of the nonlinear system in (1.1) unobservable through state feedback.

The problem of measurable disturbances has been tackled also in the framework of feedback linearization. In general, for a nonlinear system affected by a measurable disturbance d:

$$\dot{x} = f(x) + g(x) + p(x)d$$
$$y = h(x) \quad (1.8)$$

with $p(x)$ a smooth vector field.

Similar to the relative degree of the nonlinear system, a disturbance relative degree is defined as a value k for which the following relation holds:

$$L_p L_f^i h(x) = 0, \forall x \in U, i = 0, \dots \dots, k-2$$
$$L_p L_f^{k-1} h(x_0) \neq 0 \quad (1.9)$$

Thus, a comparison between the input relative degree and the disturbance relative degree gives a measure of the effect that each external signal has on the output (Daoutidis and Kravaris, 1989). If $k < r$, , the disturbance will have a more direct effect upon the output, as compared to the input signal, and therefore a simple control law as given in (1.6) cannot ensure the disturbance rejection (Henson and Seborg, 1997). In this case complex feedforward structures are required and effective control must involve anticipatory action for the disturbance. The control law in (1.6) is modified to include a dynamic feedforward/ state feedback component which differentiates a state- and disturbance-dependent signal up to $r - k$ times, in addition to the pure static state feedback component. In the particular case that $r = k$, both the disturbance and the manipulated input affect the output in the same way. Therefore, a feed-forward/state feedback element which is static in the disturbance is necessary in the control law in addition to the pure state feedback element (Daoutidis and Kravaris, 1989):

$$u = \frac{1}{L_g L_f^{r-1} h(x)} (-L_f^r h(x) + v - L_p L_f^{r-1} p(x)d) \quad (1.10)$$

I.1.2 Feedback linearization for MIMO systems

The feedback linearization method can be extended to multiple input multiple output nonlinear square systems (Sastry, 1999). For a MIMO nonlinear system having n states and m inputs/outputs the following representation is used:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x) \qquad (1.11)$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input vector and $y \in R^m$ is the output vector.

Similar to the SISO case, a vector relative degree is defined for the MIMO system in (1.11). The problem of finding the vector relative degree implies differentiation of each output signal until one of the input signals appears explicitly in the differentiation. For each output signal, we define r_j as the smallest integer such that at least one of the inputs appears in $y_j^{r_j}$:

$$y_j^{r_j} = L_f^{r_j} h_j + \sum_{i=1}^{m} L_{g_i}(L_f^{r_j-1} h_j) u_i \qquad (1.12)$$

and at least one term $L_{g_i}(L_f^{r_j-1} h_j) u_i \neq 0$ for some x (Sastry, 1999). In what follows we assume that the sum of the relative degrees of each output is equal to the number of states of the nonlinear system. Such an assumption implies that the feedback linearization method is exact. Thus, neither of the state variables of the original nonlinear system is rendered unobservable through feedback linearization. The matrix$M(x)$, defined as the decoupling matrix of the system, is given as:

$$M = \begin{bmatrix} L_{g_1}(L_f^{r_1-1} h_1) & \cdots & L_{g_m}(L_f^{r_1-1} h_1) \\ \vdots & \ddots & \vdots \\ L_{g_1}(L_f^{r_m-1} h_m) & \cdots & L_{g_m}(L_f^{r_m-1} h_m) \end{bmatrix} \qquad (1.13)$$

The nonlinear system in (11) has a defined vector relative degree $r_1, r_2, ..., r_m$ at the po11int x_0 if $L_{g_i}\left(L_f^k h_j(x)\right) = 0,\ 0 \le k \le r_i - 2$ for $i = 1, \ldots, m$ and the matrix

$M(x_0)$ is nonsingular, If the vector relative degree $r_1, r_2, \ldots\ldots, r_m$ is well defined, then (1.12) can be written as:

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} (L_f^{r_1} h_1(x)) \\ \vdots \\ (L_f^{r_m} h_m(x)) \end{bmatrix} + M(x) \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \quad (1.14)$$

Since $M(x_0)$ is nonsingular, then $M(x) \in R^{m \times m}$ is nonsingular for each $x \in U$. As a consequence, the control signal vector can be written as:

$$u = -M^{-1}(x) \begin{bmatrix} L_f^{r_1} h_1 \\ \\ L_f^{r_m} h_m \end{bmatrix} + M^{-1}(x) v = \alpha_c(x) + \beta_c(x) v \quad (1.15)$$

yielding the linearized system as:

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \quad (1.16)$$

The states x undergo a change of coordinates given by:

$$x_c = \left[y_1 \ \ldots \ L_f^{r_1 - 1} y_1 \ \ldots \ldots \ldots \ldots \ y_m \ \ldots \ L_f^{r_m - 1} y_m \right]^T \quad (1.17)$$

The nonlinear MIMO system in (1.11) is linearized to give:

$$\dot{x}_c = A_c x_c + B_c v \quad (1.18)$$

with:

$$A_c = \begin{bmatrix} A_{c_1} & 0_{r_1 \times r_2} & \ldots . & 0_{r_1 \times r_m} \\ 0_{r_2 \times r_1} & A_{c_2} & \ldots . & 0_{r_2 \times m} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{r_m \times r_1} & 0_{r_m \times r_2} & 0_{r_m \times r_3} & A_{c_m} \end{bmatrix}, B_c = \begin{bmatrix} B_{c_1} & 0_{r_1 \times r_2} & \ldots . & 0_{r_1 \times r_m} \\ 0_{r_2 \times r_1} & B_{c_2} & \ldots & 0_{r_2 \times r_m} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{r_m \times r_1} & 0_{r_m \times r_2} & 0_{r_m \times r_3} & B_{c_m} \end{bmatrix}$$

where each term individually is given by:

$$A_{ei} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B_{e_i} = [0 \quad 0 \quad \cdots \quad 0 \quad 1]^T.$$

In the classical approach, the feedback linearization is achieved through a feedback control law and a state transformation, leading to a linearized system in the form of a chain of integrators (Isidori, 1995). Thus the design of the linear controller is difficult, since the linearized system obtained bears no physical meaning similar to the initial nonlinear system (Pop *et al.*, 2009). In fact, two nonlinear systems having the same degree will lead to the same feedback linearized system.

I.2 Robust feedback linearization

Contrarily to the classical feedback linearization which transforms the original nonlinear system (1.11) into a Brunovsky form, the present method consists in transforming it into its tangent linearized system around an operating point.

Consider the multivariable nonlinear system with disturbance vector d given in the following equation:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)d \\ y &= h(x) \end{aligned} \qquad (1.19)$$

with $x \in R^n$ is the state, $u \in R^m$ is the control input vector, $y \in R^m$ is the output vector, f and g are smooth vector fields on R^n and h is a smooth nonlinear function. Choosing the operating point as $x = 0$.

$$\dot{z} = Az + Bw \qquad (1.20)$$

with $A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$ and $B = \left.\frac{\partial g}{\partial x}\right|_{x=0}$

In what follows, we assume that the feedback linearization conditions (Isidori, 1995) are satisfied and that the output of the nonlinear system given in (1.19) can be chosen as:

$y(x) = \lambda(x)$, where $\lambda(x) = [\lambda_1(x)\ \lambda_2(x) \ldots\ldots \lambda_m(x)]$ is a vector formed by real-valued functions $\lambda_i(x)$ defined on a neighborhood U of $x = 0$ satisfying, for numbers $r_1, r_2, \ldots\ldots, r_m$ such that $r_1 + r_2 + \ldots\ldots + r_m = n$

i) for all $i \in [1, m]$, all j $\in [1, m]$ and all $x \in u$

$$L_{g_i}\lambda_j(x) = L_{g_i}L_f\lambda_j(x) = \cdots = L_{g_i}L_f^{r_j-2}\lambda_j(x) = 0$$

ii) the $m \times m$ matrix :

$$M = \begin{bmatrix} L_{g_1}L_f^{r_1-1}\lambda_1 & \cdots & L_{g_m}L_f^{r_1-1}\lambda_1 \\ \vdots & \ddots & \vdots \\ L_{g_1}L_f^{r_m-1}\lambda_m & \cdots & L_{g_m}L_f^{r_m-1}\lambda_m \end{bmatrix}$$

is nonsingular at $x = 0$, we will denote $M \triangleq M(0)$.

Consider on this basis the associated classical linearizing stete feedback

$$u_c(x, v) = \alpha_c(x) + \beta_c(x)v \tag{1.21}$$

with

$$\alpha_c(x) \triangleq -M^{-1}(x)N(x), \quad B_c(x) \triangleq -M^{-1}(x)$$

$$N(x) \triangleq \left[L_f^{r_1}\lambda_1(x)\ \ L_f^{r_2}\lambda_2(x)\ \ \ldots\ldots L_f^{r_m}\lambda_m(x)\right]^T$$

and change of coordinates

$$x_c = \alpha_c(x) \tag{1.22}$$

given by

$$\alpha_c(x) = \left[\alpha_{c_1}(x) \;\; \alpha_{c_2}(x) \;\; \dots\dots\dots \alpha_{c_m}(x)\right]^T$$

$$\alpha_{c_i}(x) = \left[\lambda_i(x) \;\; L_f\lambda_i(x) \;\; \dots\dots L_f^{r_i-1}\lambda_1(x)\right]^T$$

Then under state feedback

$$u(x,w) = \alpha(x) + \beta(x)w$$

and change of coordinate

$$z = \emptyset(x)$$

defined by

$$\begin{aligned} \alpha(x) &\triangleq \alpha_c(x) + \beta_c(x)LT^{-1}\emptyset_c(x) \\ \beta(x) &\triangleq \beta_c(x)R^{-1} \\ \emptyset(x) &\triangleq T^{-1}\emptyset_c(x) \end{aligned} \qquad (1.23)$$

where

$$L \triangleq -m.\frac{\partial\alpha_c}{\partial x}\bigg|_{x=0}, \quad T \triangleq -m.\frac{\partial\emptyset_c}{\partial x}\bigg|_{x=0} \quad \text{and} \;\; R \triangleq M^{-1},$$

system (1.11) is transformed into system (1.20).

II.1 Preliminar notions

The quadrotor, an aircraft made up of four engines, holds the electronic board in the middle and the engines at four extremities. Before describing the mathematical model of a quadrotor, it is necessary to introduce the reference coordinates in which we describe the structure and the position. For the quadrotor, it is possible to use two reference systems. The first is fixed and the second is mobile.

The fixed coordinate system, called also inertial, is a system where the first Newton's law is considered valid. As fixed coordinate system, we use the O_{NED}systems, where NED is for North-East-Down. As we can observe from the following Figure (), its vectors are directed to Nord, East and to the center of the Earth.

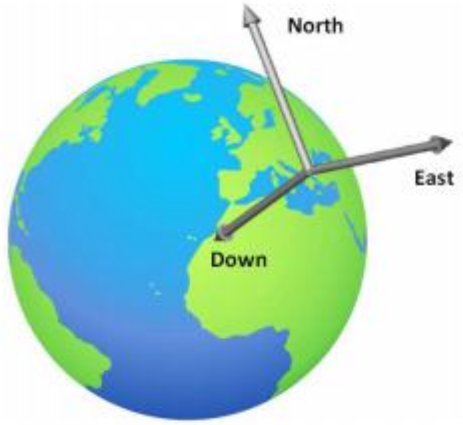

Figure 2.1: O_{NED} fixed reference system.

The mobile reference system that we have previously mentioned is united with the barycenter of the quadrotor. In the scientific literature it is called OABC system, where ABC is for Aircraft Body Center. Figure 2.2 illustrates underlines the two coordinate systems.

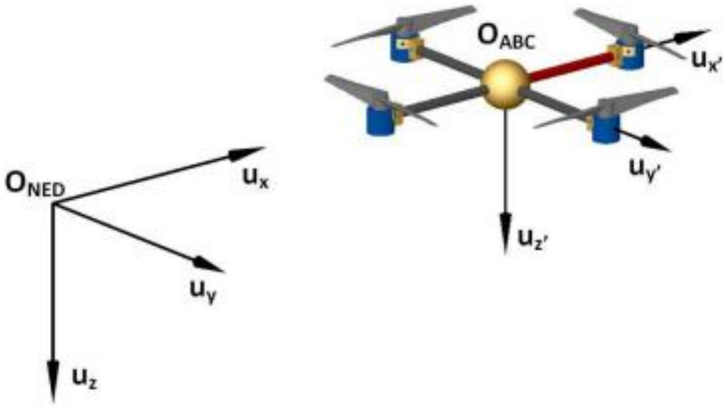

Figure 2.2: Mobile reference system and fixed reference system.

The quadrotor helicopter is shown in figure (). Two diagonal motors (1) and (3) are running in the same direction (counter-clockwise) whereas the two others (2) and (4) in the clockwise direction to eliminate the anti-torque. On varying the rotor speeds altogether with the same quantity the lift forces will change affecting in this case the altitude z of the system and enabling vertical take-Off/On landing. Yaw angle is obtained by speeding up the clockwise motors or slowing down depending on the desired angle direction. Tilting around x (roll angle) axis allows the quadrotor to move toward y direction. The sense of direction depends on the sense of angle whether it is positive or negative. Tilting around y (pitch angle) axis allows the quadrotor to move toward x direction.

Figure 2.3: A quadcopter model

II.2 Euler angles:

The Euler angles are three angles introduced by Leonhard Euler to describe the orientation of a rigid body. To describe such an orientation in the 3-dimensional Euclidean space, three parameters are required. They are also used to describe the orientation of a frame of reference relative to another and they transform the coordinates of a point in a reference frame in the coordinates of the same point in another reference frame. The Euler angles are typically denoted as $\emptyset \in [-\pi, \pi]$, $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $\psi \in [-\pi, \pi]$. Euler angles represent a sequence of three elemental rotations, i.e. rotations about the axes of a coordinate system, since any orientation can be achieved by composing three elemental rotations. These rotations start from a known standard orientation. This combination used is described by the following rotation matrices:

$$\mathbf{R}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\phi) & -s(\phi) \\ 0 & s(\phi) & c(\phi) \end{bmatrix}, \quad (2.1)$$

$$\mathbf{R}_y(\theta) = \begin{bmatrix} c(\theta) & 0 & s(\theta) \\ 0 & 1 & 0 \\ -s(\theta) & 0 & c(\theta) \end{bmatrix}, \quad (2.2)$$

$$\mathbf{R}_z(\psi) = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (2.3)$$

where $c(\ \ ) = \cos(\ \ )$ *and* $s(\ \ ) = \sin(\ \ )$. So, the inertial position coordinates and the body reference coordinates are related by the rotation matrix $R_{zyx}(\emptyset, \theta, \psi)$.

$$R_{zyx}(\emptyset, \theta, \psi) = R_z(\psi).R_y(\theta).R_x(\emptyset)$$

$$= \begin{bmatrix} c(\theta)c(\psi) & s(\phi)s(\theta)c(\psi) - c(\phi)s(\psi) & c(\phi)s(\theta)c(\psi) + s(\phi)s(\psi) \\ c(\theta)s(\psi) & s(\phi)s(\theta)s(\psi) + c(\phi)c(\psi) & c(\phi)s(\theta)s(\psi) - s(\phi)c(\psi) \\ -s(\theta) & s(\phi)c(\theta) & c(\phi)c(\theta) \end{bmatrix} \quad (2.4)$$

II.3 Quadrotor mathematical model

We provide here a mathematical model of the quadrotor, exploiting Newton and Euler equations for the 3D motion of a rigid body. The goal of this section is to obtain a deeper understanding of the dynamics of the quadrotor and to provide a model that is sufficiently reliable for simulating and controlling its behavior.

Let us call $[X, Y, Z, \emptyset, \theta, \psi]^T$ the vector containing the linear and angular position of the quadrotor in the earth frame and $[u, v, w, p, q, r]^T$ the vector containing the linear and angular velocities in the body frame. From 3D body dynamics, it follows that the two reference frames are linked by the following relations:

$$V = R.V_B \tag{2.5}$$

$$W = T.W_B \tag{2.6}$$

where $= [\dot{x}\ \dot{y}\ \dot{z}]^T \in R^3$, $W = [\dot{\emptyset}\ \dot{\theta}\ \dot{\psi}]^T \in R^3$, $V_B = [u\ v\ w]^T \in R^3$ and $W_B = [p\ q\ r]^T \in R^3$ and T is a matrix for angular transformations

$$\mathbf{T} = \begin{bmatrix} 1 & s(\phi)t(\theta) & c(\phi)t(\theta) \\ 0 & c(\phi) & -s(\phi) \\ 0 & \dfrac{s(\phi)}{c(\theta)} & \dfrac{c(\phi)}{c(\theta)} \end{bmatrix} \tag{2.7}$$

where $t(\theta) = \tan(\theta)$. So, the kinematic model of the quadrotor is:

$$\begin{cases} \dot{x} = w[s(\phi)s(\psi) + c(\phi)c(\psi)s(\theta)] - v[c(\phi)s(\psi) - c(\psi)s(\phi)s(\theta)] + u[c(\psi)c(\theta)] \\ \dot{y} = v[c(\phi)c(\psi) + s(\phi)s(\psi)s(\theta)] - w[c(\psi)s(\phi) - c(\phi)s(\psi)s(\theta)] + u[c(\theta)s(\psi)] \\ \dot{z} = w[c(\phi)c(\theta)] - u[s(\theta)] + v[c(\theta)s(\phi)] \\ \dot{\phi} = p + r[c(\phi)t(\theta)] + q[s(\phi)t(\theta)] \\ \dot{\theta} = q[c(\phi)] - r[s(\phi)] \\ \dot{\psi} = r\dfrac{c(\phi)}{c(\theta)} + q\dfrac{s(\phi)}{c(\theta)} \end{cases} \tag{2.8}$$

Newton's law states the following matrix relation for the total force acting on the quadrotor:

$$m\left(W_B \wedge V_B + \dot{V}_B\right) = f_B \tag{2.9}$$

where m is the mass of the quadrotor, $\wedge$ is the cross product and $f_B = \left[f_x \; f_y \;\; f_z\right]^T \in R^3$ is the total force.

Euler's equation gives the total torque applied to the quadrotor:

$$\mathrm{I}.\dot{W}_B + W_B \wedge (\mathrm{I}.W_B) = m_B \tag{2.10}$$

Where $m_B = \left[m_x \; m_y \;\; m_z\right]^T \in R^3$ is the total torque and I is the diagonal inertia matrix:

$$\mathbf{I} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix} \in \mathbb{R}^{3\times3}$$

So, the dynamic model of the quadrotor in the body frame is:

$$\begin{Bmatrix} f_x & = m(\dot{u} + qw - rv) \\ f_y & = m(\dot{v} - pw + ru) \\ f_z & = m(\dot{w} + pv - qu) \\ m_x & = \dot{p}I_x - qrI_y + qrI_z \\ m_y & = \dot{q}I_y + prI_x - prI_z \\ m_z & = \dot{r}I_z - pqI_x + pqI_y \end{Bmatrix} \tag{2.11}$$

The equations stand as long as we assume that the origin and the axes of the body frame coincide with the barycenter of the quadrotor and the principal axes.

II.4 Forces and moments

The external forces in the body frame, F_B given by

$$f_B = mgR^T.\hat{e}_Z - f_t\hat{e}_3 + f_w \qquad (2.12)$$

Where $\hat{e}_Z$ is the unit vector in the inertial Z axis, $\hat{e}_3$ is the unit vector in the body z axis, g is the gravitational acceleration, f_t is the total thrust generated by rotors and $f_w = [f_{wx}\ f_{wy}\ \ f_{wz}]^T \in R^3$ are the forces produced by wind on the quadrotors. The external moments in the body frame, m_B are given by

$$m_B = \tau_B + \tau_w$$

where $\tau_B = [\tau_x\ \tau_y\ \ \tau_z]^T \in R^3$ are the control torques generated by differences in the rotor speeds and $\tau_w = [\tau_{wx}\ \tau_{wy}\ \ \tau_{wz}]^T \in R^3$ are the torques produced by wind on the quadrotors. So, the complete dynamic model of the quadrotor in the body frame is obteined substituting the force expression in (2.11):

$$\left\{\begin{array}{c} -mg[s(\theta)] + f_{wx} = m(\dot{u} + qw - rv) \\ mg[c(\theta)s(\phi)] + f_{wy} = m(\dot{v} - pw + ru) \\ mg[c(\theta)c(\phi)] + f_{wz} - f_t = m(\dot{w} + pv - qu) \\ \tau_x + \tau_{wx} = \dot{p}I_x - qrI_y + qrI_z \\ \tau_y + \tau_{wy} = \dot{q}I_y + prI_x - prI_z \\ \tau_z + \tau_{wz} = \dot{r}I_z - pqI_x + pqI_y \end{array}\right\} \qquad (2.15)$$

II.5 Actuator dynamics

Here we consider the inputs that can be applied to the system in order to control the behavior of the quadrotor. The rotors are four and the degrees of freedom we control are as many: commonly, the control inputs that are considered are one for the vertical thrust and one for each of the angular motions. Let us consider the values of the input forces and torques proportional to the squared speeds of the rotors; their values are the following:

$$\begin{aligned} f_t &= K_b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) \\ \tau_x &= K_b l\,(\Omega_3^2 - \Omega_1^2) \\ \tau_y &= K_b l\,(\Omega_4^2 - \Omega_2^2) \\ \tau_z &= K_d l\,(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 - \Omega_3^2) \end{aligned} \tag{2.16}$$

where l is the distance between any rotor and the center of the drone, K_b is the thrust factor and K_d is the drag factor. Substituting (2.16) in (2.15), we have: the dynamic model of the quadrotor in the body frame is:

$$\left\{ \begin{aligned} -mg[s(\theta)] + f_{wx} &= m(\dot{u} + qw - rv) \\ mg[c(\theta)s(\phi)] + f_{wy} &= m(\dot{v} - pw + ru) \\ mg[c(\theta)c(\phi)] + f_{wz} - b(\Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2) &= m(\dot{w} + pv - qu) \\ bl(\Omega_3^2 - \Omega_1^2) + \tau_{wx} &= \dot{p}I_x - qrI_y + qrI_z \\ bl(\Omega_4^2 - \Omega_2^2) + \tau_{wy} &= \dot{q}I_y + prI_x - prI_z \\ d(\Omega_2^2 + \Omega_4^2 - \Omega_1^2 + \Omega_3^2) + \tau_{wz} &= \dot{r}I_z - pqI_x + pqI_y \end{aligned} \right\} \tag{2.17}$$

II.6 State-space model

$$\mathbf{x} = [\phi \quad \theta \quad \psi \quad p \quad q \quad r \quad u \quad v \quad w \quad x \quad y \quad z]^T \in \mathbb{R}^{12} \qquad (2.18)$$

It is possible to rewrite the equations of the dynamics of the quadrotor in the state space from (2.8) and (2.15):

$$\left\{\begin{array}{c}
\dot{\phi} = p + r[c(\phi)t(\theta)] + q[s(\phi)t(\theta)] \\
\dot{\theta} = q[c(\phi)] - r[s(\phi)] \\
\dot{\psi} = r\frac{c(\phi)}{c(\theta)} + q\frac{s(\phi)}{c(\theta)} \\
\dot{p} = \frac{I_y - I_z}{I_x} rq + \frac{\tau_x + \tau_{wx}}{I_x} \\
\dot{q} = \frac{I_z - I_x}{I_y} pr + \frac{\tau_y + \tau_{wy}}{I_y} \\
\dot{r} = \frac{I_x - I_y}{I_z} pq + \frac{\tau_z + \tau_{wz}}{I_z} \\
\dot{u} = rv - qw - g[s(\theta)] + \frac{f_{wx}}{m} \\
\dot{v} = pw - ru + g[s(\phi)c(\theta)] + \frac{f_{wy}}{m} \\
\dot{w} = qu - pv + g[c(\theta)c(\phi)] + \frac{f_{wz} - f_t}{m} \\
\dot{x} = w[s(\phi)s(\psi) + c(\phi)c(\psi)s(\theta)] - v[c(\phi)s(\psi) - c(\psi)s(\phi)s(\theta)] + u[c(\psi)c(\theta)] \\
\dot{y} = v[c(\phi)c(\psi) + s(\phi)s(\psi)s(\theta)] - w[c(\psi)s(\phi) - c(\phi)s(\psi)s(\theta)] + u[c(\theta)s(\psi)] \\
\dot{z} = w[c(\phi)c(\theta)] - u[s(\theta)] + v[c(\theta)s(\phi)]
\end{array}\right\} \qquad (2.19)$$

Below we obtain two alternative forms of the dynamical model useful for studying the control. From Newton's law we can write:

$$m\dot{v} = R.f_B = mg\hat{e}_Z - f_t R.\hat{e}_3 \qquad (2.20)$$

Therefore:

$$\begin{cases} \ddot{x} = -\frac{f_t}{m}[s(\phi)s(\psi) + c(\phi)c(\psi)s(\theta)] \\ \ddot{y} = -\frac{f_t}{m}[c(\phi)s(\psi)s(\theta) - c(\psi)s(\phi)] \\ \ddot{z} = g - \frac{f_t}{m}[c(\phi)c(\theta)] \end{cases} \tag{2.21}$$

Now a simplification is made by setting $[\dot{\phi}\ \dot{\theta}\ \dot{\psi}]^T = [p\ q\ r\ ]^T$.This assumption holds true for small angles of movement. So, the dynamic model of the quadrotor in the inertial frame is:

$$\begin{cases} \ddot{x} = -\frac{f_t}{m}[s(\phi)s(\psi) + c(\phi)c(\psi)s(\theta)] \\ \ddot{y} = -\frac{f_t}{m}[c(\phi)s(\psi)s(\theta) - c(\psi)s(\phi)] \\ \ddot{z} = g - \frac{f_t}{m}[c(\phi)c(\theta)] \\ \ddot{\phi} = \frac{I_y - I_z}{I_x}\dot{\theta}\dot{\psi} + \frac{\tau_x}{I_x} \\ \ddot{\theta} = \frac{I_z - I_x}{I_y}\dot{\phi}\dot{\psi} + \frac{\tau_y}{I_y} \\ \ddot{\psi} = \frac{I_x - I_y}{I_z}\dot{\phi}\dot{\theta} + \frac{\tau_z}{I_z} \end{cases} \tag{2.22}$$

Redefining the state's vector as:

$$\mathbf{x} = [x \quad y \quad z \quad \psi \quad \theta \quad \phi \quad \dot{x} \quad \dot{y} \quad \dot{z} \quad p \quad q \quad r]^T \in \mathbb{R}^{12} \tag{2.23}$$

It is possible to rewrite the equations of the quadrotor in the spate-space:

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}) + \sum_{i=1}^{4} \mathrm{g_i}(\mathrm{x})u_i \tag{2.24}$$

Where:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ q\frac{s(\phi)}{c(\theta)} + r\frac{c(\phi)}{c(\theta)} \\ q[c(\phi)] - r[s(\phi)] \\ p + q[s(\phi)t(\theta)] + r[c(\phi)t(\theta)] \\ 0 \\ 0 \\ g \\ \frac{(I_y - I_z)}{I_x}qr \\ \frac{(I_z - I_x)}{I_y}pr \\ \frac{(I_x - I_y)}{I_z}pq \end{bmatrix} \tag{2.25}$$

And

$$\mathrm{g}_1(\mathrm{x}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad g_1^7 \quad g_1^8 \quad g_1^9 \quad 0 \quad 0 \quad 0]^T \in \mathbb{R}^{12}$$
$$\mathrm{g}_2(\mathrm{x}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{I_x} \quad 0 \quad 0]^T \in \mathbb{R}^{12}$$
$$\mathrm{g}_3(\mathrm{x}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{I_y} \quad 0]^T \in \mathbb{R}^{12}$$
$$\mathrm{g}_4(\mathrm{x}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{I_z}]^T \in \mathbb{R}^{12}$$

With

$$g_1^7 = -\tfrac{1}{m}[s(\phi)s(\psi) + c(\phi)c(\psi)s(\theta)]$$
$$g_1^8 = -\tfrac{1}{m}[c(\psi)s(\phi) - c(\phi)s(\psi)s(\theta)]$$
$$g_1^9 = -\tfrac{1}{m}[c(\phi)c(\theta)]$$

II.7 Control strategies

In this section we discuss two control strategies, both of them are nonlinear, the first is the classical feedback linearization method, whereas the second one is the robust feedback linearization approach. Some comparisons about these control strategies are done.

II.7.1 Feedback linearization control

The quadrotor has six outputs $y = [x\ y\ z\ \emptyset\ \theta\ \psi]^T$ and the vehicle has four inputs. There are two degrees of freedom that are left uncontrollable. A solution to this problem is to use dynamic feedback control (Exact linearization and non-interacting control via dynamic feedback). Such control structures are based on the input-output linearization described earlier.

First, it is necessary to define the control objective by choosing an output function for the system (2.24). To avoid unnecessary complications, we set the number of input channels equal to the number of output channels. We would like to control the absolute position of the quadrotor $[x\ y\ z]^T$ and the angle ψ. Therefore, the output function is chosen as:

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) = [x \quad y \quad z \quad \psi]^T \qquad (2.26)$$

We assume the state x of the system being fully available for measurements and we seek a static state feedback control law of the form:

$$\mathrm{u} = \alpha(\mathrm{x}) + \beta(\mathrm{x}) \cdot \mathrm{v} \qquad (2.27)$$

Where V is an external reference input to be defined later,

$\alpha(x) = [\alpha_1(x)\ \alpha_2(x)\ \alpha_3(x)\alpha_4(x)\ ]^T$ and $\beta(x) \in R^{4\times 4}$

Let $[r_1\ r_2\ r_3\ r_4]^T$ be the relative degree vector of the system (2.24). We have:

$$[y_1^{(r_1)} \quad y_2^{(r_2)} \quad y_3^{(r_3)} \quad y_4^{(r_4)}]^T = \mathrm{b}(\mathrm{x}) + \Delta(\mathrm{x}) \cdot \mathrm{u} \qquad (2.28)$$

Where

$$\Delta(\mathrm{x}) = \begin{bmatrix} L_{g1}L_f^{r_1-1}h_1(\mathrm{x}) & L_{g2}L_f^{r_1-1}h_1(\mathrm{x}) & L_{g3}L_f^{r_1-1}h_1(\mathrm{x}) & L_{g4}L_f^{r_1-1}h_1(\mathrm{x}) \\ L_{g1}L_f^{r_2-1}h_2(\mathrm{x}) & L_{g2}L_f^{r_2-1}h_2(\mathrm{x}) & L_{g3}L_f^{r_2-1}h_2(\mathrm{x}) & L_{g4}L_f^{r_2-1}h_2(\mathrm{x}) \\ L_{g1}L_f^{r_3-1}h_3(\mathrm{x}) & L_{g2}L_f^{r_3-1}h_3(\mathrm{x}) & L_{g3}L_f^{r_3-1}h_3(\mathrm{x}) & L_{g4}L_f^{r_3-1}h_3(\mathrm{x}) \\ L_{g1}L_f^{r_4-1}h_4(\mathrm{x}) & L_{g2}L_f^{r_4-1}h_4(\mathrm{x}) & L_{g3}L_f^{r_4-1}h_4(\mathrm{x}) & L_{g4}L_f^{r_4-1}h_4(\mathrm{x}) \end{bmatrix} \tag{2.29}$$

$$\mathbf{b}(\mathbf{x}) = \begin{bmatrix} L_f^{r_1}h_1(\mathbf{x}) \\ L_f^{r_2}h_2(\mathbf{x}) \\ L_f^{r_3}h_4(\mathbf{x}) \\ L_f^{r_4}h_4(\mathrm{x}) \end{bmatrix} \tag{2.30}$$

The input-output decoupling problem is solvable if and only if the matrix $\Delta(x)$ is nonsingular. In this case, the static state feedback with:

$$\begin{Bmatrix} \alpha(\mathrm{x}) = -\Delta^{-1}(\mathrm{x}) \cdot \mathrm{b}(\mathrm{x}) \\ \beta(\mathrm{x}) = \Delta^{-1}(\mathrm{x}) \end{Bmatrix} \tag{2.31}$$

renders the closed loop system linear and decoupled from an input-output point of view. More precisely, we have

$$y_i^{(r_i)} = v_i \text{ for all } i, 1 \leq i \leq 4 \tag{2.32}$$

However, for the nonlinear system (2.24), we have

$$r_1 = r_2 = r_3 = r_4 = 2$$

and

$$\Delta(\mathrm{x}) = \begin{bmatrix} \delta_{1,1} & 0 & 0 & 0 \\ \delta_{2,1} & 0 & 0 & 0 \\ \delta_{3,1} & 0 & 0 & 0 \\ 0 & 0 & \delta_{4,3} & \delta_{4,4} \end{bmatrix}$$

with

$$\delta_{1,1} = g_1^7$$
$$\delta_{2,1} = g_1^8$$
$$\delta_{3,1} = g_1^9$$
$$\delta_{4,3} = \frac{s(\phi)}{I_y c(\theta)}$$
$$\delta_{4,4} = \frac{c(\phi)}{I_z c(\theta)}$$

Obviously $\Delta(x)$ is singular for all x and therefore the input-output decoupling problem is not solvable for the system (2.24) by means of a static state feedback control law. Setting u_1 equal to the output of a double integrator driven by $\overline{u}_1$, i.e

$$\begin{Bmatrix} u_1 = \zeta \\ \dot{\zeta} = \xi \\ \dot{\xi} = \bar{u}_1 \end{Bmatrix} \tag{2.33}$$

For consistency of notation we also set, for the other input channels which have been left unchanged, the following

$$\begin{Bmatrix} u_2 = \bar{u}_2 \\ u_3 = \bar{u}_3 \\ u_4 = \bar{u}_4 \end{Bmatrix} \tag{2.34}$$

Note that u_1 is not anymore an input for the system (2.24) but becomes the internal state ξ for the new dynamical system (2.33). The extended system obtained is described by equations of the form:

$$\dot{\bar{\mathrm{x}}} = \bar{\mathrm{f}}(\bar{\mathrm{x}}) + \sum_{i=1}^{4} \bar{\mathrm{g}}_{\mathrm{i}}(\bar{\mathrm{x}})\bar{\mathbf{u}}_i \tag{2.35}$$

In which

$$\bar{\mathrm{x}} = [x \quad y \quad z \quad \psi \quad \theta \quad \phi \quad \dot{x} \quad \dot{y} \quad \dot{z} \quad \zeta \quad \xi \quad p \quad q \quad r]^T \in \mathbb{R}^{14} \tag{2.36}$$

$$\bar{\mathbf{f}}(\bar{\mathbf{x}}) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ q\frac{s(\phi)}{c(\theta)} + r\frac{c(\phi)}{c(\theta)} \\ q[c(\phi)] - r[s(\phi)] \\ p + q[s(\phi)t(\theta)] + r[c(\phi)t(\theta)] \\ g_1^7(\psi,\theta,\phi)\zeta \\ g_1^8(\psi,\theta,\phi)\zeta \\ g_1^9(\psi,\theta,\phi)\zeta \\ \xi \\ 0 \\ \frac{(I_y - I_z)}{I_x}qr \\ \frac{(I_z - I_x)}{I_y}pr \\ \frac{(I_x - I_y)}{I_z}pq \end{bmatrix} \tag{2.37}$$

And

$$\bar{g}_1(\bar{x}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T \in \mathbb{R}^{14}$$

$$\bar{g}_2(\bar{x}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{I_x} \quad 0 \quad 0]^T \in \mathbb{R}^{14}$$

$$\bar{g}_3(\bar{x}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{I_y} \quad 0]^T \in \mathbb{R}^{14}$$

$$\bar{g}_4(\bar{x}) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{I_z}]^T \in \mathbb{R}^{14}$$

Now, the input-output decoupling problem is solvable for the nonlinear system (2.24) by means of a dynamic feedback control law if it is solvable via a static feedback for the extended system (2.35). For the nonlinear system, the relative degree vector $[r_1\ r_2\ r_3\ r_4]$ is given by

$$r_1 = r_2 = r_3 = 4, \qquad r_4 = 2$$

And we have

$$[y_1^{(r_1)} \quad y_2^{(r_2)} \quad y_3^{(r_3)} \quad y_4^{(r_4)}]^T = \mathbf{b}(\bar{\mathbf{x}}) + \Delta(\bar{\mathbf{x}})\mathbf{u} \tag{2.38}$$

where $\Delta(\bar{x})$ and $b(\bar{x})$ are computed using equations (2.29) and (2.30). The matrix $\Delta(\bar{x})$ is nonsingular.

Therefore, the input-output decoupling problem is solvable for the system (2.24) by means of a dynamic feedback control law of the form:

$$\bar{u} = \alpha(\bar{x}) + \beta(\bar{x}) \cdot v \tag{2.39}$$

where $\alpha(\bar{x})$ and $\beta(\bar{x})$ are computed using (2.31) Recall the relation between u and $\bar{u}$ (2.33) and (2.34), we get the structure in Figure 2.4 for the control law of the original system (2.24).

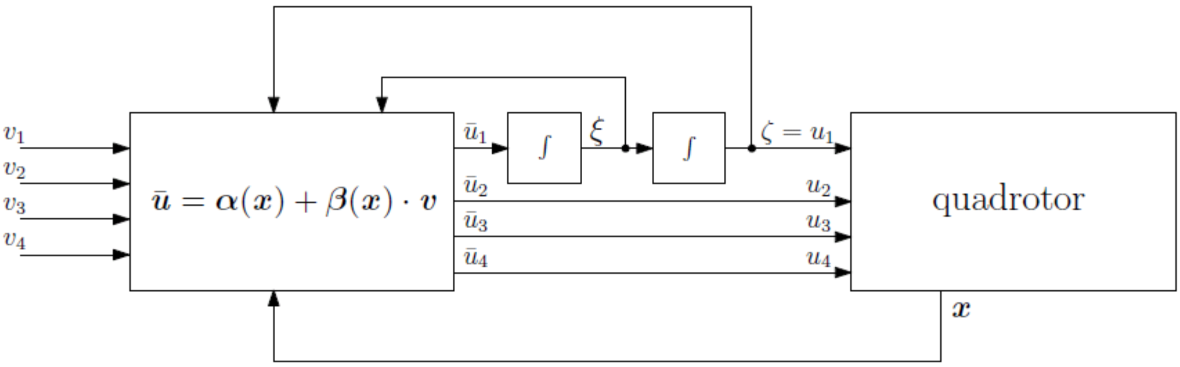

Figure 2.4: Block diagram of the control law.

Moreover, since the extended system (2.35) has dimension n = 14, the condition is fulfilled and, therefore, the system can be transformed via a dynamic feedback into a system which, in suitable coordinates, is fully linear and controllable. The change of coordinates $z = \Phi(\bar{x})$ is given by:

$$\left\{\begin{array}{c} z_1 = h_1(\mathrm{x}) = x \\ z_2 = L_f h_1(\mathrm{x}) = \dot{x} \\ z_3 = L_f^2 h_1(\mathrm{x}) = \ddot{x} \\ z_4 = L_f^3 h_1(\mathrm{x}) = x^{(3)} \\ z_5 = h_2(\mathrm{x}) = y \\ z_6 = L_f h_2(\mathrm{x}) = \dot{y} \\ z_7 = L_f^2 h_2(\mathrm{x}) = \ddot{y} \\ z_8 = L_f^3 h_2(\mathrm{x}) = y^{(3)} \\ z_9 = h_3(\mathrm{x}) = z \\ z_{10} = L_f h_3(\mathrm{x}) = \dot{z} \\ z_{11} = L_f^2 h_3(\mathrm{x}) = \ddot{z} \\ z_{12} = L_f^3 h_3(\mathrm{x}) = z^{(3)} \\ z_{13} = h_4(\mathrm{x}) = \psi \\ z_{14} = L_f h_4(\mathrm{x}) = \dot{\psi} \end{array}\right\} \tag{2.40}$$

In the new coordinates, the system appears as

$$\left\{\begin{array}{c} \dot{z} = Az + Bv \\ y = Cz \end{array}\right\} \tag{2.41}$$

where

$$\mathbf{Z} = [z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6 \quad z_7 \quad z_8 \quad z_9 \quad z_{10} \quad z_{11} \quad z_{12} \quad z_{13} \quad z_{14}]^T$$

$$\mathbf{V} = [v_1 \quad v_2 \quad v_3 \quad v_4]^T \in \mathbb{R}^4$$

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_1 & 0 & 0 & 0 \\ 0 & \mathrm{A}_1 & 0 & 0 \\ 0 & 0 & \mathrm{A}_1 & 0 \\ 0 & 0 & 0 & \mathrm{A}_2 \end{bmatrix} \in \mathbb{R}^{14\times 14}$$

$$\mathrm{B} = \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \\ \mathrm{B}_3 \\ \mathrm{B}_4 \end{bmatrix} \in \mathbb{R}^{14\times 4}$$

$$\mathrm{C} = \begin{bmatrix} \mathbf{c}_1^T & 0 & 0 & 0 \\ 0 & \mathbf{c}_1^T & 0 & 0 \\ 0 & 0 & \mathbf{c}_1^T & 0 \\ 0 & 0 & 0 & \mathbf{c}_2^T \end{bmatrix} \in \mathbb{R}^{4\times 14}$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \qquad \mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4} \qquad \mathbf{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$$\mathbf{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} \qquad \mathbf{B}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 4}$$

$$\mathbf{c}_1 = [1 \quad 0 \quad 0 \quad 0]^T \in \mathbb{R}^4 \qquad \mathbf{c}_2 = [1 \quad 0]^T \in \mathbb{R}^2$$

In the following figure, the scheme of the linear system is shown.

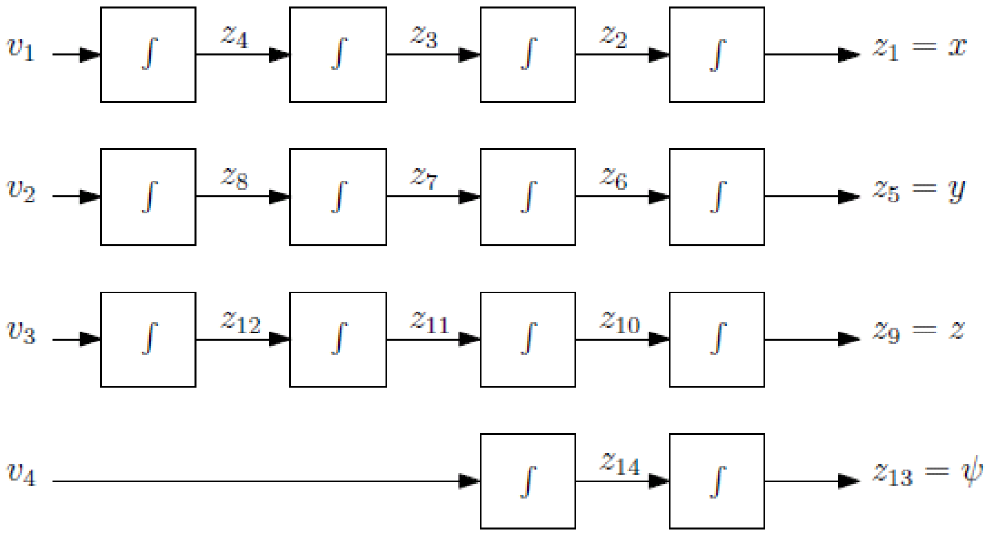

Figure 2.5: Block diagram of the closed loop system.

On the linear system (2.41), it is possible to impose a further control, using for example PD controller.

II.8 Robust Feedback Linearization :

Consider the multivariable nonlinear system with disturbance vector d given in (1.19):

$$\dot{x} = f(x) + g(x)u + p(x)\mathrm{d}$$
$$y = h(x)$$

where d $\in R^n$ is the noise and unknown perturbation vector, and d $= [\mathrm{d}_B \ \ \mathrm{d}_P]^T$, and where d_B is the noise vector of size 14. d_P is composed of aerodynamic forces disturbances $[A_x, A_y, A_z]^T$ and aerodynamic moment disturbances $[A_p, A_q, A_r]^T$. They act on the UAV and are computed from the aerodynamic coefficients C_i as $A_i = \rho_{air}\, C_i\, \Omega^2$ (ρ_{air} the air density, Ω is the velocity of the UAV with respect to the air), (C_i depends on several parameters like the angle between airspeed and the body fixed reference system, the aerodynamic and geometric form of the wing).
The robust feedback linearization method used in this context is based on Sobolev norm defined as:

$$\| h \|_W = [\int_0^\infty h^T(t)h(t)dt + \int_0^\infty \dot{h}^T(t)\dot{h}(t)dt]^{\frac{1}{2}} \qquad (2.42)$$

It transforms a nonlinear system into its tangent linearized system around an operating point. Then, under state feedback

$$u(x, w) = \alpha(x) + \beta(x)w \qquad (2.43)$$

and change of coordinates

$$z = \phi(x) \qquad (2.44)$$

defined by

$$\begin{aligned} \alpha(x) &= \alpha_c(x) + \beta_c(x)LT\phi_c(x) \\ \beta(x) &= \beta_c(x)R^{-1} \\ \phi(x) &= T^{-1}\phi_c(x) \end{aligned} \qquad (2.45)$$

where

$$L = -\Delta.\frac{\partial \alpha_c}{\partial x}|_{x=0}, T = \frac{\partial \phi_c}{\partial x}|_{x=0}, R = \Delta^{-1}$$
$$\alpha_c(x) = -\Delta^{-1}(x)b(x), \beta_c(x) = \Delta^{-1}(x)$$

then the nonlinear system is transformed into a following one

$$\dot{z} = Az + B_2 w + [\frac{\partial \emptyset}{\partial x} p(x)]_{x=\emptyset^{-1}(z)} \tag{2.46}$$

with

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}, \quad B_2 = g(0)$$

For the quadrotor helicopter the input-output decoupling problem is solvable for the nonlinear system by means of static feedback. The vector relative degree {r_1, r_2, r_3, r_4} is given by:

$$r_1 = r_2 = r_3 = 4; r_4 = 2$$

and we have

$$b(x) = [L_f^{r_1} h_1(x) \quad L_f^{r_2} h_2(x) \quad L_f^{r_3} h_3(x) \quad L_f^{r_4} h_4(x)]^T$$
$$\phi_c(x) = [\phi_{c1}(x), \phi_{c2}(x), \phi_{c3}(x), \phi_{c4}(x)]^T$$

$$\phi_{c1}(x) = \begin{bmatrix} h_1(x) = x_0 \\ L_f h_1(x) = x_7 = \dot{x}_0 \\ L_f^2 h_1(x) = \frac{A_x}{m} + g_1^7 x_{10} = \ddot{x}_0 \\ L_f^3 h_1(x) = \dddot{x}_0 \end{bmatrix}$$

$$\phi_{c2}(x) = \begin{bmatrix} h_2(x) = y_0 \\ L_f h_2(x) = x_8 = \dot{y}_0 \\ L_f^2 h_2(x) = \frac{A_y}{m} + g_1^8(x_4, x_5, x_6) x_{10} = \ddot{y}_0 \\ L_f^3 h_2(x) = \dddot{y}_0 \end{bmatrix}$$

$$\phi_{c3}(x) = \begin{bmatrix} h_3(x) = z_0 \\ L_f h_3(x) = x_9 = \dot{z}_0 \\ L_f^2 h_3(x) = \frac{A_z}{m} + g + g_1^9 x_{10} = \ddot{z}_0 \\ L_f^3 h_3(x) = \dddot{z}_0 \end{bmatrix}$$

$$\phi_{c4}(x) = \begin{bmatrix} h_4(x) = x_4 \\ L_f h_4(x) = \dot{x}_4 \end{bmatrix}$$

$$\Delta(x) = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \Delta_{24} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & \Delta_{44} \end{bmatrix}$$

with

$$\Delta_{11} = L_{g11} L_f^{r_1-1} h_1(x)$$

$$= -\frac{1}{m}(Cx_6 Cx_4 Sx_5 + Sx_6 Sx_4)$$

$$\Delta_{12} = L_{g2} L_f^{r_1-1} h_1(x)$$

$$= \frac{d}{mI_x}(x_{10} Sx_6 Cx_4 Sx_5 - x_{10} Cx_6 Sx_4)$$

$$\Delta_{13} = L_{g3} L_f^{r_1-1} h_1(x) = \frac{d}{mI_y}(-x_{10} Cx_4 Cx_5)$$

$$\Delta_{14} = 0$$

$$\Delta_{21} = L_{g1} L_f^{r_2-1} h_2(x)$$

$$= -\frac{1}{m}(Cx_6 Sx_5 Sx_4 - Cx_4 Sx_6)$$

$$\Delta_{22} = L_{g2} L_f^{r_2-1} h_2(x)$$

$$= \frac{d}{mI_x}(x_{10}Sx_6Sx_4Sx_5 + x_{10}Cx_6Cx_4)$$

$$\Delta_{23} = L_{g3}L_f^{r_2-1}h_2(x) = \frac{d}{mI_y}(-x_{10}Sx_4Cx_5)$$

$$\Delta_{24} = L_{g4}L_f^{r_2-1}h_2(x) = 0$$

$$\Delta_{31} = L_{g1}L_f^{r_3-1}h_3(x) = -\frac{1}{m}(Cx_5Cx_6)$$

$$\Delta_{32} = L_{g2}L_f^{r_3-1}h_3(x) = \frac{d}{mI_x}(x_{10}Sx_6Cx_5)$$

$$\Delta_{33} = L_{g3}L_f^{r_3-1}h_3(x) = \frac{d}{mI_y}(x_{10}Sx_5)$$

$$\Delta_{34} = L_{g4}L_f^{r_3-1}h_3(x) = 0$$

$$\Delta_{41} = L_{g1}L_f^{r_4-1}h_4(x) = 0$$

$$\Delta_{42} = L_{g2}L_f^{r_4-1}h_4(x) = 0$$

$$\Delta_{43} = L_{g3}L_f^{r_4-1}h_4(x) = \frac{d}{I_y}(Sx_6S_ex_5)$$

$$\Delta_{44} = L_{g4}L_f^{r_4-1}h_4(x) = \frac{1}{I_z}(Cx_6S_ex_5)$$

In fact the system in equation (2.45) is still nonlinear because of w vector. One seeks a controller which ensures the compensated system to be internally asymptotically stable and its output to tend asymptotically toward a desired trajectory even in the presence of external disturbance.

In this context the linear H_∞ is proposed.

Robust H∞ controller design

To ensure stability and performance against modelling errors, we have chosen the method of McFarlane-Glover to design a robust linear controller for the feedback linearized system. The method of loop-shaping is chosen due to its ability to address robust performance and robust stability in two different stages of controller design (McFarlane and Glover, 1990).

The method of loopshaping consists of three steps:

Step 1. Open loop shaping

Using a pre-weighting matrix W_I and/or a post-weighting matrix W_0 , the minimum and maxiumum singular values are modified to shape the response. This step results in an augmented matrix of the process transfer function:$P_S = W_0 P W_I$

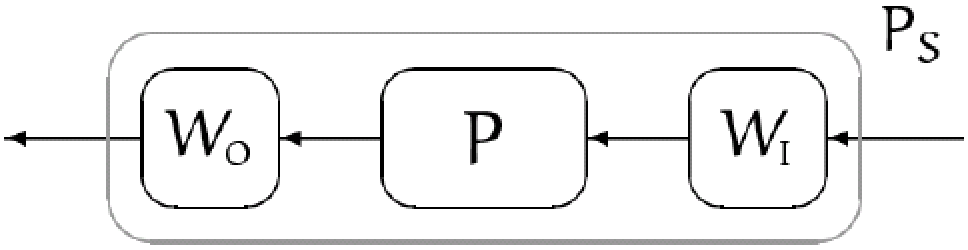

Figure 2.6 : Augmented matrix of the process transfer function

The stability margin is computed as

$$\frac{1}{\varepsilon_{\max}} = \inf_{K \text{ stabilizator}} \| \begin{bmatrix} I \\ K \end{bmatrix} (I - P_s K)^{-1} \tilde{M}_s^{-1} \|_\infty \quad (2.47)$$

Where

$$P_s = \tilde{M}_s^{-1} \tilde{N}_s$$

which is the normalized left coprime factorization of the process transfer function matrix. If $\mathcal{E}_{max} \ll 1$, the pre and post weighting matrices have to be modified by relaxing the constraints imposed on the open loop shaping. If the value of $\mathcal{E}_{max}$ is acceptable, for a value max $\mathcal{E} < \mathcal{E}_{max}$ the resulting controller - *Ka* - is computed in order to sati1sfy the following relation:

$$\left\| \begin{bmatrix} I \\ K_a \end{bmatrix} (I - P_s K_a)^{-1} \tilde{M}_s^{-1} \right\|_\infty \leq \varepsilon \tag{2.48}$$

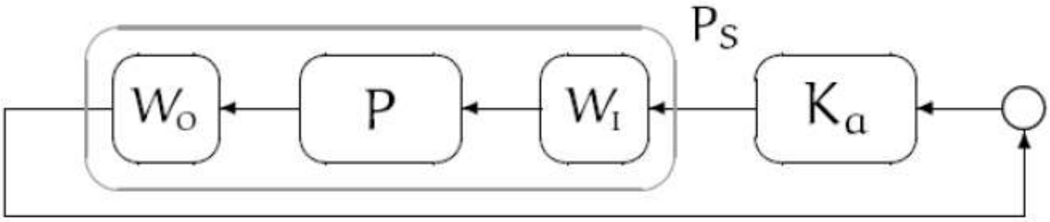

Figure 2.7: Robust closed loop control scheme

Step 3. Final robust controller

The final resulting controller is given by the sub-optimal controller *Ka* weighted with the matrices W_I and/or W_I : *K* = *WIKaWo* .

Using the McFarlane-Glover method, the loop shaping is done without considering the problem of robust stability, which is explcitily taken into account at the second design step, by imposing a stability margin for the closed loop system. This stability margin max ε is an indicator of the efficiency of the loopshaping technique.

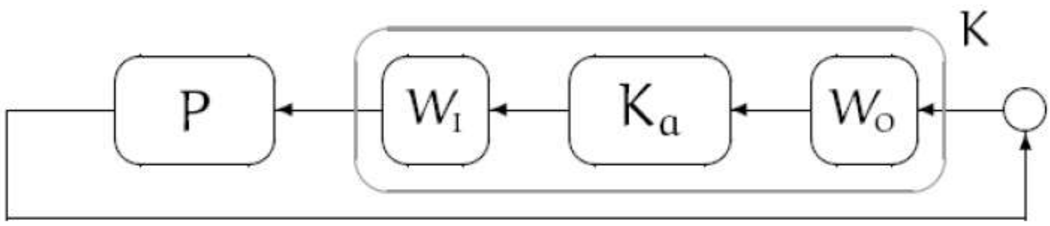

Figure2.8: Optimal controller obtained with the pre and post weighting matrices

Results and discussion

In order to verify the effectiveness of the proposed control law, the overall system is tested in numerical simulations. The physical parameters for quadrotor are:

Ix = 0.62 N m s^2, Iy = 0.62 N m s^2, Iz = 1.24 N m s^2, m = 1 kg, g = 9.81 m/s^2.

The reference trajectory chosen for $x_d(t), y_d(t), z_d(t) and\ \psi_d(t)$ is

$$x_d(t) = \cos(0.5t)$$
$$y_d(t) = \sin(0.5t)$$
$$z_d(t) = 0.5t$$
$$\psi_d(t) = 0$$

The initial conditions are: $x_d(0) = 0.5m$.

$y_d(0) = 0$ m , $z\ d$(0) = 0 m and ψ_d(0) = 0 rad. All other initial conditions are zero. To test the robustness of the controller, disturbances have been introduced. The most likely disturbance acting on the quadrotor is wind in horizontal plane, which can be modeled by forces d_{mx}, d_{my} chosen as

$$d_{mx}(t) = 1.5 + 2.5sin(4t)$$
$$d_{my} = 2.5 + 1.5\sin(3t)$$

All other external disturbances are set to zero.

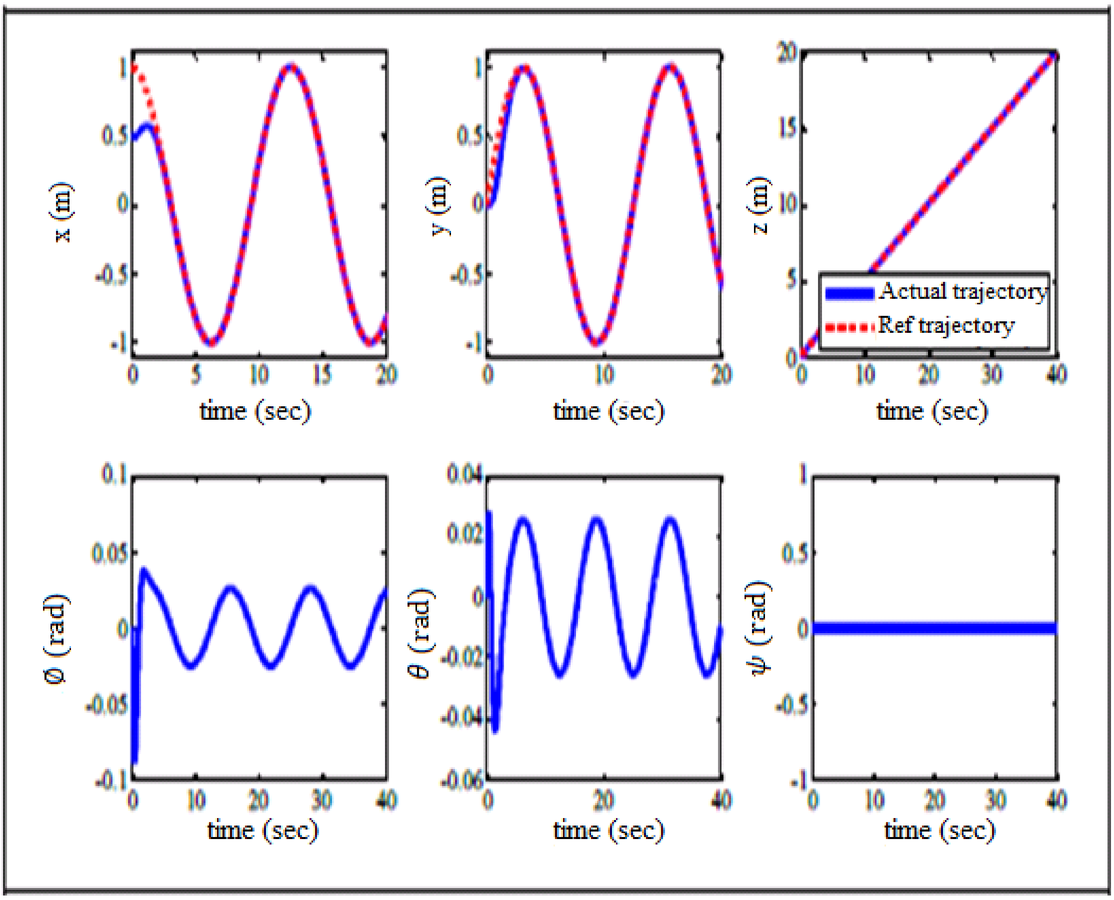

Figure 3,1: The position and attitude of quadrotor in the closed-loop with Feedback Linearization control (the case without external disturbances)

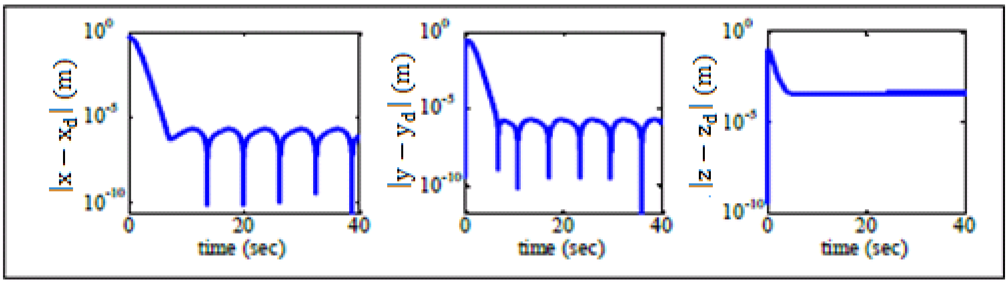

Figure 3.2: The position errors of quadrotor in the closed-loop with feedback Linearization (the case without external disturbances).

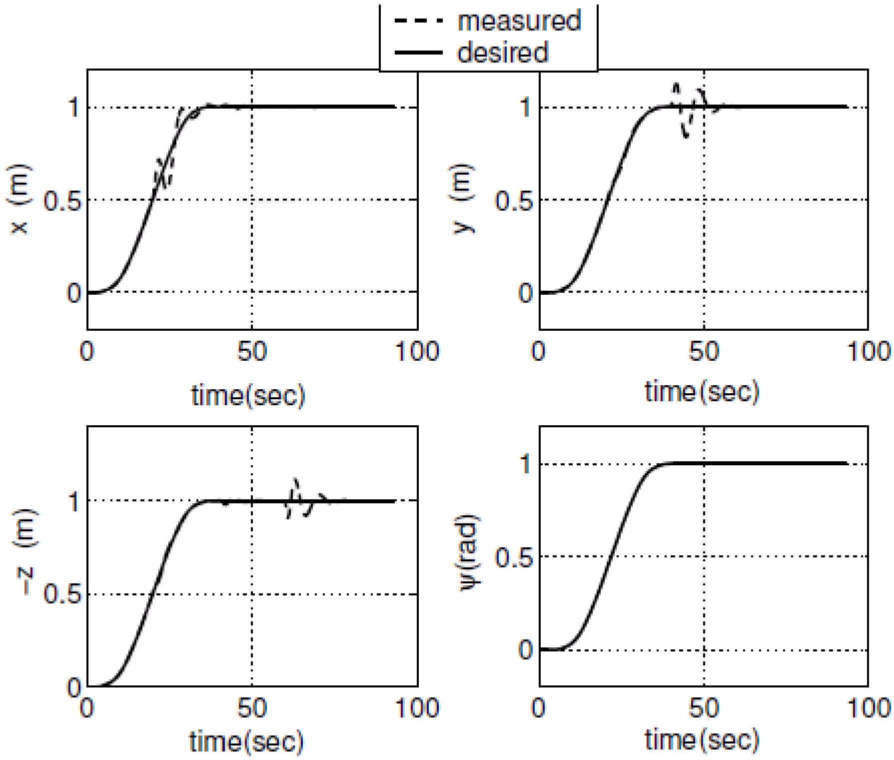

Figure 3.3: Trajectories x, y, z *and* ψ with the robust feedback linearization control.

We can see that tracking simulation results for both classical and robust feedback linearization approach for the case without external disturbances show convergence toward reference trajectory.
Choosing another trajectory for x_0 that has initial point 0.55, and see how much our system is sensitive to parameter variation in both classical and robust feedback linearization approach.

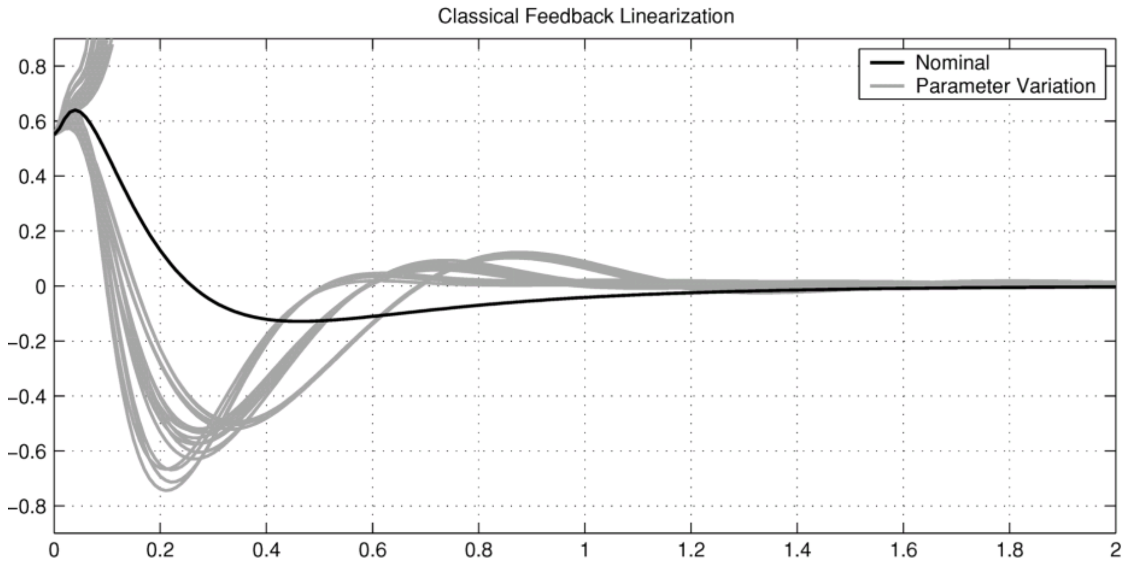

Figure 3.4: Position x_0 for the classical feedback linearization.

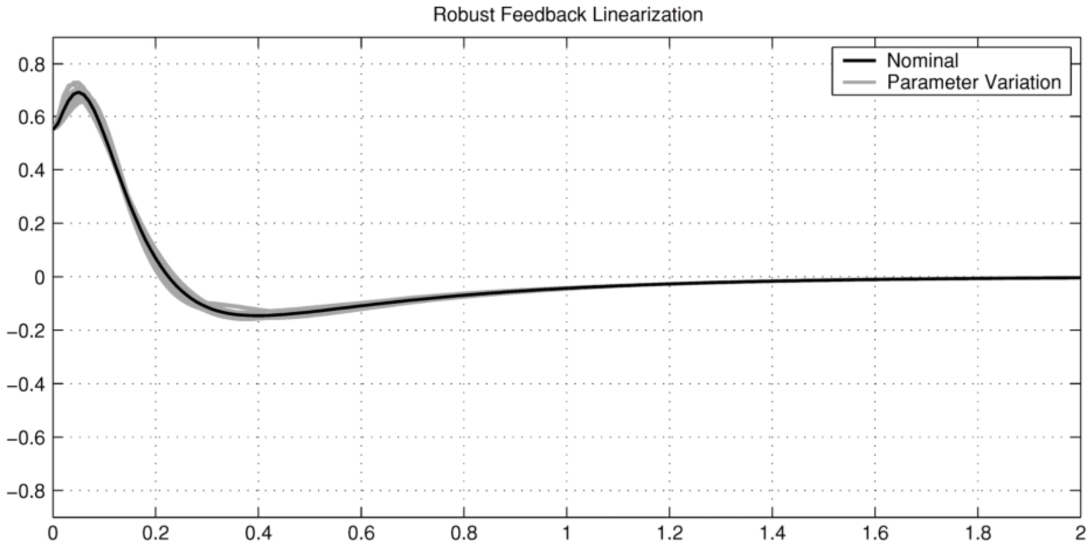

Figure 3.5: Position x for the robust feedback linearization.

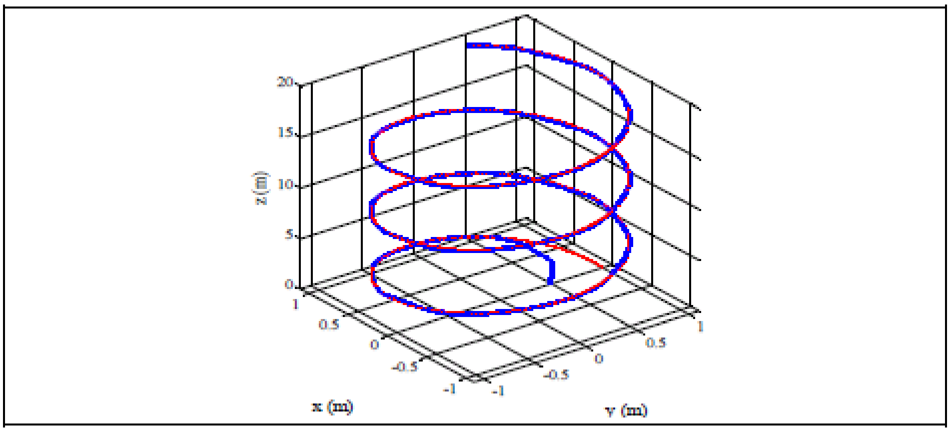

Figure 3.6: The quadrotor and reference trajectory for case with Robust Feedback Linearization, with external disturbances

It is clear that with robust feedback linearization our system is more robust to external disturbances.

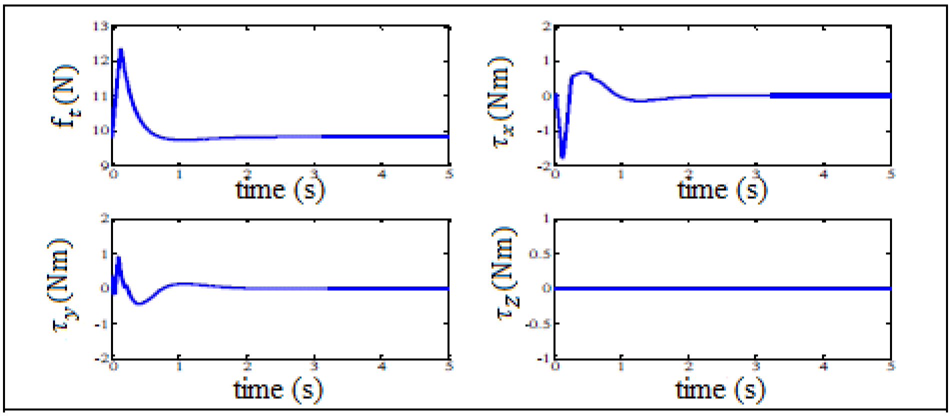

Figure3.7: The force and torques of quadrotor in the closed-loop without external disturbances.

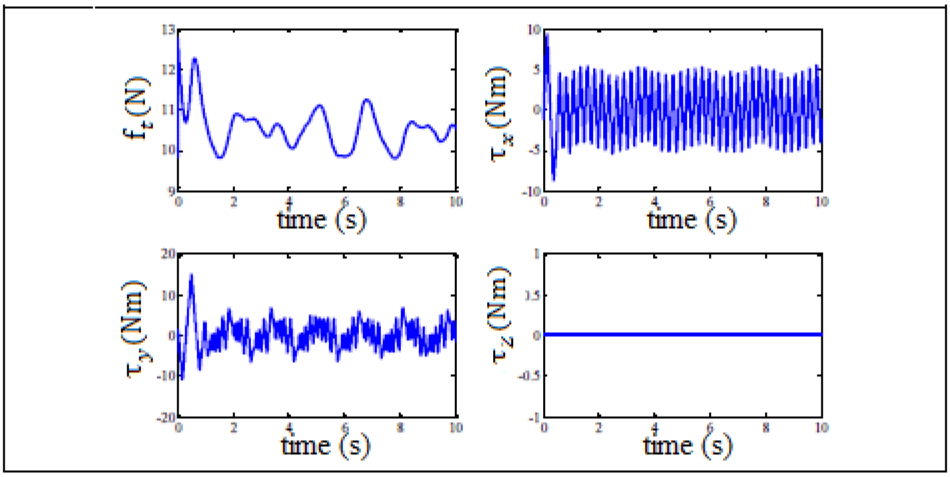

Figure3.8: The force and torques of quadrotor in the closed-loop with external disturbances.

The last two figures show the force as well as the torques of quadcopter with both no and with external disturbances.

Discussion:

As shown theoretically in this thesis and illustrated by the simulations in the last chapter about Simulation and Results, a robust nonlinear controller for the nonlinear system is obtained by using the robust feedback linearization associated with a McFarlane–Glover H_∞ controller. This does not hold when the classical feedback linearization is used due to the fact the linearized system obtained by feedback linearization is in the Brunovsky form, a non robust form whose dynamics is completely different from that of the original system and which is highly vulnerable to uncertainties

Conclusion

As it has been previously demonstrated theoretically through mathematical computations (Guillard, *et al*., 2000), the results in this project prove that by combining the robust method of feedback linearization with a robust linear controller, the robustness properties are kept when simulating the closed loop nonlinear uncertain system. Additionally, the design of the loop-shaping controller is significantly simplified as compared to the classical linearization technique, since the final linearized model bears significant information regarding the initial nonlinear model. Finally, it is shown that robust nonlinear controller - designed by combining this new method for feedback linearization (Guillard & Bourles, 2000) with a linear $H\infty$ controller - offers a simple and efficient solution, both in terms of reference tracking and input disturbance rejection.

The dynamics of a quadrotor is a simplified form of helicopter dynamics that exhibits the basic problems including underactuation, strong coupling, multi-input/multi-output. The derived controller is capable of dealing with such problems simultaneously and satisfactorily. As the quadrotor model discussed in this thesis is similar to a full-scale, unmanned helicopter model, the same control configuration derived for a quadrotor is also applicable for a helicopter model. The simulation results with and without input disturbances are shown in this project. Some aspects still remain untouched. The controller shows high sensitivity to state disturbances, which may be in the interest of future research that may be considered as an enhancement to what is discussed in this project.

List of figures:

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People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

University M'Hamed BOUGARA - Boumerdes

Institute of Electrical and Electronic Engineering

Department of Power and Control

Authorization for Final Year Project Defense

Academic year: 2016/2017

The undersigned supervisor: KESSAL FARIDA
authorizes the student(s):

SAIDI Lakhdar Option: Control

Option:

Option:

to defend his / her / their final year Master programme project entitled:

Robust Feedback Linearization of a quadrotor

during the ☒ June ☐ September session.

Date: 04/06/2017.

The Supervisor

The Department Head

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