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First Order Linear Equations On Time Scales

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DEDICATION

I dedicate this thesis to the whole Nechikwira family: Mr Watson Nechikwira my father, Chipo Murau my mother and my lovely siblings Rudo, Ndiriwenyu, Nyasha and Vimbai. They have shaped who I am today and have allowed me to pursue my intellectual dreams by providing unwavering support and love.

We must know, we shall know.

– David Hilbert

ABSTRACT OF THE THESIS

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Unification and extension of the concepts of difference calculus with that of differential calculus led to the development of time scale calculus, by which we can recover, as special cases the former and the latter concepts, plus more. We consider First Order Linear Dynamic Equations (F.O.L.D.Es) on an arbitrary time scale. These F.O.L.D.Es have as special cases the ordinary differential equations, difference equations and many others e.g *q-difference equations*. We begin by taking a step back and explain what a time scale is, covering the main theorems and special properties of functions on a time scale. We then introduce the exponential function for a time scale including its properties, and use these properties to derive solutions for F.O.L.D.Es with constant coefficients. Several examples and applications, among them an insect population model, and the logistics equation are considered.

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INTRODUCTION

Mathematicians are interested generalizing seemingly different mathematical concepts into a unifying framework. Hence it is not unexpected that, after noticing a lot of well-known multiple analogies in the concepts of difference calculus with the difference operator

$$(\Delta_h f)(t) = \frac{f(t+h) - f(t)}{h}$$

on one hand and the differential calculus with the differential operator

$$(\frac{d}{dt} f)(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

on the other hand, mathematicians developed a higher ranging calculus called time scale calculus or measure chains. The theory of time scales was introduced by Stefan Hilger in his PhD thesis [2] in 1988 (supervised by Bernd Aulbach) inorder to unify continuous and discrete calculus. Since then the work has received a lot of attention from researchers and now it not only unifies the continuous with the discrete, but also extends them both, for example in modeling hybrid systems (for example a bouncing ball) that are a combination of both discrete and continuous calculus.

In this thesis, we focus on solving First Order Linear Dynamic Equations that we abbreviate (F.O.L.D.Es) on time scales. To do this we first develop the necessary background which means defining what a time scale is, its fundamental theorems and an extensive treatment of examples that apply these theorems. After setting the ground we dive into F.O.L.D.Es and prove results that work for an arbitrary time scale. Here we see the main advantage of time scale calculus in play– it allows us to prove results for dynamic equations once, instead of proving for the discrete and continuous cases separately. To summarize, the goal is to establish a result for a dynamic equation in which the domain of the unknown function is a time scale, **which is an any nonempty closed subset of the reals**. By making the time scale a set of real numbers, the general result offers a result for an ordinary differential equation, as studied in a first differential equations course. Choosing the time scale to be the set of integers, on the other hand, obtains the same basic result for difference equations. However, because there are many more time scales than just the set of real numbers or the set of integers, the result is far more broad.

We have set up the thesis as follows: In **Chapter 1** *the time scale calculus* is introduced covering the main definitions and theorems. Chapter 1 comprises of 7 sections as seen in the table of contents. We begin by defining a time scale $\mathbb{T}$ as an arbitrary nonempty closed subset of the real numbers. In order to rigorously describe the distribution of points on time scales we define the forward jump $\sigma(t)$ and the backward jump $\rho(t)$ that define the 'closest' point to the right and left respectively on a time scale. For functions $f : \mathbb{T} \to \mathbb{R}$ we introduce the so-called Hilger derivative on an arbitrary time scale. Fundamental results on Hilger derivatives are presented e.g. the product rule and the quotient rule. To demonstrate these theorems in action, we treat important examples of time scales in section 1.3. Such examples contain of course $\mathbb{R}$ (which gives rise to differential equations), the set of all integer multiples of a number $h > 0$ ($h\mathbb{Z}$) and the set of Harmonic numbers H_k. Other examples include the quantum numbers which are integer powers of a number $q > 1$, including 0. These give rise to quantum calculus or q-difference equations see, e.g., [9].

In section 1.4 we introduce an integral on a time scale and treat results concerning integrability as they are needed in Chapter 2 in order to solve F.O.L.D.Es. As expected we treat interesting examples for example an integral on disjoint intervals that reveals that time scales can extend continuous calculus to more sophisticated sets.

In section 1.5 we derives analogues of the chain rule on a time scale, where we realize that the ordinary chain rule from calculus does not apply. We end the chapter by presenting Taylor's formula and L'Hôpital's Rule, the former being helpful in the study of boundary value problems.

In **Chapter 2** we consider F.O.L.D.Es. In order to solve a F.O.L.D.E, we introduce the Hilger complex plane which generalizes the common complex plane. allows us to define the so-called cylinder transformation. The cylinder transformation plays a crucial role since it is used to define the exponential function on time scales. It turns out that this exponential function indeed verifies an initial value problem involving a F.O.L.D.E. The natural thing is to extensively study this exponential function hence we prove its interesting properties and use them to solve all initial value problems involving *first order linear dynamic equations.* We cover both homogeneous and non-homogeneous F.O.L.D.Es, and we employ a time scale equivalent of the variation of constants technique.

We finish the chapter by applying the theory to real life applications, mainly using the theory to solve the time scale version of the popular logistic growth equation $y' = ry\left(1 - \frac{y}{K}\right)$.

CHAPTER 1

Time scales essentials

1.1 Basic Definitions

Before delving deep into solving first order linear equations on time scales, we need to know what a time scale is, including some basic properties of functions on them. As a result, we begin this chapter by giving the basic definitions that will be used in this work, and we consider the main theorems that will be used.

Definition 1.1. *A time scale is an arbitrary nonempty closed subset of the real numbers.*

Thus $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, i.e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales, as are

$$[0,1] \cup [2,3],\ [0,1] \cup \mathbb{N},\ \text{and the Cantor set},$$

while

$$\mathbb{Q},\ \mathbb{R}\backslash\mathbb{Q},\ \mathbb{C},\ (0,1),$$

i.e, the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1 are *not* time scales. Throughout this thesis, we will denote a time scale by the symbol $\mathbb{T}$. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

The calculus of time scales was initiated by Stefan Hilger [2] in order to create a theory that can unify discrete and continuous analysis. Indeed, we will introduce the delta (Hilger) derivative f^{Δ} for a function f defined on $\mathbb{T}$, and it turns out that

(i) $f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$.

(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

In this section we introduce the basic notions connected to time scales and differentiability of functions on them. However, the general theory is of course applicable to many more time scales $\mathbb{T}$, and we will treat some examples of such time scales in Section 1.3. So let us start by defining the forward and backward jump operators.

The *forward jump* and *backward jump* operators represent the closest point in the time scale on the right and left of a given point t, respectively. Formally:

Definition 1.2. *Let* $\mathbb{T}$ *be a time scale. For* $t \in \mathbb{T}$ *we define the forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. If $\sigma(t) > t$ we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time

Table 1.1. Classification of points.

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

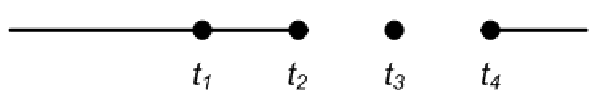

Figure 1.1. Classification of points

are called *isolated*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-dense and left-dense at the same time are called *dense*. Finally, the *forward graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

See Table 1.1 for a classification of points in $\mathbb{T}$. As illustrated by Figure 1.1 (a schematic classification of points in $\mathbb{T}$):

- Point t_1 is *dense*

- Point t_2 is *left-dense* and *right-scattered*
- Point t_3 is *isolated*
- Point t_4 is *left-scattered* and *right-dense*

Note that in the definition above both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$. We also need below the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. In summary

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Similarly if $\mathbb{T}$ has a right-scattered minimum m, then we can define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$.

Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we defined the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \text{ for all } t \in \mathbb{T},$$

i.e., $f^\sigma = f \circ \sigma$.

Example 1.3. *Let us briefly consider the two examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf(t, \infty) = t$$

and

$$\rho(t) = \sup\{s \in \mathbb{R} : s < t\} = \sup(-\infty, t) = t$$

. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) \equiv 0 \text{ for all } t \in \mathbb{T}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, t+3, \cdots\} = t+1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$$\mu(t) \equiv 1 \text{ for all } t \in \mathbb{T}.$$

For the two cases discussed above, the graininess function is a constant function.

Of course, for the case of a general time scale, the graininess function might very well be a function of $t \in \mathbb{T}$, as can be seen in a few examples treated below.

Example 1.4. *For each of the following time scales* $\mathbb{T}$, *find* σ, ρ *and* μ, *and classify each point* $t \in \mathbb{T}$ *as left-dense, left-scattered, right-dense or right-scattered:*

(i) $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$;

(ii) $\mathbb{T} = \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$

Solution

(i) The set $\mathbb{T} = \{2^n : n \in \mathbb{Z}\} \cup \{0\}$ contains the following elements: all multiples of 2, fractions of the form $\frac{1}{2^n}$ for negative integers and 0. It can be easily seen that the forward jump $\sigma(2^n) = 2^{n+1}$ e.g $\sigma(2^2) = 2^{2+1} = 8$, while the backward jump $\rho(2^n) = 2^{n-1}$. Taking $t = 2^n$ we get

$$\sigma(t) = 2^{n+1} = 2 \cdot 2^n = 2t$$

Similarly $\rho(t) = \frac{t}{2}$ and finally for the point $t = 0$ we have $\sigma(0) = 0 = \rho(0)$. Hence the point $t = 0$ is both left-dense and right-dense whilst the rest of the points are isolated.

(ii) With the same reasoning as in (i), the set $\mathbb{T} = \{\sqrt[3]{n} : n \in \mathbb{N}_0\} = \{0, \sqrt[3]{1}, \sqrt[3]{2}, \ldots\}$. We find that $\sigma(\sqrt[3]{n}) = \sqrt[3]{n+1}$. And taking $t = \sqrt[3]{n}$ we get $\sigma(t) = \sqrt[3]{t^3+1}$. The graininess $\mu(t) = \sqrt[3]{t^n+1} - t$. One can easily find a formula for the backward jump and finally we see that all the points in this set are isolated.

In time scales,we have the following *induction principle* which is a useful tool when proving some important theorems as we shall see in later sections.

Theorem 1.5. *(Induction Principle). Let* $t \in \mathbb{T}$ *and assume that*

$$\{S(t) : t \in [t_0, \infty)\}$$

is a family of statements satisfying:

(i) The statement $S(t_0)$ *is true.*

(ii) If $t \in [t_0, \infty)$ *is right-scattered and* $S(t)$ *is true, then* $S(\sigma(t))$ *is also true.*

(iii) If $t \in [t_0, \infty)$ *is right-dense and* $S(t)$ *is true, then there is a neighborhood* U *of* t *such that* $S(s)$ *is true for all* $s \in U \cap (t, \infty)$.

(iv) If $t \in (t_0, \infty)$ is left-dense and $S(s)$ is true for all $s \in [t_0, \infty)$, then $S(t)$ is true.

Then $S(t)$ is true for all $t \in [t_0, \infty)$

Proof. Let

$$S^* := \{t \in [t_0, \infty) : S(t) \text{ is not true}\}.$$

We want to show $S^* = \emptyset$. To achieve a contradiction we assume $S^* \neq \emptyset$. But since S^* is closed and nonempty, we have

$$\inf S^* := t^* \in \mathbb{T}.$$

We claim that $S(t^*)$ is true. If $t^* = t_0$, then $S(t^*)$ is true from (i). If $t^* \neq t_0$ and $\rho(t^*) = t^*$, i.e. if t^* is left-dense, then $S(t^*)$ is true from (iv). Finally if $\rho(t^*) < t^*$ then $S(\rho(t^*))$ is true because t^* is defined as the minimum of all t for which $S(t)$ is false. Then from property (ii) it follows that $S(\sigma(\rho(t^*))$ is true (note $\rho(t^*)$ is right-scattered because $\rho(t^*) < t^*$). But $\sigma(\rho(t^*)) = t^*$ Hence, in any case,

$$t^* \notin S^*.$$

Thus, t^* cannot be right-scattered, and $t^* \neq \max \mathbb{T}$ either. Hence t^* is right-dense. But now (iii) leads to a contradiction since we can find an s such that $S(s)$ is true and at the same time $s \in S^*$. □

Remark 1.6. *A dual version of the induction principle also holds for a family of statements $S(t)$ for t in an interval of the form $(-\infty, t_0]$. The proof is similar to that of the previous theorem.*

1.2 Differentiation

We define what is a derivative of a function on an arbitrary time scale. The derivative can be defined using the forward jump operator σ as well as the backward jump operator ρ. However, in this thesis we will be using the definition with the forward jump operator only since the results will be analogous with the backward jump operator.

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called *delta* (or *Hilger*) *derivative* of f at a point $t \in \mathbb{T}^\kappa$.

Definition 1.7. *(Bohner and Peterson [1]) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define f^Δ to be the number (provided it exists) with the property that* *given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \text{ for all } s \in U.$$

We call f^{Δ} the delta (or Hilger) derivative of f at t.

Moreover, we say that f is delta (or Hilger) differentiable (or in short: differentiable) on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. The function $f^{\Delta} : \mathbb{T} \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^{\kappa}$.

Example 1.8. *(i) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \alpha$ for all $t \in \mathbb{T}$, where $\alpha \in \mathbb{R}$ is a constant, then $f^{\Delta}(t) \equiv 0$. This is clear because for any $\varepsilon > 0$,*

$$|f(\sigma(t)) - f(s) - 0.[\sigma(t) - s]| = |\alpha - \alpha| \leq \varepsilon|\sigma(t) - s|$$

holds for all $s \in \mathbb{T}$.

(ii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) \equiv 1$. This follows since for any $\varepsilon > 0$,

$$|f(\sigma(t)) - f(s) - 1 \cdot [\sigma(t) - s]| = |\sigma(t) - s - (\sigma(t) - s)| = 0 \leq \varepsilon|\sigma(t) - s|$$

holds for all $s \in \mathbb{T}$.

Remark 1.9. *It should be noted that the forward jump operator σ and the backward jump operator ρ are not always differentiable. We will show as an example that the forward jump operator is not always differentiable at the end of Section* 1.3

Some easy and useful relationships concerning the delta derivative are given next.

Theorem 1.10. *(Bohner and Peterson [1]) Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

Proof. See [[1], Theorem 1.16, p. 5] □

Example 1.11. *Again we consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.*

(i) If $\mathbb{T} = \mathbb{R}$, then Theorem 1.10 (iii) yields that $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \quad \text{exists,}$$

i.e., if and only if f is differentiable (in the ordinary sense) at t. In this case we then have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

by Theorem 1.10 (iii).

(ii) If $\mathbb{T} = \mathbb{Z}$, then Theorem 1.10 property (ii) yields that $f : \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t),$$

where Δ is the usual forward difference operator defined by the last equation above.

Let us treat two examples that apply Theorem 1.10 to find f^{Δ} for:

(i) $f(t) = t^2$ for $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{2}} := \{\sqrt{n} : n \in \mathbb{N}_0\}$;

(ii) $f(t) = t^3$ for $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{3}} := \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$.

Solutions

(i) For $f(t) = t^2$, we have $\sigma(t) = \sqrt{t^2 + 1}$. Hence

$$\begin{aligned} f^{\Delta}(t) &= \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \\ &= \frac{f(\sqrt{t^2+1}) - f(t)}{\sqrt{t^2+1} - t} \\ &= \frac{t^2 + 1 - t^2}{\sqrt{t^2+1} - t} \\ &= \sqrt{t^2+1} + t, \ t \in \mathbb{N}_0^{\frac{1}{2}}, \text{after rationalising the denominator} \end{aligned}$$

(ii) By a similar approach we get $f^{\Delta} = (t^3+1)^{\frac{2}{3}} + t(t^3+1)^{\frac{1}{3}} + t^2,\ t \in \mathbb{N}_0^{\frac{1}{3}}$.

As expected, we would like to be able to find the derivatives of sums, products, and quotients of differentiable functions. This is possible according to the following theorem.

Theorem 1.12. *(Bohner and Peterson [1]) Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) For any constant $\alpha, \alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then g is differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Proof. We will prove parts (ii) and (iv). Assume that f and g are delta differentiable at $t \in \mathbb{T}^{\kappa}$.

Part (ii) If $\alpha = 0$ then we are done. Suppose $\alpha \neq 0$. Let $\varepsilon > 0$. Define $\varepsilon^* = \frac{\varepsilon}{|\alpha|}$. Then $\varepsilon^* > 0$ and hence there exist a neighborhood U of t such that for all $s \in U$

$$\begin{aligned}
&\left|(\alpha f)(\sigma(t)) - (\alpha f)(s) - \alpha f^{\Delta}(t)(\sigma(t) - s)\right| \\
&\quad = |\alpha| \left|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| \\
&\quad \leq |\alpha| \varepsilon^* |\sigma(t) - s| \\
&\quad = \varepsilon |\sigma(t) - s|
\end{aligned}$$

Therefore αf is differentiable at t and $(\alpha f)^{\Delta} = \alpha f^{\Delta}$ holds at t.

Part (iv) Is a simple application of part (iii) and using the fact that $1^\Delta = 0$. In fact, since $(\frac{1}{f} \times f) = 1$ we use the product rule to get

$$1^\Delta = 0 = \left(\frac{1}{f}\right)^\Delta (t)f(t) + \frac{1}{f(\sigma(t))} f^\Delta(t)$$

and we get

$$\left(\frac{1}{f}\right)^\Delta (t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}$$

hence the result.

□

To illustrate the theorem 1.12 in action, we solve a few examples below.

Example 1.13. 1. *Prove that if x, y, and z are delta differentiable at t, then*

$$(xyz)^\Delta = x^\Delta yz + x^\sigma y^\Delta z + x^\sigma y^\sigma z^\Delta$$

holds at t. Write down the generalization of this formula for n functions.

2. *We have by Theorem 1.12 (iii)*

$$\left(f^2\right)^\Delta = (f \cdot f)^\Delta = f^\Delta f + f^\sigma f^\Delta = (f + f^\sigma) f^\Delta. \tag{1.1}$$

Give the generalization of this formula for the derivative of the $(n+1)$ st power of $f, n \in \mathbb{N}$, i.e., for $(f^{n+1})^\Delta$.

Solutions

1. If x, y, and z are delta differentiable at t, we simply apply the product rule in two stages. We know that for two differentiable functions, $(yz)^\Delta = y^\Delta z + y^\sigma z^\Delta$ hence;

$$\begin{aligned}(xyz)^\Delta &= x^\Delta (yz) + x^\sigma (yz)^\Delta \\ &= x^\Delta yz + x^\sigma [y^\Delta z + y^\sigma z^\Delta]\end{aligned}$$

and we get the desired result. Now for n functions, the trick is to write down the product formula as we show below. To see the pattern we write the formula for 4 functions $(f_1 \cdots f_4)$. We have

$$(f_1 \cdots f_4)^\Delta = f_1^\Delta f_2 f_3 f_4 + f_1^\sigma f_2^\Delta f_3 f_4 + f_1^\sigma f_2^\sigma f_3^\Delta f_4 + f_1^\sigma f_2^\sigma f_3^\sigma f_4^\Delta.$$

Its easy to notice that we get f_j^σ's for $j < i$ where i is the indice in the function we are taking the delta derivative i.e f_i^Δ, and f_i's when $j > i$. Putting this in symbols, for this example ($n = 4$) we get (note that the indice j should start from 1)

$$(f_1 \cdots f_4)^\Delta = \sum_{i=1}^{4} f_i^\Delta \left(\prod_{j<i, j\neq i}^{4} f_j^\sigma \right) \left(\prod_{j>i, j\neq i}^{4} f_j \right)$$

By direct mathematical induction on n we get, for n functions,

$$(f_1 \cdots f_n)^\Delta = \sum_{i=1}^{n} f_i^\Delta \left(\prod_{j<i, j\neq i}^{n} f_j^\sigma \right) \left(\prod_{j>i, j\neq i}^{n} f_j \right)$$

2. The trick to tackle these type of exercises is writing down the formulae for a few powers of f and then noticing the pattern. We write down the formula for $(f^4)^\Delta$ which is $(f^4)^\Delta = f^\Delta(f^3 + f^2 f^\sigma + f(f^\sigma)^2 + (f^\sigma)^3)$. For this case we realize that

$$(f^{3+1})^\Delta = f^\Delta \left\{ \sum_{k=0}^{3} f^k (f^\sigma)^{3-k} \right\}$$

which can be generalized (and proved by induction on n) to

$$(f^{n+1})^\Delta = f^\Delta \left\{ \sum_{k=0}^{n} f^k (f^\sigma)^{n-k} \right\}, n \in \mathbb{N}$$

and we are done.

Up to now, we have defined and considered differentiability theorems using the forwad jump σ. The name for these derivatives being Delta (Δ) derivatives. But it is worth knowing that the same theorems and definitions can be analogously defined using the backward jump ρ and we call these the Nabla (∇) derivative. In fact, we provide these definitions and analogues of the differentiability theorems in this section in the Appendix of this thesis, basing on the work of Duke and Elizabeth R in [11].

The following theorem provides the delta derivative of a simple polynomial on an arbitrary time scale.

Theorem 1.14. *(Bohner and Peterson [1]) Let α be constant and $m \in \mathbb{N}$.*

(i) For f defined by $f(t) = (t-\alpha)^m$ we have

$$f^\Delta(t) = \sum_{\nu=0}^{m-1} (\sigma(t) - \alpha)^\nu (t-\alpha)^{m-1-\nu}.$$

(ii) For g defined by $g(t) = \frac{1}{(t-\alpha)^m}$ we have

$$g^\Delta(t) = -\sum_{\nu=0}^{m-1} \frac{1}{(\sigma(t)-\alpha)^{m-\nu}(t-\alpha)^{\nu+1}},$$

provided $(t-\alpha)(\sigma(t)-\alpha) \neq 0$.

Proof. See ([1], Theorem 1.24, p. 9) □

Remark 1.15. *Theorem 1.14 show that polynomials on time scales are likely to have at most 1 delta derivative due to the fact that $\sigma(t)$ is not always differentiable.*

Example 1.16. *The derivative of t^2 is*

$$t + \sigma(t).$$

and the derivative of $1/t$ is

$$-\frac{1}{t\sigma(t)}.$$

Below is a theorem that one can use to find the nth derivative under certain conditions. However this will not be our focus since the main goal is solving first order equations hence derivative of at most 1 order!

Theorem 1.17. *(Bohner and Peterson [1]). Let $S_k^{(n)}$ be the set consisting of all possible strings of length n, containing exactly k times σ and $n-k$ times Δ. If*

$$f^{\Lambda} \quad \text{exists for all } \Lambda \in S_k^{(n)},$$

then

$$(fg)^{\Delta^n} = \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\Delta^k} \tag{1.2}$$

holds for all $n \in \mathbb{N}$.

The formula 1.2 is referred to as the Leibniz formula on time scales.

Proof. See ([1], Theorem 1.32, p. 11) □

Example 1.18. *If $\mathbb{T} = \mathbb{R}$, then*

$$f^{\Lambda} = f^{(n-k)} \quad \text{for all} \quad \Lambda \in S_k^{(n)},$$

where $f^{(n)}$ denotes the nth (usual) derivative of f, if it exists, and since

$$\left| S_k^{(n)} \right| = \binom{n}{k},$$

where $|M|$ denotes the cardinality of the set M, we have

$$\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} = \sum_{\Lambda \in S_k^{(n)}} f^{(n-k)} = f^{(n-k)} \sum_{\Lambda \in S_k^{(n)}} 1 = \binom{n}{k} f^{(n-k)}$$

and therefore

$$(fg)^{\Delta^n} = \sum_{k=0}^{n} \left(\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right) g^{\Delta^k} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) f^{(n-k)} g^{(k)}.$$

This is the usual Leibniz formula that we know from calculus.

Remark 1.19. *We can use the notation* $\Delta^n(fg)$ *to represent* $(fg)^{\Delta^n}$

1.3 Examples of Time Scales

We treat a few examples to motivate what has been covered so far, mainly delta derivative on some commonly encountered time scales.

Example 1.20. *Let* $h > 0$ *and*

$$\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}.$$

Then we have for $t \in \mathbb{T}$

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} = \inf\{t + nh : n \in \mathbb{N}\} = t + h$$

and similarly $\rho(t) = t - h$. *Hence every point* $t \in \mathbb{T}$ *is isolated and*

$$\mu(t) = \sigma(t) - t = t + h - t \equiv h \text{ for all } t \in \mathbb{T}$$

so that μ *in this example is constant. For a function* $f : \mathbb{T} \to \mathbb{R}$ *we have*

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h} \quad \text{for all} \quad t \in \mathbb{T}.$$

Next,

$$\begin{aligned} f^{\Delta\Delta}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\mu(t)} \\ &= \frac{f^{\Delta}(t+h) - f^{\Delta}(t)}{h} \\ &= \frac{\frac{f(t+2h)-f(t+h)}{h} - \frac{f(t+h)-f(t)}{h}}{h} \\ &= \frac{f(t+2h) - f(t+h) - f(t+h) + f(t)}{h^2} \\ &= \frac{f(t+2h) - 2f(t+h) + f(t)}{h^2}. \end{aligned}$$

Example 1.21. *Let H_n be the so-called harmonic numbers*

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n \in \mathbb{N}.$$

Consider the time scale

$$\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$$

Then $\sigma(H_n) = H_{n+1}$ for all $n \in \mathbb{N}_0, \rho(H_n) = H_{n-1}$ when $n \in \mathbb{N}$, and $\rho(H_0) = H_0$. The graininess is given by

$$\mu(H_n) = \sigma(H_n) - H_n = H_{n+1} - H_n = \frac{1}{n+1}$$

for all $n \in \mathbb{N}_0$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then

$$f^{\Delta}(H_n) = \frac{f(H_{n+1}) - f(H_n)}{\mu(H_n)} = (n+1)\Delta f(H_n).$$

Table 1.2. Examples of Time Scales.

$\mathbb{T}$	$\mu(t)$	$\sigma(t)$	$\rho(t)$
$\mathbb{R}$	0	t	t
$\mathbb{Z}$	1	$t+1$	$t-1$
$h\mathbb{Z}$	h	$t+h$	$t-h$
$q^{\mathbf{N}}$	$(q-1)t$	qt	$\frac{t}{q}$
$2^{\mathbb{N}}$	t	$2t$	$\frac{t}{2}$
$\mathbb{N}_0^2$	$2\sqrt{t}+1$	$(\sqrt{t}+1)^2$	$(\sqrt{t}-1)^2$

We now discuss some examples for the time scale $\mathbb{T} = \mathbb{Z}$.

Definition 1.22. *(Bohner and Peterson [1]) Let $t \in \mathbb{C}$ (i.e., t is a complex number) and $k \in \mathbb{Z}$. The factorial function $t^{(k)}$ is defined as follows:*

i) If $k \in \mathbb{N}$, then

$$t^{(k)} = t(t-1)\cdots(t-k+1).$$

ii) If $k = 0$, then

$$t^{(0)} = 1.$$

iii) If $-k \in \mathbb{N}$, then

$$t^{(k)} = \frac{1}{(t+1)(t+2)\cdots(t-k)}$$

for $t \neq -1, -2, \cdots, k$.

In general

$$t^{(k)} := \frac{\Gamma(t+1)}{\Gamma(t-k+1)} \tag{1.3}$$

for all $t, k \in \mathbb{C}$ *such that the right-hand side of* 1.3 *makes sense, where* Γ *is the gamma function.*

Definition 1.23. *(Bohner and Peterson [1]) We define the binomial coefficient* $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ *by*

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha^{(\beta)}}{\Gamma(\beta+1)}$$

for all $\alpha, \beta \in \mathbb{C}$ *such that the right-hand side of this equation makes sense.*

1.4 Integration on Time Scales

In this section we introduce the notion of antiderivatives on time scales that will allow us to define an integral on time scales. So, in order to describe classes of functions that are "integrable", we introduce the following two concepts.

Definition 1.24. *(Bohner and Peterson [1]) A function $f : \mathbb{T} \to \mathbb{R}$ is called* ***regulated*** *provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Definition 1.25. *(Bohner and Peterson [1])A function $f : \mathbb{T} \to \mathbb{R}$ is called* ***rd-continuous*** *provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*

The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted in this thesis by

$$\mathrm{C}_{\mathrm{rd}} = \mathrm{C}_{\mathrm{rd}}(\mathbb{T}) = \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$\mathrm{C}^1_{\mathrm{rd}} = \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}) = \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$$

Some results concerning rd-continuous and regulated functions are contained in the following simple theorem.

Theorem 1.26. *(Bohner and Peterson [1]) Assume $f : \mathbb{T} \to \mathbb{R}$.*

(i) If f is continuous, then f is rd-continuous.

(ii) If f is rd-continuous, then f is regulated.

(iii) The jump operator σ is rd-continuous.

(iv) If f is regulated or rd-continuous, then so is f^σ.

(v) Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.

Proof. The proof of the statements follows easily from the definitions. □

Definition 1.27. *(Bohner and Peterson [1]) A continuous function $f : T \to \mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided $D \subset \mathbb{T}^\kappa, \mathbb{T}^\kappa \backslash D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.*

Example 1.28. *Let $\mathbb{T} := \mathbb{P}_{2,1}$ and let $f : \mathbb{T} \to \mathbb{R}$ be defined by*

$$f(t) = \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[3k, 3k+1] \\ t - 3k - 1 & \text{if} \quad t \in [3k+1, 3k+2], k \in \mathbb{N}_0. \end{cases}$$

Then f is pre-differentiable with

$$D := \mathbb{T} \backslash \bigcup_{k=0}^{\infty} \{3k+1\}$$

One can see that $\mathbb{T}^\kappa \backslash D$ is the union of singletons of the form $\{3k+1\}$ hence it is countable and does not contain any right-scattered elements of $\mathbb{T}$.

Let us treat a few examples of some interesting functions to understand how they can or fail to be pre-differentiable.

Example 1.29. *For each of the following determine if f is regulated on T, if f is rd-continuous on $\mathbb{T}$, and if f is pre-differentiable. If f is pre-differentiable, find its region of differentiability D.*

(i) The function f is defined on a time scale $\mathbb{T}$ and every point $t \in \mathbb{T}$ is isolated.

(ii) Assume $\mathbb{T} = \mathbb{R}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ \frac{1}{t} & \text{if} \quad t \in \mathbb{R} \backslash \{0\}. \end{cases}$$

Solution

(i) Since every $t \in \mathbb{T}$ is isolated, then we cannot talk about limits on right-dense and left-dense points hence we can say that the condition of f being regulated is vacuously true. The same reasoning applies for rd-continuity of f. We conclude that f is pre-differentiable with $D = \mathbb{T}^\kappa$ since $\mathbb{T}$ has no right-scattered points.

(ii) In this case the right-side limit at the point $t = 0$ is ∞ so the function fails to be regulated. From theorem 1.26, this implies f is not rd-continous on $\mathbb{T}$, and finally not pre-differentiable.

Remark 1.30. *If f is regulated or even if $f \in \mathrm{C}_{\mathrm{rd}}, \max_{a \le t \le b} f(t)$ and $\min_{a \le t \le b} f(t)$ need not exist.*

In what follows we introduce the mean value theorem which holds for pre-differentiable functions. The theorem will be used to prove the main existence theorems for pre-antiderivatives and antiderivatives later on in this section. Its proof is an application of the induction principle that we proved earlier.

Theorem 1.31. (Mean Value Theorem, Bohner and Peterson [1]). *Let f and g be real-valued functions defined on T, both pre-differentiable with D. Then*

$$\left|f^{\Delta}(t)\right| \leq g^{\Delta}(t) \quad \textit{for all} \quad t \in D$$

implies

$$|f(s) - f(r)| \leq g(s) - g(r) \quad \textit{for all} \quad r, s \in \mathbb{T}, r \leq s$$

Proof. See ([1], Theorem 1.67, p. 23) □

Corollary 1.32. *Suppose f and g are pre-differentiable with D.*

(i) If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U^{\kappa} \cap D} \left|f^{\Delta}(t)\right| \right\} |s - r|.$$

(ii) If $f^{\Delta}(t) = 0$ for all $t \in D$, then f is a constant function.

(iii) If $f^{\Delta}(t) = g^{\Delta}(t)$ for all $t \in D$, then

$$g(t) = f(t) + C \quad \textit{for all } t \in \mathbb{T},$$

where C is a constant.

Proof. (i) Suppose f is pre-differentiable with D and let $r, s \in \mathbb{T}$ with $r \leq s$. If we define

$$g(t) := \left\{ \sup_{\tau \in [r,s]^{\kappa} \cap D} \left|f^{\Delta}(\tau)\right| \right\} (t - r) \quad \text{for} \quad t \in \mathbb{T}$$

then

$$g^{\Delta}(t) = \sup_{\tau \in [r,s]^{\kappa} \cap D} \left|f^{\Delta}(\tau)\right| \geq \left|f^{\Delta}(t)\right| \quad \text{for all} \quad t \in D \cap [r, s]^{\kappa}.$$

By Theorem 1.31,

$$g(t) - g(r) \geq |f(t) - f(r)| \quad \text{for all} \quad t \in [r, s]$$

so that

$$|f(s) - f(r)| \leq g(s) - g(r) = g(s) = \left\{ \sup_{r \in [r,s]^{\kappa} \cap D} \left|f^{\Delta}(\tau)\right| \right\} (s - r).$$

This completes the proof of part (i).

(ii) Part (ii) follows immediately from (i). Suppose that $f^{\Delta}(t) = 0$ for all $t \in D$. By part (i) we have

$$|f(s) - f(r)| \le \left\{ \sup_{t \in U^{\kappa} \cap D} \left| f^{\Delta}(t) \right| \right\} |s - r| = 0,$$

which implies that $f(s) = f(r)$ and hence f is a constant function on D.

(iii) If we define $h(t) = g(t) - f(t)$ for all $t \in D$, then $h^{\Delta}(t) = 0$ for all $t \in D$. By part (ii), $h(t)$ is a constant function, that is $g(t) = f(t) + C$ for all $t \in \mathbb{T}$ where C is a constant.

□

The main existence theorem for pre-antiderivatives now reads as follows.

Theorem 1.33. *(Bohner and Peterson [1]). Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t) \quad \text{holds for all} \quad t \in D.$$

The generalisation of this theorem which involves more generalized measure chains (time scales) can be found in Chapter 8 of [1].

Definition 1.34. *(Bohner and Peterson [1]) Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 1.33 is called a **pre-antiderivative** of f. We define the **indefinite integral** of a regulated function f by*

$$\int f(t)\Delta t = F(t) + C$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the Cauchy integral by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \quad \text{holds for all } t \in \mathbb{T}^{\kappa}.$$

Example 1.35. *If $\mathbb{T} = \mathbb{Z}$, evaluate the indefinite integral*

$$\int a^t \Delta t$$

where $a \neq 1$ is a constant. Since

$$\left(\frac{a^t}{a-1}\right)^{\Delta} = \Delta\left(\frac{a^t}{a-1}\right) = \frac{a^{t+1} - a^t}{a-1} = a^t$$

we get that

$$\int a^t \Delta t = \frac{a^t}{a-1} + C$$

where C is an arbitrary constant.

Here are useful formulars that can be derived exactly in the same manner as the previous example: If $\mathbb{T} = \mathbb{Z}, k \neq -1$, and $\alpha \in \mathbb{R}$, then

(i) $\int (t+\alpha)^{(k)} \Delta t = \frac{(t+\alpha)^{(k+1)}}{k+1} + C$;

(ii) $\int \begin{pmatrix} t \\ \alpha \end{pmatrix} \Delta t = \begin{pmatrix} t \\ \alpha+1 \end{pmatrix} + C.$

Theorem 1.36. (Bohner and Peterson [1]). *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \quad \text{for} \quad t \in \mathbb{T}$$

is an antiderivative of f.

Proof. See ([1], Theorem 1.74, p. 27) □

Table 1.3. The two most important examples.

Time scale $\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$
Backward jump operator $\rho(t)$	t	$t-1$
Forward jump operator $\sigma(t)$	t	$t+1$
Graininess $\mu(t)$	0	1
Derivative $f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$
Integral $\int_a^b f(t)\Delta t$	$\int_a^b f(t)dt$	$\sum_{t=a}^{b-1} f(t)$ (if $a < b$)
Rd-continuous f	continuous f	any f

Theorem 1.37. *(Bohner and Peterson [1]) If $f \in \mathrm{C_{rd}}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)$$

Proof. By Theorem 1.36, there exists an antiderivative F of f, and

$$\begin{aligned}\int_t^{\sigma(t)} f(\tau)\Delta\tau &= F(\sigma(t)) - F(t) \\ &= \mu(t)F^{\Delta}(t) \\ &= \mu(t)f(t),\end{aligned}$$

where the second equation holds because of Theorem 1.10 (iv). □

Theorem 1.38. *(Bohner and Peterson [1]) If $f^\Delta \geq 0$, then f is nondecreasing.*

Proof. Let $f^\Delta \geq 0$ on $[a,b]$ and let $s,t \in \mathbb{T}$ with $a \leq s \leq t \leq b$. Then

$$f(t) = f(s) + \int_s^t f^\Delta(\tau)\Delta\tau \geq f(s)$$

so that the conclusion follows. □

Theorem 1.39. *(Bohner and Peterson [1]) If $a,b,c \in \mathbb{T}, \alpha \in \mathbb{R}$, and $f,g \in \mathrm{C}_{\mathbf{rd}}$, then*

(i) $\int_a^b [f(t)+g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$

(v) $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t;$

(vi) $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t;$

(vii) $\int_a^a f(t)\Delta t = 0$

(viii) *If $|f(t)| \leq g(t)$ on $[a,b)$, then*

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t$$

(ix) *if $g(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b g(t)\Delta t \geq 0$.*

Proof. See ([1], Theorem 1.77, p. 28) □

Note that the formulas in Theorem 1.39 (v) and (vi) are called integration by parts formulas. Also note that all of the formulas given in Theorem 1.39 also hold for the case that f and g are only regulated functions because rd-continuity implies that the function is regulated.

Theorem 1.40. *(Bohner and Peterson) [1] Let $a,b \in \mathbb{T}$ and $f \in \mathrm{C}_{\mathrm{rd}}$.*

(i) *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If $[a,b]$ *consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if} \quad a<b \\ 0 & \text{if} \quad a=b \\ -\sum_{t\in(b,a)} \mu(t)f(t) & \text{if} \quad a>b. \end{cases}$$

(iii) If $\mathbf{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$*, where* $h > 0$*, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if } a<b \\ 0 & \text{if } a=b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & \text{if } a>b. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$*, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if} \quad a<b \\ 0 & \text{if} \quad a=b \\ -\sum_{t=b}^{a-1} f(t) & \text{if} \quad a>b. \end{cases}$$

Let us treat an exercise that illustrates integration techniques on an arbitrary time scale

Example 1.41. *Evaluate* $\int_0^t s\Delta s$ *for* $t \in \mathbb{T}$ *for* $\mathbb{T} = [0,1] \cup [2,3]$

Response

Care must be taken since $\mathbb{T}$ in this case is a union of disjoint intervals. So we evaluate the integral for each interval and also at the discontinuity at 1.

If $t \in [0,1]$, notice that points in this interval are dense.

Since we know that $(s^2)^\Delta = s + \sigma(s)$, we get $(s^2)^\Delta = 2s$ since each point in $[0,1]$ is dense. We deduce that

$$\int_0^t s\Delta s = \frac{1}{2}[s^2]_0^t = \frac{1}{2}t^2$$

when $t \in [0,1]$

If $t \in [2,3]$, in order to calculate the integral from 0 up to t in $[2,3]$, we first add the integral from 0 to 1 then add the part from 1 to t. Note that at $t=1$, $\sigma(1) = 2$ and so our integral becomes:

$$\int_0^t s\Delta s = \int_0^1 s\Delta s + \int_1^2 s\Delta s + \int_2^t s\Delta s.$$

The second integral can be expressed as $\int_1^{\sigma(1)} s\Delta s$, which by Theorem 1.37 is equal to $\mu(1)(1) = 1$.

(Note that we can subdivide integrals because of theorem 1.39).

So for $t \in [2,3]$, by adding the 3 integrands we get:

$$\int_0^t s\Delta s = \frac{t^2}{2} - \frac{1}{2}$$

We next define the improper integral $\int_a^\infty f(t)\Delta t$ as one would expect.

Definition 1.42. *(Bohner and Peterson [1]) If $a \in \mathbb{T}$, sup $\mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)$, then we define the improper integral by*

$$\int_a^\infty f(t)\Delta t := \lim_{b\to\infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

We now give an exercise concerning improper integrals.

Example 1.43. *(i) Assume $a \in \mathbb{T}, a > 0$ and sup $\mathbb{T} = \infty$. Evaluate*

$$\int_a^\infty \frac{1}{t\sigma(t)}\Delta t$$

Solutions

(i) Clearly this is an improper integral. However, by realizing that $-\frac{1}{t}$ is an anti-derivative of $\frac{1}{t\sigma(t)}$, we simply put the limits on the anti-derivative to find the answer $\frac{1}{a}$

1.5 Chain Rules

When it comes to time scales, the usual chain rule from calculus does not hold for all time scales as we will see in this section. However, there are several versions of the chain rule that have been developed over the course of the development of the theory.

If $f, g : \mathbb{R} \to \mathbb{R}$, then the chain rule from calculus is that if g is differentiable at t and if f is differentiable at $g(t)$, then

$$(f \circ g)'(t) = f'(g(t))g'(t).$$

The next example shows that the chain rule as we know it in calculus does not hold for all time scales.

Example 1.44. *Assume $f, g : \mathbb{Z} \to \mathbb{Z}$ are defined by $f(t) = t^2, g(t) = 2t$ and our time scale is $\mathbb{T} = \mathbb{Z}$. It is easy to see that*

$$(f \circ g)^\Delta(t) = 8t + 4 \neq 8t + 2 = f^\Delta(g(t))g^\Delta(t) \text{ for all } t \in \mathbb{Z}.$$

Hence the chain rule as we know it in calculus does not hold in this setting.

Sometimes the following substitute of the "continuous" chain rule is useful.

Theorem 1.45. (Chain Rule, Bohner and Peterson [1]). *Assume* $g : \mathbb{R} \to \mathbb{R}$ *is continuous,* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}^\kappa$*, and* $f : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable. Then there exists* c *in the real interval* $[t, \sigma(t)]$ *with*

$$(f \circ g)^\Delta(t) = f'(g(c))g^\Delta(t). \tag{1.4}$$

Proof. See ([1], Theorem 1.87, p. 32-33) □

Example 1.46. *Given* $\mathbb{T} = \mathbb{Z}, f(t) = t^2, g(t) = 2t$*, find directly the value* c *guaranteed by Theorem 1.45 so that*

$$(f \circ g)^\Delta(3) = f'(g(c))g^\Delta(3)$$

and show that c *is in the interval guaranteed by Theorem 1.45. Using the calculations we made in Example 1.44 we get that this last equation becomes*

$$28 = (4c)2.$$

Solving for c *we get that* $c = \frac{7}{2}$ *which is in the real interval* $[3, \sigma(3)] = [3, 4]$ *as we are guaranteed by Theorem 1.45.*

Now we present a chain rule which calculates $(f \circ g)^\Delta$, where

$$g : \mathbb{T} \to \mathbb{R} \text{ and } f : \mathbb{R} \to \mathbb{R}$$

This chain rule is due to Christian Pötzsche, who derived it first in 1998 (see also Stefan Keller's PhD thesis [3] and [4]).

Theorem 1.47. (Chain Rule). *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable and suppose* $g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable. Then* $f \circ g : \mathbb{T} \to \mathbb{R}$ *is delta differentiable and the formula*

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t)g^\Delta(t)\right) dh \right\} g^\Delta(t)$$

holds.

Proof. First of all we apply the ordinary substitution rule from calculus by letting $\tau = hg(\sigma(t)) + (1-h)g(s)$, and also noting that $g(t) + h\mu(t)g^\Delta(t) = hg(\sigma(t)) + (1-h)g(t)$, to find

$$\begin{aligned} f(g(\sigma(t))) - f(g(s)) &= \int_{g(s)}^{g(\sigma(t))} f'(\tau)d\tau \\ &= [g(\sigma(t)) - g(s)] \int_0^1 f'(hg(\sigma(t)) + (1-h)g(s))dh. \end{aligned}$$

Let $t \in \mathbb{T}^\kappa$ and $\varepsilon > 0$ be given. Since g is differentiable at t, there exists a neighborhood U_1 of t such that

$$\left|g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon^*|\sigma(t) - s| \text{ for all } \quad s \in U_1,$$

where

$$\varepsilon^* = \frac{\varepsilon}{1 + 2\int_0^1 |f'(hg(\sigma(t)) + (1-h)g(t))|\, dh}.$$

Moreover, f' is continuous on $\mathbb{R}$, and therefore it is uniformly continuous on closed subsets of $\mathbb{R}$, and (observe also that g is continuous as it is differentiable, see Theorem 1.10 (i)) hence there exists a neighborhood U_2 of t such that

$$|f'(hg(\sigma(t)) + (1-h)g(s)) - f'(hg(\sigma(t)) + (1-h)g(t))| \leq \frac{\varepsilon}{2\left(\varepsilon^* + |g^\Delta(t)|\right)}$$

for all $s \in U_2$. To see this, note also that

$$\begin{aligned} |hg(\sigma(t)) + (1-h)g(s) - (hg(\sigma(t)) + (1-h)g(t))| &= (1-h)|g(s) - g(t)| \\ &\leq |g(s) - g(t)| \end{aligned}$$

holds for all $0 \leq h \leq 1$. We then define $U = U_1 \cap U_2$ and let $s \in U$. To avoid very long formulae, we put

$$\alpha = hg(\sigma(t)) + (1-h)g(s) \quad \text{and} \quad \beta = hg(\sigma(t)) + (1-h)g(t).$$

Then we have (note we use the triangle inequality in the third last equality i.e.

$$|f'(\alpha)| = |f'(\alpha) - f'(\beta) + f'(\beta)| \leq |f'(\alpha) - f'(\beta)| + |f'(\beta)|$$

So we have

$$
\begin{aligned}
&\left|(f\circ g)(\sigma(t)) - (f\circ g)(s) - (\sigma(t)-s)g^{\Delta}(t)\int_0^1 f'(\beta)dh\right| \\
&= \left|[g(\sigma(t)) - g(s)]\int_0^1 f'(\alpha)dh - (\sigma(t)-s)g^{\Delta}(t)\int_0^1 f'(\beta)dh\right| \\
&= \Big| \left[g(\sigma(t)) - g(s) - (\sigma(t)-s)g^{\Delta}(t)\right]\int_0^1 f'(\alpha)dh \\
&\quad + (\sigma(t)-s)g^{\Delta}(t)\int_0^1 \left(f'(\alpha) - f'(\beta)\right)dh \Big| \\
&\le \quad |g(\sigma(t)) - g(s) - (\sigma(t)-s)g^{\Delta}(t)|\int_0^1 |f'(\alpha)|dh \\
&\quad + |\sigma(t)-s||g^{\Delta}(t)|\int_0^1 |f'(\alpha) - f'(\beta)|dh \\
&\le \varepsilon^*|\sigma(t)-s|\int_0^1 |f'(\alpha)|dh + |\sigma(t)-s|\|g^{\Delta}(t)|\int_0^1 |f'(\alpha) - f'(\beta)\mid dh \\
&\le \varepsilon^*|\sigma(t)-s|\int_0^1 |f'(\beta)|\, dh + \left[\varepsilon^* + \left|g^{\Delta}(t)\right|\right]|\sigma(t)-s|\int_0^1 |f'(\alpha) - f'(\beta)|\, dh \\
&\le \frac{\varepsilon}{2}|\sigma(t)-s| + \frac{\varepsilon}{2}|\sigma(t)-s| \\
&= \varepsilon|\sigma(t)-s|.
\end{aligned}
$$

Therefore $f\circ g$ is differentiable at t and the derivative is as claimed above. □

Example 1.48. *We define $g:\mathbb{Z}\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ by*

$$g(t) = t^2 \quad \text{and} \quad f(x) = \exp(x).$$

Then

$$g^{\Delta}(t) = (t+1)^2 - t^2 = 2t+1 \quad \text{and} \quad f'(x) = \exp(x)$$

Hence we have by Theorem 1.47

$$
\begin{aligned}
(f\circ g)^{\Delta}(t) &= \left\{\int_0^1 f'\left(g(t) + h\mu(t)g^{\Delta}(t)\right)dh\right\} g^{\Delta}(t) \\
&= (2t+1)\int_0^1 \exp\left(t^2 + h(2t+1)\right)dh \\
&= (2t+1)\exp\left(t^2\right)\int_0^1 \exp(h(2t+1))dh \\
&= (2t+1)\exp\left(t^2\right)\frac{1}{2t+1}\left[\exp(h(2t+1))\right]_{h=0}^{h=1} \\
&= (2t+1)\exp\left(t^2\right)\frac{1}{2t+1}\left(\exp(2t+1) - 1\right) \\
&= \exp\left(t^2\right)\left(\exp(2t+1) - 1\right).
\end{aligned}
$$

On the other hand, it is easy to check that we have indeed

$$\begin{aligned}\Delta f(g(t)) &= f(g(t+1)) - f(g(t)) \\ &= \exp\left((t+1)^2\right) - \exp\left(t^2\right) \\ &= \exp\left(t^2+2t+1\right) - \exp\left(t^2\right) \\ &= \exp\left(t^2\right)(\exp(2t+1)-1).\end{aligned}$$

In the remainder of this section we present some results related to results in the paper by C. D. Ahlbrandt, M. Bohner, and J. Ridenhour [5]. Let $\mathbb{T}$ be a time scale and $\nu : \mathbb{T} \to \mathbb{R}$ be a strictly increasing function such that $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is also a time scale. By $\tilde{\sigma}$ we denote the jump function on $\tilde{\mathbb{T}}$ and by $^{\tilde{\Delta}}$ we denote the derivative on $\tilde{\mathbb{T}}$. Then $\nu \circ \sigma = \tilde{\sigma} \circ \nu$. You can prove the latter by just defining $\tilde{\sigma}(x) := \nu(\sigma(t))$ for $x = \nu(t)$.

Theorem 1.49. (Chain Rule). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(w \circ \nu)^{\Delta} = \left(w^{\tilde{\Delta}} \circ \nu\right)\nu^{\Delta}.$$

Proof. See ([1], Theorem 1.93, p. 34) □

Example 1.50. *Let $\mathrm{T} = \mathbb{N}_0$ and $\nu(t) = 4t+1$. Hence*

$$\tilde{\mathbb{T}} = \nu(\mathbb{T}) = \{4n+1 : n \in \mathbb{N}_0\} = \{1, 5, 9, 13, \ldots\}.$$

Moreover, let $w : \tilde{\mathbb{T}} \to \mathbb{R}$ be defined by $w(t) = t^2$. Then

$$(w \circ \nu)(t) = w(\nu(t)) = w(4t+1) = (4t+1)^2$$

and hence

$$\begin{aligned}(w \circ \nu)^{\Delta}(t) &= [4(t+1)+1]^2 - (4t+1)^2 \\ &= (4t+5)^2 - (4t+1)^2 \\ &= 16t^2 + 40t + 25 - 16t^2 - 8t - 1 \\ &= 32t + 24.\end{aligned}$$

Now we apply Theorem 1.49 to obtain the derivative of this composite function. We first calculate $\nu^{\Delta}(t) \equiv 4$ and then

$$w^{\tilde{\Delta}}(t) = \frac{w(\tilde{\sigma}(t)) - w(t)}{\tilde{\sigma}(t) - t} = \frac{(t+4)^2 - t^2}{t+4-t} = \frac{8t+16}{4} = 2t+4$$

and therefore

$$\left(w^{\tilde{\Delta}} \circ \nu\right)(t) = w^{\tilde{\Delta}}(\nu(t)) = w^{\tilde{\Delta}}(4t+1) = 2(4t+1)+4 = 8t+6.$$

Thus we obtain

$$\left[\left(w^{\tilde{\Delta}} \circ \nu\right) \nu^{\Delta}\right](t) = (8t+6)4 = 32t + 24 = (w \circ \nu)^{\Delta}(t).$$

As a consequence of the above Theorem 1.49 we can now write down a formula for the derivative of the inverse function.

Theorem 1.51. *(Derivative of the Inverse). Assume* $\nu : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ *is a time scale. Then*

$$\frac{1}{\nu^{\Delta}} = (\nu^{-1})^{\tilde{\Delta}} \circ \nu \tag{1.5}$$

at points where ν^{Δ} *is different from zero.*

Proof. Let $\omega = \nu^{-1} : \tilde{\mathbb{T}} \to \mathbb{T}$ in the previous theorem. □

1.6 Polynomials on Time scales

An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a closed formula (for an arbitrary time scale) of an antiderivative of t. Certainly t^2 is not the solution, as the derivative of t^2 is

$$t + \sigma(t)$$

which is, as we know is not even necessarily a differentiable function (although it is the product of two differentiable functions): Similarly, none of the "classical" polynomials are necessarily more than once differentiable, see Theorem 1.14. So the question arises which function plays the role of e.g., $t^2/2$, in the time scales calculus. It could be either

$$\int_0^t \sigma(\tau)\Delta\tau \quad \textit{or} \quad \int_0^t \tau\Delta\tau.$$

In fact, if we define

$$g_2(t,s) = \int_s^t (\sigma(\tau) - s)\Delta\tau \quad \textit{and} \quad h_2(t,s) = \int_s^t (\tau - s)\Delta\tau,$$

we find the following relation between g_2 and h_2:

$$\begin{aligned} g_2(t,s) &= \int_s^t (\sigma(\tau)-s)\Delta\tau \\ &= \int_s^t (\sigma(\tau)+\tau)\Delta\tau - \int_s^t \tau\Delta\tau - \int_s^t s\Delta\tau \\ &= \int_s^t (\tau^2)^{\Delta}\Delta\tau + \int_t^s \tau\Delta\tau - s(t-s) \\ &= \int_s^t (\tau - t)\Delta\tau \\ &= h_2(s,t). \end{aligned}$$

In this section we give a Taylor's formula for functions on a general time scale. As it turns out Taylor monomials are important as they are intimately related to Cauchy functions of certain dynamic equations which are important in variation of constants formulas. Many of the results in this section can be found in R.P. Agarwal and M.Bohner [6]. The generalized polynomials, that also occur in Taylor's formula, are the functions $g_k, h_k : \mathbb{T} \to \mathbb{R}, k \in \mathbb{N}_0$, defined recursively as follows: The functions g_0 and h_0 are

$$g_0(t,s) = h_0(t,s) \equiv 1 \quad \forall s,t \in \mathbb{T}, \tag{1.6}$$

and given g_k and h_k for $k \in \mathbb{N}_0$ the functions g_{k+1} and h_{k+1} are

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau),s)\Delta\tau \quad \forall s,t \in \mathbb{T} \tag{1.7}$$

and

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau \quad \forall s,t \in \mathbb{T}. \tag{1.8}$$

Note that the functions g_k and h_k are all well defined according to Theorem 1.26 and Theorem 1.36. If we let $h_k^{\Delta}(t,s)$ denote for each fixed s the delta derivative of $h_k(t,s)$ with respect to t, then

$$h_k^{\Delta}(t,s) = h_{k-1}(t,s) \quad \text{for} \quad k \in \mathbb{N}, t \in \mathbb{T}^{\kappa}.$$

Similarly

$$g_k^{\Delta}(t,s) = g_{k-1}(\sigma(t),s) \quad \text{for} \quad k \in \mathbb{N}, t \in \mathbb{T}^{\kappa}. \tag{1.9}$$

The above definitions obviously imply

$$g_1(t,s) = h_1(t,s) = t - s \quad \text{for all} s,t \in \mathbb{T}.$$

However, finding g_k and h_k for $k > 1$ is not easy in general. But for a particular given time scale it might be easy to find these functions. We will consider several examples first before we present Taylor's formula in general.

Example 1.52. *For the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ it is easy to find the functions g_k and h_k:*

(i) First consider $\mathbb{T} = \mathbb{R}$. Then $\sigma(t) = t$ for $t \in \mathbb{R}$ so that $g_k = h_k$ for $k \in \mathbb{N}_0$. We have

$$\begin{aligned} g_2(t,s) = h_2(t,s) &= \int_s^t (\tau - s)d\tau \\ &= \left.\frac{(\tau - s)^2}{2}\right|_{\tau=s}^{\tau=t} \\ &= \frac{(t-s)^2}{2}. \end{aligned}$$

We claim that for $k \in \mathbb{N}_0$

$$g_k(t,s) = h_k(t,s) = \frac{(t-s)^k}{k!} \quad \text{for all} \quad s,t \in \mathbb{R} \tag{1.10}$$

as we will now show using the principle fo mathematical induction. Obviously (1.10) holds for $k = 0$. Assume (1.10) holds with k replaced by some $m \in \mathbb{N}_0$. Then

$$\begin{aligned} g_{m+1}(t,s) &= h_{m+1}(r,s) \\ &= \int_s^t \frac{(\tau - s)^m}{m!} d\tau \\ &= \left.\frac{(\tau - s)^{m+1}}{(m+1)!}\right|_{\tau=s}^{\tau=t} \\ &= \frac{(t-s)^{m+1}}{(m+1)!}, \end{aligned}$$

i.e., (1.10) holds with k replaced by $m+1$. We note that, for an n-times differentiable function $f : \mathbb{R} \to \mathbb{R}$, the following well-known Taylor's formula holds: Let $\alpha \in \mathbb{R}$ be arbitrary. Then, for all $t \in \mathbb{R}$, the representations

$$\begin{aligned} f(t) &= \sum_{k=0}^{n-1} \frac{(t-\alpha)^k}{k!} f^{(k)}(\alpha) + \frac{1}{(n-1)!} \int_\alpha^t (t-\tau)^{n-1} f^{(n)}(\tau) d\tau \\ &= \sum_{k=0}^{n-1} h_k(t,\alpha) f^{(k)}\alpha + \int_\alpha^t h_{n-1}(t, \sigma(\tau)) f^{(n)}(\tau) d\tau \qquad (1.11) \\ &= \sum_{k=0}^{n-1} (-1)^k g_k(\alpha, t) f^{(k)}(\alpha) + \int_\alpha^t (-1)^{n-1} g_{n-1}(\sigma(\tau), t) f^{(n)}(\tau) d\tau \qquad (1.12) \end{aligned}$$

are valid, where $f^{(k)}$ denotes the usual kth derivative of f. Above we have used the relationship

$$(-1)^k g_k(s,t) = (-1)^k \frac{(s-t)^k}{k!} = \frac{(t-s)^k}{k!} = h_k(t,s), \tag{1.13}$$

which holds for all $k \in \mathbb{N}_0$.

(ii) Next, consider $\mathbb{T} = \mathbb{Z}$. Then $\sigma(t) = t + 1$ for $t \in \mathbb{Z}$. We have for $s, t \in \mathbb{Z}$

$$\begin{aligned} h_2(t,s) &= \int_s^t h_1(\tau, s)\Delta\tau = \int_s^t (\tau - s)^{(1)}\Delta\tau \\ &= \left[\frac{(\tau - s)^{(2)}}{2}\right]_s^t \\ &= \frac{(t-s)^{(2)}}{2!} = \binom{t-s}{2}. \end{aligned}$$

We claim that for $k \in \mathbb{N}_0$, we have

$$h_k(t,s) = \frac{(t-s)^{(k)}}{k!} = \binom{t-s}{k} \quad \text{for all} \quad s, t \in \mathbb{Z}. \tag{1.14}$$

Assume (1.14) holds for k replaced by m. Then

$$\begin{aligned} h_{m+1}(t,s) &= \int_s^t h_m(\tau, s)\Delta\tau \\ &= \int_s^t \frac{(\tau - s)^{(m)}}{m!}\Delta\tau \\ &= \frac{(t-s)^{(m+1)}}{(m+1)!}, \end{aligned}$$

which is (1.14)with k replaced by $m + 1$. Hence by mathematical induction we get that ((1.14) holds for all $k \in \mathbb{N}_0$. Similarly it is possible to show that

$$g_k(t,s) = \frac{(t-s+k-1)^{(k)}}{k!} \quad \text{for all} \quad s, t \in \mathbb{Z} \tag{1.15}$$

holds for all $k \in \mathbb{N}_0$. As before we observe that the relationship

$$\begin{aligned} (-1)^k g_k(s,t) &= (-1)^k \frac{(s-t+k-1)^{(k)}}{k!} \\ &= (-1)^k \frac{(s-t+k-1)(s-t+k-2)\cdots(s-t)}{k!} \\ &= \frac{(t-s)\cdots(t-s+2-k)(t-s+1-k)}{k!} \\ &= \frac{(t-s)^{(k)}}{k!} \\ &= h_k(t,s) \end{aligned}$$

holds for all $k \in \mathbb{N}_0$. The well-known discrete version of Taylor's formular (see e.g [7]) reads as follows: Let $f : \mathbb{R} \to \mathbb{Z}$ be a function, and let $\alpha \in \mathbb{Z}$. Then, for all

$t \in \mathbb{Z}$ with $t > \alpha + n$, the representations

$$f(t) = \sum_{k=0}^{n-1} \frac{(t-\alpha)^{(k)}}{k!} \Delta^k f(\alpha) + \frac{1}{(n+1)!} \sum_{\tau=\alpha}^{t-n} (t-\tau-1)^{(n-1)} \Delta^n f(\tau)$$

$$= \sum_{k=0}^{n-1} h_k(t,\alpha) \Delta^k f(\alpha) + \sum_{\tau=\alpha}^{t-n} h_{n-1}(t,\sigma(\tau)) \Delta^n f(\tau) \tag{1.16}$$

$$= \sum_{k=0}^{n-1} (-1)^k g_k(\alpha,t) \Delta^k f(\alpha) + \sum_{\tau=\alpha}^{t-n} (-1)^{n-1} g_{n-1}(\sigma(\tau),t) \Delta^n f(\tau) \tag{1.17}$$

hold, where Δ^k is the usual k-times iterated forward difference operator.

Now we will present and prove Taylor's formula for the case of a general time scale $\mathbb{T}$. First we need three preliminary results.

Lemma 1.53. *(Bohner and Peterson [1]) Let $n \in \mathbb{N}$. Suppose f is n-times differentiable and p_k, $0 \le k \le n-1$, are differentiable at some $t \in \mathbb{T}$ with*

$$p_{k+1}^{\Delta}(t) = p_k^{\sigma}(t) \quad \text{for all} \quad 0 \le k \le n-2. \tag{1.18}$$

Then we have at t

$$\left[\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k} p_k \right]^{\Delta} = (-1)^{n-1} f^{\Delta^n} p_{n-1}^{\sigma} + f p_0^{\Delta}$$

Proof. Using Theorem (1.12) (i), (ii), (iii) and (1.18) we find that

$$\begin{aligned}
\left[\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k} p_k \right]^{\Delta} &= \sum_{k=0}^{n-1} (-1)^k \left[f^{\Delta^k} p_k \right]^{\Delta} \\
&= \sum_{k=0}^{n-1} (-1)^k \left[f^{\Delta^{k+1}} p_k^{\sigma} + f^{\Delta^k} p_k^{\Delta} \right] \\
&= \sum_{k=0}^{n-2} (-1)^k f^{\Delta^{k+1}} p_k^{\sigma} + (-1)^{n-1} f^{\Delta^n} p_{n-1}^{\sigma} + \sum_{k=1}^{n-1} (-1)^k f^{\Delta^k} p_k^{\Delta} + f p_0^{\Delta} \\
&= \sum_{k=0}^{n-2} (-1)^k f^{\Delta^{k+1}} p_k^{\sigma} + (-1)^{n-1} f^{\Delta^n} p_{n-1}^{\sigma} - \sum_{k=0}^{n-2} (-1)^k f^{\Delta^{k+1}} p_{k+1}^{\Delta} + f p_0^{\Delta} \\
&= (-1)^{n-1} f^{\Delta^n} p_{n-1}^{\sigma} + f p_0^{\Delta}
\end{aligned}$$

holds at t. This proves the lemma. □

Lemma 1.54. *(Bohner and Peterson [1]) The functions g_k defined in (1.6) and (1.7) satisfy for all $t \in \mathbb{T}$*

$$g_n(\rho^k(t), t) = 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and all} \quad 0 \le k \le n-1.$$

Proof. We prove this result by induction. First, for $k=0$, we have

$$g_n(\rho^0(t),t) = g_n(t,t) = 0.$$

To complete the induction it suffices to show that

$$g_{n-1}(\rho^k(t),t) = g_n(\rho^k(t),t) = 0 \quad \text{with} \quad 0 \le k \le n$$

implies that

$$g_n(\rho^{k+1}(t),t) = 0.$$

If $\rho^k(t)$ is left-dense, then $\rho^{k+1}(t) = \rho^k(t)$ so that

$$g_n\left(\rho^{k+1}(t),t\right) = g_n\left(\rho^k(t),t\right) = 0.$$

If $\rho^k(t)$ is not left-dense, then it is left-scattered, and $\sigma\left(\rho^{k+1}(t)\right) = \rho^k(t)$ so that by Theorem 1.10 (iv) and (1.9)

$$\begin{aligned} g_n\left(\rho^{k+1}(t),t\right) &= g_n\left(\sigma\left(\rho^{k+1}(t)\right),t\right) - \mu\left(\rho^{k+1}(t)\right) g_n^{\Delta}\left(\rho^{k+1}(t),t\right) \\ &= g_n\left(\rho^k(t),t\right) - \mu\left(\rho^{k+1}(t)\right) g_{n-1}\left(\sigma\left(\rho^{k+1}(t)\right),t\right) \\ &= g_n\left(\rho^k(t),t\right) - \mu\left(\rho^{k+1}(t)\right) g_{n-1}\left(\rho^k(t),t\right) \\ &= 0 \end{aligned}$$

(observe $n \neq 1$). This proves our claim. □

Lemma 1.55. *(Bohner and Peterson [1]) Let $n \in \mathbb{N}, t \in \mathbb{T}$, and suppose that f is $(n-1)$ times differentiable at $\rho^{n-1}(t)$. Then we have*

$$\sum_{k=0}^{n-1}(-1)^k f^{\Delta^k}\left(\rho^{n-1}(t)\right) g_k\left(\rho^{n-1}(t),t\right) = f(t), \tag{1.19}$$

where the functions g_k are defined by (1.6) and (1.7).

Proof. See ([1], Lemma 1.110, p. 43 -44) □

Theorem 1.56. *(Taylor's Formula, Bohner and Peterson [1]). Let $n \in \mathbb{N}$. Suppose f is n times differentiable on $\mathbb{T}^{\kappa^n}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}, t \in \mathbb{T}$, and define the functions g_k by (1.6) and (1.7, i.e.,*

$$g_0(r,s) \equiv 1 \quad \text{and} \quad g_{k+1}(r,s) = \int_s^r g_k(\sigma(\tau),s)\Delta\tau \text{ for } k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t) f^{\Delta^k}(\alpha) + \int_\alpha^{\rho^{n-1}(t)} (-1)^{n-1} g_{n-1}(\sigma(\tau),t) f^{\Delta^n}(\tau)\Delta\tau.$$

Proof. By Lemma 1.53 we have

$$\left[\sum_{k=0}^{n-1}(-1)^k g_k(\cdot,t) f^{\Delta^k}\right]^{\Delta}(\tau) = (-1)^{n-1} g_{n-1}(\sigma(\tau),t) f^{\Delta^n}(\tau)$$

for all $\tau \in \mathbb{T}^{\kappa^n}$. Since $\alpha, \rho^{n-1}(t) \in \mathbb{T}^{\kappa^{n-1}}$, we may integrate the above equation from α to $\rho^{n-1}(t)$ to obtain

$$\int_{\alpha}^{\rho^{n-1}(t)} (-1)^{n-1} g_{n-1}(\sigma(\tau),t) f^{\Delta^n}(\tau)\Delta\tau = \int_{\alpha}^{\rho^{n-1}(t)} \left[\sum_{k=0}^{n-1}(-1)^k g_k(\cdot,t) f^{\Delta^k}\right]^{\Delta}(\tau)\Delta\tau$$

$$= \sum_{k=0}^{n-1}(-1)^k g_k\left(\rho^{n-1}(t),t\right) f^{\Delta^k}\left(\rho^{n-1}(t)\right) - \sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t) f^{\Delta^k}(\alpha)$$

$$= f(t) - \sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t) f^{\Delta^k}(\alpha),$$

where we used formula (1.19) from Lemma 1.53. □

1.7 L'Hôpital's Rule on Time Scales

In this short section we present a few basic results that have proofs analogous to their respective results in ordinary calculus. These include the intermediate value theorem, and L'Hôpital's Rule. The proofs can be found on pages (46-49) in [1]. We now state the intermediate value theorem for a continuous function on a time scale.

Theorem 1.57. *(Intermediate Value Theorem). Assume $x : \mathbb{T} \to \mathbb{R}$ is continuous, $a < b$ are points in $\mathbb{T}$, and*

$$x(a)x(b) < 0$$

Then there exists $c \in [a, b)$ such that either $x(c) = 0$ or

$$x(c)x^{\sigma}(c) < 0$$

By $f^{\Delta}(t,\tau)$ in the following theorem we mean for each fixed τ the derivative of $f(t,\tau)$ with respect to t.

Theorem 1.58. *(Bohner and Peterson [1])Let $a \in \mathbb{T}^{\kappa}, b \in \mathbb{T}$ and assume $f : \mathbb{T} \times \mathbb{T}^{\kappa} \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^{\kappa}$ with $t > a$. Also assume that $f^{\Delta}(t,\cdot)$ is rd-continuous on $[a,\sigma(t)]$. Suppose that for each $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\tau \in [a,\sigma(t)]$, such that*

$$\left|f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t)-s)\right| \le \varepsilon|\sigma(t)-s| \quad \text{for all } s \in U,$$

where f^{Δ} *denotes the derivative of* f *with respect to the first variable. Then*

(i) $g(t) := \int_a^t f(t,\tau)\Delta\tau$ *implies* $g^{\Delta}(t) = \int_a^t f^{\Delta}(t,\tau)\Delta\tau + f(\sigma(t),t)$;

(ii) $h(t) := \int_t^b f(t,\tau)\Delta\tau$ *implies* $h^{\Delta}(t) = \int_t^b f^{\Delta}(t,\tau)\Delta\tau - f(\sigma(t),t)$.

Finally, we present several versions of L'Hôpital's rule. We let

$$\overline{\mathbb{T}} = \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}$$

If $\infty \in \overline{\mathbb{T}}$, we call ∞ left-dense, and $-\infty$ is called right-dense provided $-\infty \in \overline{\mathbb{T}}$. For any left-dense $t_0 \in \mathbb{T}$ and any $\varepsilon > 0$, the set

$$L_{\varepsilon}(t_0) = \{t \in \mathbb{T} : 0 < t_0 - t < \varepsilon\}$$

is nonempty, and so is $L_{\varepsilon}(\infty) = \left\{t \in \mathbb{T} : t > \frac{1}{\varepsilon}\right\}$ if $\infty \in \overline{\mathbb{T}}$. The sets $R_{\varepsilon}(t_0)$ for right-dense $t_0 \in \overline{\mathbb{T}}$ and $\varepsilon > 0$ are defined accordingly. For a function $h : \mathbb{T} \to \mathbb{R}$ we define

$$\liminf_{t \to t_0^-} h(t) = \lim_{t \to 0^+} \inf_{t \in L_{\varepsilon}(t_0)} h(t) \quad \text{for left-dense} \quad t_0 \in \overline{\mathbb{T}},$$

and $\liminf_{t \to t_0^+} h(t), \limsup_{t \to t_0^-} h(t), \limsup_{t \to t_0^+} h(t)$ are defined analogously.

Theorem 1.59. *(L'Hôpital's Rule, Bohner and Peterson [1]). Assume* f *and* g *are differentiable on* T *with*

$$\lim_{t \to t_0^-} f(t) = \lim_{t \to t_0^-} g(t) = 0 \quad \textit{for some left-dense} \quad t_0 \in \overline{\mathbb{T}}. \tag{1.20}$$

Suppose there exists $\varepsilon > 0$ *with*

$$g(t) > 0, g^{\Delta}(t) < 0 \quad \textit{for all} \quad t \in L_{\varepsilon}(t_0). \tag{1.21}$$

Then we have

$$\liminf_{t \to t_0^-} \frac{f^{\Delta}(t)}{g^{\Delta}(t)} \le \liminf_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f^{\Delta}(t)}{g^{\Delta}(t)}$$

Proof. See ([1], Theorem 1.119, p. 48) □

Theorem 1.60. *(L'Hôpital's Rule, Bohner and Peterson [1]). Assume* f *and* g *are differentiable on* T *with*

$$\lim_{t \to t_0^-} g(t) = \infty \quad \textit{for some left-dense} \quad t_0 \in \overline{\mathbb{T}}. \tag{1.22}$$

Suppose there exists $\varepsilon > 0$ *with*

$$g(t) > 0, g^{\Delta}(t) > 0 \quad \textit{for all} \quad t \in L_{\varepsilon}(t_0). \tag{1.23}$$

Then $\lim_{t \to t_0^-} \frac{f^{\Delta}(t)}{g^{\Delta}(t)} = r \in \overline{\mathbb{R}}$ *implies* $\lim_{t \to t_0^-} \frac{f(t)}{g(t)} = r$.

Proof. See ([1], Theorem 1.120, p. 49) □

CHAPTER 2

First Order Linear Equations

After having set up the ground work in the previous chapter, we now turn to the main goal of this thesis, that is applying the theory of time scale calculus to solve first order linear dynamic equations. We proceed by defining a new version of the exponential function on time scales, study its properties and prove that it is indeed a solution of the dynamic equations that we will define. We will also prove existence and uniqueness of solutions defined in terms of the exponential functions or first order equations.

This chapter provides an introduction to first order linear dynamic equations, extended by its applications. A first order dynamic equation is of the form

$$y^{\Delta}(t) = f(t, y, y^{\sigma}), \tag{2.1}$$

for $y : \mathbb{T} \to \mathbb{R}^n$ and $f : \mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$. A first order initial value problem (short: IVP) is then given by (2.1) with an initial condition $y(t_0) = y_0 \in \mathbb{R}^n$ for $t_0 \in \mathbb{T}$. A function $y : \mathbb{T} \to \mathbb{R}^n$ is called a *solution* of (2.1) if y satisfies the equation for all $t \in \mathbb{T}^{\kappa}$.

We call (2.1) linear if

$$f(t, y, y^{\sigma}) = f_1(t)y + f_2(t), \text{ or } f(t, y, y^{\sigma}) = f_1(t)y^{\sigma} + f_2(t),$$

where $f_1, f_2 : \mathbb{T} \to \mathbb{R}^n$. We say the linear dynamic equation is homogeneous, if $f_2 \equiv 0$.

The theorems and definitions in this chapter are analogous to the corresponding results on the classical time scale. We therefore follow the original proofs, which are given in the monograph by Martin Bohner and Allan Peterson [1]. So the theorems and definitions in this chapter are from [1] unless stated otherwise. Also when we mention the term exponential in this chapter we refer to the Hilger exponential function unless stated otherwise.

We will first start by constructing the solution of the initial value problem

$$y^{\Delta} = p(t)y, \; y(t_0) = 1$$

explicitly, and the solution will be called the *exponential function* associated with the given time scale, and then proceed to the more general initial condition $y(t_0) = t_0$. Let's start by introducing the so-called *Hilger complex plane.*

2.1 The Hilger complex plane

Definition 2.1. *For $h > 0$, define the Hilger complex numbers, the Hilger real axis, the Hilger alternating axis, and the Hilger imaginary circle by*

$$\mathbb{C}_h := \left\{z \in \mathbb{C} : z \neq -\tfrac{1}{h}\right\}, \quad \mathbb{R}_h := \left\{z \in \mathbb{R} : z > -\tfrac{1}{h}\right\},$$
$$\mathbb{A}_h := \left\{z \in \mathbb{R} : z < -\tfrac{1}{h}\right\}, \quad \mathbb{I}_h := \left\{z \in \mathbb{C} : \left|z + \tfrac{1}{h}\right| = \tfrac{1}{h}\right\},$$

respectively. For $h = 0$, let $\mathbb{C}_0 := \mathbb{C}, \mathbb{R}_0 := \mathbb{R}, \mathbb{A}_0 := \emptyset$, and $\mathbb{I}_0 := i\mathbb{R}$. See Figure 2.1 .

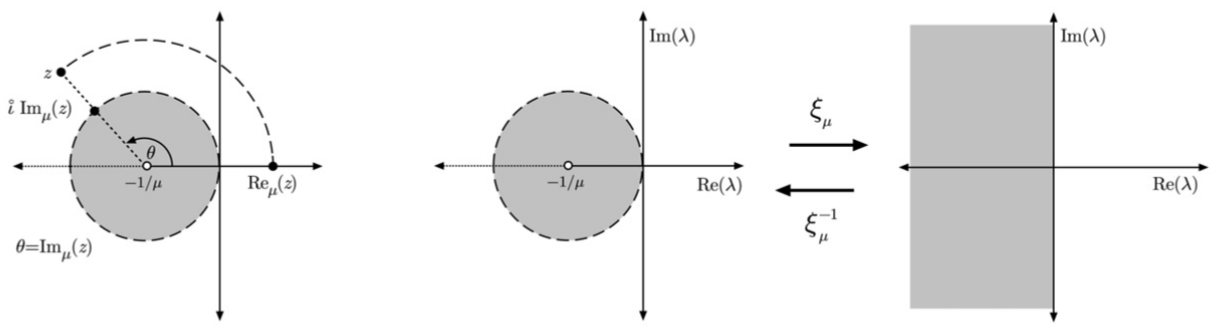

Figure 2.1. (Left) The Hilger complex plane. (Right) The cylinder and inverse cylinder transformations map the familiar stability region in the continuous case to the interior of the Hilger circle in the general time scale case

The figure above was sourced from [13].

The Hilger complex plane, pictured on the left in Figure 2.1 , is akin to the s plane of the one-sided Laplace transforms and the z plane of the one-sided z transform. Stability is associated with the shaded interior of the Hilger circle. The circle, centered at $-1/\mu(t)$, opens to the entire left plane as $\mu \to 0^+$, i.e. when the time scale becomes continuous. We are, in essence, in the s domain. When $\mu = 1$, the Hilger circle is the shifted unit circle in the z plane [1].

Definition 2.2. *Let $h > 0$ and $z \in \mathbb{C}_h$. We define the Hilger real part of z by*

$$\mathrm{Re}_h(z) := \frac{|zh + 1| - 1}{h}$$

and the Hilger imaginary part of z by

$$\mathrm{Im}_h(z) := \frac{\mathrm{Arg}(zh + 1)}{h}$$

[1] The unit circle in the z plane is typically centered around the origin. The circle's shift comes from our consideration of equations of the form $x^{\Delta}(t) = px(t)$ rather than of the form of the equivalent difference type equation of the form $x(t + \mu(t)) = (\mu(t)p + 1)x(t)$

where $\operatorname{Arg}(z)$ *denotes the principal argument of* z *(i.e.,* $-\pi < \operatorname{Arg}(z) \leq \pi$ *). Note that* $\operatorname{Re}_h(z)$ *and* $\operatorname{Im}_h(z)$ *satisfy*

$$-\frac{1}{h} < \operatorname{Re}_h(z) < \infty \quad \text{and} \quad -\frac{\pi}{h} < \operatorname{Im}_h(z) \leq \frac{\pi}{h}$$

respectively. In particular, $\operatorname{Re}_h(z) \in \mathbb{R}_h$

Definition 2.3. *Let* $-\frac{\pi}{h} < \omega \leq \frac{\pi}{h}$*. We define the Hilger purely imaginary number* $\mathring{\imath}\omega$ *by*

$$\mathring{\imath}\omega = \frac{e^{i\omega h} - 1}{h}.$$

For $z \in \mathbb{C}_h$*, we have that* $\mathring{\imath}\operatorname{Im}_h(z) \in \mathbb{I}_h$*.*

2.1.1 The cylinder transformation

Definition 2.4. *For* $h > 0$*, define the strip* $\mathbb{Z}_h := \left\{z \in \mathbb{C} : -\frac{\pi}{h} < \operatorname{Im}(z) \leq \frac{\pi}{h}\right\}$*, and for* $h = 0$*, set* $\mathbb{Z}_0 := \mathbb{C}$*. Then we can define the cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ *by*

$$\xi_h(z) = \frac{1}{h}\log(1 + zh), h > 0$$

where Log is the principal logarithm function. When $h = 0$*, we define* $\xi_0(z) = z$*, for all* $z \in \mathbb{C}$*. The cylinder transformation maps the interior of the Hilger circle to the left half plane.*

The inverse cylinder transformation $\xi_h^{-1} : \mathbb{Z}_h \to \mathbb{C}_h$ *is given by*

$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h}$$

We call ξ_h the cylinder transformation because when $h > 0$ we can view the strip $\mathbb{Z}_h$ as a cylinder if we glue the bordering lines $\operatorname{Im}_h(z) = -\frac{\pi}{h}$ and $\operatorname{Im}_h(z) = \frac{\pi}{h}$ together to form a cylinder.

2.2 Algebra on $\mathbb{C}_h$

We give a brief overview of the algebraic structures attached to the Hilger complex plane $\mathbb{C}_h$. As we will see latter, some of the algebraic properties we will define will be used to simplify long expressions involving the exponential function. More interestingly, these algebraic operations allow the definition of abelian (topological) groups and hence the emergence of group homomorphisms(i.e. a map between two groups such that the group operation is preserved), one of which the cylinder transformation is. We remind the reader again that theorems and definitions in this and following sections are from the monograph by Bohner and Peterson [1] unless explicitly stated. Some proofs of the latter can also be found in the monograph in Chapter 2.

Theorem 2.5. *If we define the "circle plus" addition* $\oplus$ *on* $\mathbb{C}_h$ *by*

$$z \oplus w := z + w + zwh$$

then $(\mathbb{C}_h, \oplus)$ *is an Abelian group.*

Remark 2.6. *The additive inverse of* z *under the operation* $\oplus$ *is*

$$\ominus := \frac{-z}{1+zh} \tag{2.2}$$

Definition 2.7. *We also define the "circle minus" substraction* $\ominus$ *on* $\mathbb{C}_h$ *by*

$$z \ominus w := z \oplus (\ominus w). \tag{2.3}$$

Theorem 2.8. *For* $z \in \mathbb{C}_h$ *we have*

$$z = \operatorname{Re}_h z \oplus \mathring{\imath} \operatorname{Im}_h z$$

Proof. Let $z \in \mathbb{C}_h$. Then

$$\begin{aligned}
\operatorname{Re}_h z \oplus i \operatorname{Im}_h z =& \frac{|zh+1|-1}{h} \oplus \mathring{\imath} \frac{\operatorname{Arg}(zh+1)}{h} \\
=& \frac{|zh+1|-1}{h} \oplus \frac{\exp(i \operatorname{Arg}(zh+1)) - 1}{h} \\
=& \frac{|zh+1|-1}{h} + \frac{\exp(i \operatorname{Arg}(zh+1)) - 1}{h} \\
&+ \frac{|zh+1|-1}{h} \frac{\exp(i \operatorname{Arg}(zh+1)) - 1}{h} h \\
=& \frac{1}{h}\{|zh+1| - 1 + \exp(i \operatorname{Arg}(zh+1)) - 1 \\
&+ [|zh+1|-1][\exp(i \operatorname{Arg}(zh+1)) - 1]\} \\
=& \frac{1}{h}\{|zh+1| \exp(i \operatorname{Arg}(zh+1)) - 1\} \\
=& \frac{(zh+1)-1}{h} = z,
\end{aligned}$$

which proves the claim. □

Theorem 2.9. *The cylinder transformation* ξ_h *is a group homomorphism from* $(\mathbb{C}_h, \oplus)$ *onto* $(\mathbb{Z}_h, +)$*, where the addition* $+$ *on* $\mathbb{Z}_h$ *is defined by*

$$z + w := z + w \left(\bmod \frac{2\pi i}{h}\right) \text{ for } z, w \in \mathbb{Z}_h. \tag{2.4}$$

Proof. Let $h > 0$ and $z, w \in \mathbb{C}_h$ and consider

$$\begin{aligned}
\xi_h(z \oplus w) &= \frac{1}{h} \log(1 + (z \oplus w)h) \\
&= \frac{1}{h} \log\left(1 + zh + wh + zwh^2\right) \\
&= \frac{1}{h} \log[(1 + zh)(1 + wh)] \\
&= \frac{1}{h} \log(1 + zh) + \frac{1}{h} \log(1 + wh) \\
&= \xi_h(z) + \xi_h(w).
\end{aligned}$$

This proves our result for $h > 0$. The case $h = 0$ is trivial. □

2.3 Explicit representation of the Exponential function

In this section we use the cylinder transformation introduced in Section 2.1.1 to define a generalized exponential function for an arbitrary time scale $\mathbb{T}$. First we make some preliminary definitions. Things get interesting as this exponential function lies at the heart of solving F.O.L.D.Es!

Definition 2.10. *We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided*

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted in this book by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$$

We also define the set $\mathcal{R}^+$of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$$

Remark 2.11. *It can be easily showed that $\mathcal{R}$ is an Abelian group under the "circle plus" addition $\oplus$ defined by*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \text{for all} \quad t \in \mathbb{T}^\kappa,$$

$p, q \in \mathcal{R}$. This group is called the regressive group. If $p, q \in \mathcal{R}$, then the function $p \oplus q$ and the function $\ominus p$ defined by

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)} \quad \text{for all} \quad t \in \mathbb{T}^\kappa$$

are also elements of $\mathcal{R}$.'

Definition 2.12. *We define the "circle minus" subtraction $\ominus$ on $\mathcal{R}$ by*

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t) \quad \text{for all } t \in \mathbb{T}^{\kappa}. \tag{2.5}$$

Lemma 2.13. *Suppose $p, q \in \mathcal{R}$, then*

(i) $p \ominus p = 0$;

(ii) $\ominus(\ominus p) = p$;

(iii) $p \ominus q \in \mathcal{R}$;

(iv) $p \ominus q = \frac{p-q}{1+\mu q}$

(v) $\ominus(p \ominus q) = q \ominus p$

(vi) $\ominus(p \oplus q) = (\ominus p) \oplus (\ominus q)$.

Proof. Follows easily by substituting the p formula and simple algebraic manipulations. ∎

Definition 2.14. *If $p \in \mathcal{R}$, then we define the exponential function by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for} \quad s, t \in \mathbb{T}, \tag{2.6}$$

where the cylinder transformation $\xi_h(z)$ is introduced in Definition 2.4.

Lemma 2.15. *If $p \in \mathcal{R}$, then the semigroup property*

$$e_p(t, r)e_p(r, s) = e_p(t, s) \quad \text{for all} \quad r, s, t \in \mathbb{T}$$

is satisfied.

Proof. Suppose $p \in \mathcal{R}$. Let $r, s, t \in \mathbb{T}$. Then we have by Definition 2.14

$$\begin{aligned} e_p(t, r)e_p(r, s) &= \exp\left(\int_r^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)\exp\left(\int_s^r \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\ &= \exp\left(\int_r^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau + \int_s^r \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\ &= \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \\ &= e_p(t, s) \end{aligned}$$

where we have used Theorem 1.39 (iv). ∎

Definition 2.16. *If $p \in \mathcal{R}$, then the first order linear dynamic equation*

$$y^{\Delta} = p(t)y \tag{2.7}$$

is called ***regressive****.*

Up to now we have defined what regressive functions are, but why are they important? Well, we find out that regressivity plays a crucial role in developing the fundamental theory of linear dynamic equations. When we consider the generalized time scale exponential function, we need the concept of regressiveness since the exponential function $e_p(t, t_0)$ is defined only for $p(t)$ satisfying the regressivity condition, i.e. $1 + \mu(t)p(t) \neq 0$.

Let us prove a new lemma that was posed as a research exercise by Ronald Mathsen (see [1] Exercise 2.34 p. 61), that is useful in the following existence theorem for the solution of an initial value problem of a linear dynamic equation.

Lemma 2.17. *Let $p \in \mathcal{R}$ and $t \in \mathbb{T}^{\kappa}$ be such that $\sigma(t) = t$. Then*

$$\lim_{r \to t} \xi_{\mu(r)}(p(r)) = \xi_0(p(t)). \tag{2.8}$$

Proof. Let $h := \mu(r)p(r)$ for $r \in \mathbb{T}^{\kappa}$. Since σ is rd-continuous, then $h \to 0$ when $r \to t$ in $\mathbb{T}^{\kappa}$. This implies that

$$\begin{aligned} \lim_{r \to t} \xi_{\mu(r)}(p(r)) &= \lim_{r \to t} \frac{\log(1 + \mu(r)p(r))}{\mu(r)p(r)}.p(r) \\ &= \lim_{h \to 0} \frac{\log(1 + h)}{h}.p(t) \\ &= p(t) \\ &= \xi_0(p(t)). \end{aligned}$$

□

Theorem 2.18. *Suppose (2.1) is regressive and fix $t_0 \in \mathbb{T}$. Then $e_p(\cdot, t_0)$ is a solution of the initial value problem*

$$y^{\Delta} = p(t)y, \quad y(t_0) = 1 \tag{2.9}$$

on $\mathbb{T}$.

Proof. Fix $t_0 \in \mathbb{T}^{\kappa}$ and assume (2.7) is regressive. First note that

$$e_p(t_0, t_0) = 1$$

It remains to show that $e_p(t,t_0)$ satisfies the dynamic equation $y^{\Delta} = p(t)y$. Fix $t \in \mathbb{T}^{\kappa}$. There are two cases.

Case 1. Assume $\sigma(t) > t$. By the inverse cylinder transformation formula in definition 2.4 and Lemma 2.15,

$$\begin{aligned} e_p^{\Delta}(t,t_0) &= \frac{\exp\left(\int_{t_0}^{\sigma(t)} \xi_{\mu(r)}(p(r))\Delta r\right) - \exp\left(\int_{t_0}^{t} \xi_{\mu(r)}(p(r))\Delta r\right)}{\mu(t)} \\ &= \frac{\exp\left(\int_{t}^{\sigma(t)} \xi_{\mu(r)}(p(r))\Delta r\right) - 1}{\mu(t)} e_p(t,t_0) \ \text{(semigroup property)} \\ &= \frac{e^{\xi_{\mu(t)}(p(t))\mu(t)} - 1}{\mu(t)} e_p(t,t_0) \\ &= \xi_{\mu(t)}^{-1}\left(\xi_{\mu(t)}(p(t))\right) \cdot e_p(t,t_0) \ \text{(inverse cylinder transformation)} \\ &= p(t) \cdot e_p(t,t_0). \end{aligned}$$

Case 2. Assume $\sigma(t) = t$. If $y(t) := e_p(t,t_0)$, then we want to show that $y^{\Delta}(t) = p(t)y(t)$. Using Lemma 2.15 we obtain

$$\begin{aligned} &|y(t) - y(s) - p(t)y(t)(t-s)| = |e_p(t,t_0) - e_p(s,t_0) - p(t)e_p(t,t_0)(t-s)| \\ &= |e_p(t,t_0)| \cdot |1 - e_p(s,t) - p(t)(t-s)| \\ &= |e_p(t,t_0)| \left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t) + \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - p(t)(t-s)\right| \\ &\le |e_p(t,t_0)| \cdot \left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)\right| \\ &+ |e_p(t,t_0)| \cdot \left|\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - p(t)(t-s)\right| \\ &\le |e_p(t,t_0)| \cdot \left|1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)\right| \\ &+ |e_p(t,t_0)| \cdot \left|\int_s^t \left[\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))\right]\Delta\tau\right|. \end{aligned}$$

Let $\varepsilon > 0$ be given. We now show that there is a neighborhood U of t so that the right-hand side of the last inequality is less than $\varepsilon|t-s|$, and the proof will be complete. Since $\sigma(t) = t$ and $p \in \mathrm{C_{rd}}$, it follows that (see lemma 2.17)

$$\lim_{r \to t} \xi_{\mu(r)}(p(r)) = \xi_0(p(t)).$$

This implies that there is a neighborhood U_1 of t such that

$$\left|\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t))\right| < \frac{\varepsilon}{3|e_p(t,t_0)|} \quad \text{for all} \quad \tau \in U_1.$$

Let $s \in U_1$. Then

$$|e_p(t,t_0)| \cdot \left| \int_s^t \left[\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t)) \right] \Delta\tau \right| < \frac{\varepsilon}{3}|t-s|. \tag{2.10}$$

Next, by L'Hôpital's rule

$$\lim_{z\to 0} \frac{1-z-e^{-z}}{z} = 0$$

so there is a neighborhood U_2 of t so that if $s \in U_2$, then

$$\left| \frac{1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t)}{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau} \right| < \varepsilon^*,$$

where

$$\varepsilon^* = \min\left\{1, \frac{\varepsilon}{1+3\,|p(t)e_p(t,t_0)|}\right\}.$$

Let $s \in U := U_1 \cap U_2$. Then

$$\begin{aligned}
|e_p(t,t_0)| \cdot \left| 1 - \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau - e_p(s,t) \right| &< |e_p(t,t_0)|\,\varepsilon^* \left| \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right| \\
&\le |e_p(t,t_0)| \cdot \varepsilon^* \left\{ \left| \int_s^t \left\{ \xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t)) \right] \Delta\tau \right| + |p(t)||t-s| \right\} \\
&\le |e_p(t,t_0)| \cdot \left| \int_s^t \left[\xi_{\mu(\tau)}(p(\tau)) - \xi_0(p(t)) \right] \Delta\tau \right| + |e_p(t,t_0)|\,\varepsilon^* |p(t)||t-s| \\
&\le \frac{\varepsilon}{3}|t-s| + |e_p(t,t_0)|\,\varepsilon^*|p(t)||t-s| \\
&\le \frac{\varepsilon}{3}|t-s| + \frac{\varepsilon}{3}|t-s| \\
&= \frac{2\varepsilon}{3}|t-s|
\end{aligned}$$

using (2.10). □

As we have just confirmed, the generalized exponential function $e_p(t,t_0)$ **exists** as a solution to the initial value problem (2.9). The natural thing to do next in the theory of equations in general is to verify if indeed this solution is unique. The result is affirmative, as we see in the following theorem.

Theorem 2.19. *If (2.7) is regressive, then the only solution of (2.9) is given by* $e_p(\cdot,t_0)$

Proof. Assume y is a solution of (2.9) and consider the quotient $y/e_p(\cdot, t_0)$ (note that by Definition 2.14 we have that $e_p(t,s) \neq 0$ for all $t, s \in \mathbb{T}$). By Theorem 1.12 (v) we have

$$\begin{aligned} \left(\frac{y}{e_p(\cdot, t_0)}\right)^{\Delta}(t) &= \frac{y^{\Delta}(t)e_p(t,t_0) - y(t)e_p^{\Delta}(t,t_0)}{e_p(t,t_0)\, e_p(\sigma(t),t_0)} \\ &= \frac{p(t)y(t)e_p(t,t_0) - y(t)p(t)e_p(t,t_0)}{e_p(t,t_0)\, e_p(\sigma(t),t_0)} \\ &= 0 \end{aligned}$$

so that $y/e_p(\cdot, t_0)$ is constant according to Corollary1.32 (ii). Hence

$$\frac{y(t)}{e_p(t,t_0)} \equiv \frac{y(t_0)}{e_p(t_0,t_0)} = \frac{1}{1} = 1$$

and therefore $y = e_p(\cdot, t_0)$. □

We now know that there exists a solution for a first order linear dynamic equation (F.O.L.D.E) that is homogenous, that is given explicitly as the generalized exponential function over an arbitrary time scale. Interestingly enough, this solution is unique. Before we proceed to non-homogenous first order linear dynamic equations, we state and prove some important properties of the exponential function that will make it easy when solving typical F.O.L.D.Es. We proceed by collecting some important properties of the exponential function.

Theorem 2.20. *If $p, q \in \mathcal{R}$, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

(viii) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

Proof. Note that the proofs are a straightforward application of the definition of the exponential function and an application of the existence and uniqueness theorems we have just proved. Thus we prove (ii), (v) and (vii), for brevity sake as other parts of the proof can be found in [1].

Part (ii). By Theorem 1.10 (iv) we have

$$\begin{aligned} e_p(\sigma(t), s) &= e_p^\sigma(t, s) \\ &= e_p(t, s) + \mu(t) e_p^\Delta(t, s) \\ &= e_p(t, s) + \mu(t)p(t)e_p(t, s) \\ &= (1 + \mu(t)p(t))e_p(t, s), \end{aligned}$$

where we have used Theorem 2.18.

Part (v). This semigroup property was already shown in Lemma 2.15, based on the definition of the exponential function in terms of the cylinder transformation. However, let us here deduce the semigroup property using only the fact that exponential functions are unique solutions of initial value problems (2.18). Consider the initial value problem

$$y^\Delta = p(t)y, \quad y(r) = 1. \tag{2.11}$$

We show that $y(t) := e_p(t, s)e_p(s, r)$ satisfies (2.11), and then the claim follows from Theorem 2.18 and Theorem 2.19. It is obvious that $y^\Delta(t) = p(t)y(t)$, and

$$y(r) = e_p(r, s)e_p(s, r) = 1$$

follows from part (iv).

Part (vii). One can see that

$$\begin{aligned} \frac{e_p(t,s)}{e_q(t,s)} &= \exp\left(\int_s^t \frac{\log(1+\mu(\tau)p(\tau))}{\mu(\tau)} \Delta\tau\right) \cdot \exp\left(-\int_s^t \frac{\log(1+\mu(\tau)q(\tau))}{\mu(\tau)} \Delta\tau\right) \\ &= \exp\left(\int_s^t \frac{1}{\mu(\tau)} \log \frac{1+\mu(\tau)p(\tau)}{1+\mu(\tau)q(\tau)} \Delta\tau\right) \\ &= \exp\left(\int_s^t \frac{1}{\mu(\tau)} \log \left(1+\mu(\tau)\frac{p(\tau)-q(\tau)}{1+\mu(\tau)q(\tau)}\right) \Delta\tau\right) \\ &= e_{p\ominus q}(t,s) \end{aligned}$$

□

The following results simplify calculations that involve integrals of exponential functions and we will use this technique in an example to follow.

Theorem 2.21. *If $p, q \in \mathcal{R}$, then*

$$e_{p\ominus q}^\Delta(\cdot, t_0) = (p - q)\frac{e_p(\cdot, t_0)}{e_q^\sigma(\cdot, t_0)}$$

Proof. We have, and use Theorem 2.20 (ii) and (vii)

$$\begin{aligned} e^{\Delta}_{p\ominus q}(t,t_0) &= (p\ominus q)(t)e_{p\ominus q}(t,t_0) \\ &= \frac{p(t)-q(t)}{1+\mu(t)q(t)}\frac{e_p(t,t_0)}{e_q(t,t_0)} \\ &= \frac{(p(t)-q(t))e_p(t,t_0)}{e_q(\sigma(t),t_0)} \end{aligned}$$

□

Theorem 2.22. *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c,\cdot)]^{\Delta} = -p\,[e_p(c,\cdot)]^{\sigma}$$

and

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b).$$

Proof. We use many of the properties from Theorem 2.20 to find

$$\begin{aligned} p(t)e_p(c,\sigma(t)) &= p(t)e_{\ominus p}(\sigma(t),c) \\ &= p(t)[1+\mu(t)(\ominus p)(t)]e_{\ominus p}(t,c) \\ &= p(t)\left[1-\frac{\mu(t)p(t)}{1+\mu(t)p(t)}\right]e_{\ominus p}(t,c) \\ &= p(t)\frac{1}{1+\mu(t)p(t)}e_{\ominus p}(t,c) \\ &= -(\ominus p)(t)e_{\ominus p}(t,c) \\ &= -e^{\Delta}_{\ominus p}(t,c) \\ &= -\left[e_p(c,t)\right]^{\Delta}, \end{aligned}$$

where $^{\Delta}$ denotes differentiation with respect to t. Thus

$$\begin{aligned} \int_a^b p(t)e_p(c,\sigma(t))\Delta t &= -\int_a^b \left[e_p(c,\cdot)\right]^{\Delta}(t)\Delta t \\ &= e_p(c,a) - e_p(c,b) \end{aligned}$$

which proves our desired identity. □

And now an example to illustrate how we can use the previous theorem to quickly solve integrals involving the exponential function

Example 2.23. *Assume $\frac{2}{t}, \frac{5}{t}$ are regressive on $\mathbb{T}\cap(0,\infty)$ and let $t_0 \in \mathbb{T}\cap(0,\infty)$. Evaluate the integral*

$$\int_{t_0}^{t} \frac{e_{\frac{5}{s}}(s,t_0)}{s e^{\sigma}_{\frac{2}{s}}(s,t_0)}\Delta s.$$

We use Theorem 2.21 by choosing $p = \frac{5}{s}$ and $q = \frac{2}{s}$.

$$\begin{aligned}\int_{t_0}^{t} \frac{e_{\frac{5}{s}}(s,t_0)}{s e^{\sigma}_{\frac{2}{s}}(s,t_0)}\Delta s &= \frac{1}{3}\int_{t_0}^{t} \frac{3}{s}\frac{e_{\frac{5}{s}}(s,t_0)}{e^{\sigma}_{\frac{2}{s}}(s,t_0)}\Delta s\\ &= \frac{1}{3}\int_{t_0}^{t} e^{\Delta}_{p\ominus q}(s,t_0)\,\Delta s\\ &= \frac{1}{3}\big(e_{p\ominus q}(t,t_0) - e_{p\ominus q}(t_0,t_0)\big)\end{aligned}$$

and now using Lemma 2.13, $p \ominus q = \frac{3}{t+2\mu t}$ and the fact that $e_{p\ominus q}(t_0,t_0) = 1$ we finally get

$$\int_{t_0}^{t} \frac{e_{\frac{5}{s}}(s,t_0)}{s e^{\sigma}_{\frac{2}{s}}(s,t_0)}\Delta s = \frac{1}{3}e_{\frac{3}{t+2\mu t}}(t,t_0) - \frac{1}{3}.$$

Unlike the classical exponential function which is never negative, the Hilger exponential function can be negative sometimes. In some time scales like $\mathbb{T} = \mathbb{Z}$ the exponential function changes sign at every point and is said to be **oscillatory**. However, In particular, the Hilger exponential function $e_p(,,t_0)$ is a real-valued function that is never equal to zero.

In the remainder of this section we study the sign of the exponential function. We state the theorems without proof which can be found in [1] and more specifically Lemma 2.24 below appear in Akın, Erbe, Kaymakçalan, and Peterson [8].

Lemma 2.24. *Let $p \in \mathcal{R}$. Suppose there exists a sequence of distinct points $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{T}^\kappa$ such that*

$$1 + \mu(t_n)\,p(t_n) < 0 \quad \textit{for all} \quad n \in \mathbb{N}.$$

Then $\lim_{n\to\infty}|t_n| = \infty$. In particular, if there exists a bounded interval $J \subset \mathbb{T}^\kappa$ such that $1 + \mu(t)p(t) < 0$ for all $t \in J$, then $|J| < \infty$.

Proof. See Lemma 13 in [8] □

Theorem 2.25. *Assume $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$.*

(i) If $1 + \mu p > 0$ on $\mathbb{T}^\kappa$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.

(ii) If $1 + \mu p < 0$ on $\mathbb{T}^\kappa$, then $e_p(t,t_0) = \alpha(t,t_0)(-1)^{n_t}$ for all $t \in \mathbb{T}$, where

$$\alpha(t,t_0) := \exp\left(\int_{t_0}^{t} \frac{\log|1+\mu(\tau)p(\tau)|}{\mu(\tau)}\Delta\tau\right) > 0$$

and

$$n_t = \begin{cases} |[t_0,t)| & \textit{if} \quad t \geq t_0\\ |[t,t_0)| & \textit{if} \quad t < t_0\end{cases}$$

Proof. See Theorem 2.44 in [1] □

In view of Theorem 2.25 (i), we notice that for positively regressive elements of $\mathcal{R}$, the exponential function is strictly positive.

We now turn to examples of exponential functions where we try to derive the exponential function equivalent of an arbitrary time scale. For some time scales the resulting exponential function is rather unexpected and involve therefore factorials, binomial coefficients and even the Wallis product! And hence it is interesting to take a look at several of these:

2.4 Examples of Exponential Functions on Arbitrary Time Scales

Example 2.26. *Let $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. Let $\alpha \in \mathcal{R}$ be constant, i.e.,*

$$\alpha \in \mathbb{C} \backslash \left\{ -\frac{1}{h} \right\}.$$

Then

$$e_\alpha(t, 0) = (1 + \alpha h)^{\frac{t}{h}} \quad \text{for all} \quad t \in \mathbb{T}. \tag{2.12}$$

We will first derive 2.12 from first principles i.e. by using the properties of the time scale $h\mathbb{Z}$ and the definition of the exponential function. And then we go on to show it is indeed the exponential function (solution) for F.O.L.D.Es (first order linear dynamic equations) defined on $h\mathbb{Z}$.

Note that the graininess ($\mu(t)$) of $h\mathbb{Z}$ is h, and in this case our regressive function p is the constant function α.

$$\begin{aligned} e_\alpha(t, 0) &= e^{\int_0^t \xi_h(\alpha)\Delta\tau} \\ &= e^{\int_0^t \frac{\log(1+\alpha h)}{h}\Delta\tau} \text{ (definition of cylinder transformation)} \\ &= e^{\sum_0^{\frac{t}{h}-1} \log(1+\alpha h)} \text{ by Theorem 1.40 (iii)} \\ &= e^{\frac{t}{h}\log(1+\alpha h)} \\ &= (1 + \alpha h)^{\frac{t}{h}} \end{aligned}$$

Let us now show that this exponential function is a solution of a F.O.L.D.E on $h\mathbb{Z}$. To show this we note that y defined by the right-hand side of (2.12) satisfies

$$y(0) = (1 + \alpha h)^0 = 1$$

and

$$\begin{aligned} y^{\Delta}(t) &= \frac{y(t+h)-y(t)}{h} \\ &= \frac{(1+\alpha h)^{\frac{t+h}{h}} - (1+\alpha h)^{\frac{t}{h}}}{h} \\ &= \frac{(1+\alpha h)^{\frac{t}{h}}(1+\alpha h - 1)}{h} \\ &= \alpha(1+\alpha h)^{\frac{t}{h}} \\ &= \alpha y(t) \end{aligned}$$

for all $t \in \mathbb{T}$.

Example 2.27. *Consider the time scale*

$$\mathbb{T} = \mathbb{N}_0^2 = \left\{n^2 : n \in \mathbb{N}_0\right\}$$

We claim that

$$e_1(t,0) = 2^{\sqrt{t}}(\sqrt{t})! \quad \textit{for} \quad t \in \mathbb{T}. \tag{2.13}$$

Let y be defined by the right-hand side of (2.13). Clearly, $y(0) = 1$, and for $t \in \mathbb{T}$ we have

$$\begin{aligned} y(\sigma(t)) &= 2^{\sqrt{\sigma(t)}}(\sqrt{\sigma(t)})! \\ &= 2^{1+\sqrt{t}}(1+\sqrt{t})! \\ &= 2 \cdot 2^{\sqrt{t}}(1+\sqrt{t})(\sqrt{t})! \\ &= 2(1+\sqrt{t})y(t) \\ &= (1+\mu(t))y(t) \\ &= y(t) + \mu(t)y(t) \end{aligned}$$

so that $y^{\Delta}(t) = y(t)$

We consider an interesting time scale formed by the so called quantum numbers below, which is mainly used to solve *q-difference equations*.

Example 2.28. *We consider the time scale* $\mathbb{T} = q^{\mathbf{N}_0}$.

Let $p \in \mathcal{R}$. The problem

$$y^{\Delta} = p(t)y, \quad y(1) = 1$$

can be equivalently rewritten as

$$y^{\sigma} = (1 + (q-1)tp(t))y, \quad y(1) = 1.$$

The solution of this problem obviously (easily derived using the cylinder transformation formular and the definition of the exponential function) is

$$e_p(t,1) = \prod_{s \in \mathbb{T} \cap (0,t)} (1 + (q-1)sp(s)). \tag{2.14}$$

If $\alpha \in \mathcal{R}$ is constant, then we have

$$e_\alpha(t,1) = \prod_{s\in\mathbb{T}\cap(0,t)} (1+(q-1)\alpha s).$$

For $q = 2$ this simplifies as

$$e_\alpha(t,1) = \prod_{s\in\mathbb{T}\cap(0,t)} (1+\alpha s).$$

Now consider the special case when

$$p(t) = \frac{1-t}{(q-1)t^2} \quad \text{for} \quad t \in \mathbb{T}$$

Using (2.14), we find

$$\begin{aligned} e_p(t,1) &= \prod_{s\in\mathbf{T}\cap(0,t)} (1+(q-1)sp(s)) \\ &= \prod_{s\in\mathrm{T}\cap(0,t)} \left(1+\frac{1-s}{s}\right) \\ &= \prod_{s\in\mathbf{T}\cap(0,t)} \frac{1}{s} \\ &= \prod_{n=0}^{k-1} \frac{1}{q^n} \\ &= \frac{1}{q^{k(k-1)/2}} \\ &= q^{\frac{k}{2}} q^{-\frac{k^2}{2}} \end{aligned}$$

where we put $t = q^k$. Substituting $t = q^k$ we finally get that

$$e_p(t,1) = \sqrt{t}e^{-\frac{\ln^2(t)}{2\ln(q)}}$$

Definition 2.29. *If $q > 1$, then any dynamic equation on either of the time scales*

$$\overline{q^{\mathbf{Z}}} \text{ or } q^{\mathbf{N}_0}$$

is called a q-difference equation (see, e.g., Bézivin [9]).

We now give an application where we would want to find the exponential function $e_1(\cdot, 0)$ for $\mathbb{P}_{1,1}$. The following example seems to be the favorite among time scale researchers! Certainly because it captures the dynamic nature of nature by modelling both continuous and discrete points. More details below.

Let $N(t)$ be the number of plants of one particular kind at time t in a certain area. By experiments we know that N grows exponentially according to $N' = N$ during the months of April until September. At the beginning of October, all plants suddenly die, but the seeds remain in the ground and start growing again at the beginning of April with N now being doubled. We model this situation using the time scale

$$\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$$

where $t = 0$ is April 1 of the current year, $t = 1$ is October 1 of the current year, $t = 2$ is April 1 of the next year, $t = 3$ is October 1 of the next year, and so on. We have

$$\mu(t) = \begin{cases} 0 & \text{if} \quad 2k \leq t < 2k+1 \\ 1 & \text{if} \quad t = 2k+1 \end{cases}$$

On $[2k, 2k+1)$, we have $N' = N$, i.e., $N^{\Delta} = N$. However, we also know that $N(2k+2) = 2N(2k+1)$, i.e., $\Delta N(2k+1) = N(2k+1)$, i.e., $N^{\Delta} = N$ at $2k+1$. As a result, N is a solution of the dynamic equation

$$N^{\Delta} = N$$

Thus, if $N(0) = 1$ is given, N is exactly $e_1(\cdot, 0)$ on the time scale $\mathbb{T}$. We can calculate N as follows: If $k \in \mathbb{N}_0$ and $t \in [2k, 2k+1]$, then N satisfies $N' = N$ so that

$$N(t) = \alpha_k e^t \quad \text{for some} \quad \alpha_k \in \mathbb{R}$$

Since $N(0) = 1$, we have

$$1 = N(0) = \alpha_0 e^0 = \alpha_0 \quad \text{and} \quad N(t) = \alpha_0 e^t = e^t \text{ for } 0 \leq t \leq 1.$$

Thus $N(1) = e$ and $N(2) = 2N(1) = 2e$. Now

$$2e = N(2) = \alpha_1 e^2 \quad \text{and} \quad N(t) = \alpha_1 e^t = \frac{2}{e} e^t = 2e^{t-1} \text{ for } 2 \leq t \leq 3.$$

Hence $N(3) = 2e^2$ and $N(4) = 2N(3) = 4e^2$. Next

$$4e^2 = N(4) = \alpha_2 e^4 \quad \text{and} \quad N(t) = \alpha_2 e^t = \frac{4}{e^2} e^t = 4e^{t-2} \text{ for } 4 \leq t \leq 5.$$

We now use mathematical induction to show that

$$N(t) = \left(\frac{2}{e}\right)^k e^t \quad \text{for} \quad t \in [2k, 2k+1].$$

The statement is already shown for $k = 0$. Assume it is true for $k = m \in \mathbb{N}_0$. Then

Table 2.1. Plant Population

t	0	1	2	3	4	5	6	7
$e_1(t,0)$	1	e	$2e$	$2e^2$	$4e^2$	$4e^3$	$8e^3$	$8e^4$

$N(t) = (2/e)^m e^t$ for $t \in [2m, 2m+1]$ so that

$$N(2m+1) = \left(\frac{2}{e}\right)^m e^{2m+1} = 2^m e^{m+1}$$

and

$$N(2m+2) = 2N(2m+1) = 2 \cdot 2^m e^{m+1} = (2e)^{m+1}$$

Therefore

$$(2e)^{m+1} = N(2m+2) = \alpha_{m+1} e^{2m+2}$$

and

$$N(t) = \alpha_{m+1} e^t = \left(\frac{2}{e}\right)^{m+1} e^t \quad \text{for} \quad 2m+2 \le t \le 2m+3.$$

In Table 2.1 we collected a few values of this exponential function.

We cannot exhaust the list of exponential functions on arbitrary time scales, and below we list without demonstrating, several versions of exponential functions on mostly encountered time scales.

Table 2.2. Exponential functions

$\mathbb{T}$	$e_\alpha(t, t_0)$
$\mathbb{R}$	$e^{\alpha(t-t_0)}$
$\mathbb{Z}$	$(1+\alpha)^{t-t_0}$
$h\mathbb{Z}$	$(1+\alpha h)^{(t-t_0)/h}$
$\frac{1}{n}\mathbb{Z}$	$\left(1+\frac{\alpha}{n}\right)^{n(t-t_0)}$
$q^{\mathbb{N}_0}$	$\prod_{s\in[t_0,t)}[1+(q-1)\alpha s]$ if $t > t_0$
2^{N_0}	$\prod_{s\in[t_0,t)}(1+\alpha s)$ if $t > t_0$
$\left\{\sum_{k=1}^{n}\frac{1}{k} : n \in \mathbb{N}\right\}$	$\binom{n+\alpha-t_0}{n-t_0}$ if $t = \sum_{k=1}^{n}\frac{1}{k}$

2.5 Initial Value Problems (IVPs)

In this final section of this expository work, we study the first order non-homogeneous linear equations. This will provide a strong framework to solve most F.O.L.D.Es that are non-homogenous that one might encounter.

We study the equation

$$y^{\Delta} = p(t)y + f(t) \tag{2.15}$$

and the corresponding homogeneous equation

$$y^{\Delta} = p(t)y \tag{2.16}$$

on a time scale $\mathbb{T}$. The results from Section 2.3 immediately yield the following theorem.

Theorem 2.30. *Suppose (2.16) is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^{\Delta} = p(t)y, \quad y(t_0) = y_0 \tag{2.17}$$

is given by

$$y(t) = e_p(t, t_0)\, y_0$$

Proof. Simple delta differentiation of the function y given above verifies the Theorem directly. □

Definition 2.31. *For $p \in \mathcal{R}$ we define an operator $L_1 : \mathrm{C}^1_{\mathrm{rd}} \to \mathrm{C}_{\mathrm{rd}}$ by*

$$L_1 y(t) = y^{\Delta}(t) - p(t)y(t), \quad t \in \mathbb{T}^{\kappa}.$$

Then (2.16) can be written in the form $L_1 y(t) = 0$ and (2.15) can be written in the form $L_1 y(t) = f(t)$. Since L_1 is a linear operator we say that (2.15) is a linear equation. We say y is a solution of (2.15) on $\mathbb{T}$ provided $y \in \mathrm{C}^1_{\mathrm{rd}}$ and $L_1 y(t) = f(t)$ for $t \in \mathbb{T}^{\kappa}$.

Definition 2.32. *The adjoint operator $L_1^* : \mathrm{C}^1_{\mathrm{rd}} \to \mathrm{C}_{\mathrm{rd}}$ is defined by*

$$L_1^* x(t) = x^{\Delta}(t) + p(t)x^{\sigma}(t), \quad t \in \mathbb{T}^{\kappa}.$$

Theorem 2.33. *(Lagrange Identity). If $x, y \in \mathrm{C}^1_{\mathrm{rd}}$, then*

$$x^{\sigma} L_1 y + y L_1^* x = (xy)^{\Delta} \text{ on } \mathbb{T}^{\kappa}.$$

Proof. Assume $x, y \in \mathrm{C}_{\mathrm{rd}}^1$ and consider

$$\begin{aligned}(xy)^\Delta &= x^\sigma y^\Delta + x^\Delta y \\ &= x^\sigma \left(y^\Delta - py\right) + y\left(x^\Delta + px^\sigma\right) \\ &= x^\sigma L_1 y + y L_1^* x\end{aligned}$$

on $\mathbb{T}^\kappa$. □

The next result follows immediately from the Lagrange identity.

Corollary 2.34. *If y and x are solutions of $L_1 y = 0$ and $L_1^* x = 0$, respectively, then*

$$x(t)y(t) = C \quad \text{for} \quad t \in \mathbb{T},$$

where C is a constant.

It follows from this corollary that if a nontrivial y satisfies $L_1 y = 0$, then $x := \frac{1}{y}$ satisfies the adjoint equation $L_1^* x = 0$.

Bohner and Peterson [1] proceed to give the following existence and uniqueness result for the adjoint initial value problem.

Theorem 2.35. *Suppose $p \in \mathcal{R}$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$x^\Delta = -p(t)x^\sigma, \quad x(t_0) = x_0 \tag{2.18}$$

is given by

$$x(t) = e_{\ominus p}(t, t_0)\, x_0.$$

Proof. Theorem 2.35 follows directly from Corollary 2.34. It can also be easily proved by showing directly that the function y given in Theorem 2.35 solves the initial value problem (2.18). □

Let us focus on the non-homogeneous problem below

$$x^\Delta = -p(t)x^\sigma + f(t), \quad x(t_0) = x_0. \tag{2.19}$$

Let us assume that x is a solution of (2.19). We multiply both sides of the dynamic equation in (2.19) by the so-called integrating factor $e_p(t, t_0)$ and obtain

$$\begin{aligned}\left[e_p(\cdot, t_0)\, x\right]^\Delta(t) &= e_p(t, t_0)\, x^\Delta(t) + p(t) e_p(t, t_0)\, x^\sigma(t) \\ &= e_p(t, t_0)\left[x^\Delta(t) + p(t)x^\sigma(t)\right] \\ &= e_p(t, t_0)\, f(t),\end{aligned}$$

and now we integrate both sides from t_0 to t to conclude

$$e_p(t, t_0)\, x(t) - e_p(t_0, t_0)\, x(t_0) = \int_{t_0}^t e_p(\tau, t_0)\, f(\tau)\Delta\tau$$

This integration is possible according to Theorem 1.36 provided $f \in \mathrm{C}_{\mathbf{rd}}$. Hence the following definition is useful.

Definition 2.36. *The equation (2.15) is called regressive provided (2.16) is regressive and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous.*

We are equipped to attack the non-homogenous F.O.L.D.E defined by the adjoint equation $L_1^* x = f$. We next give the variation of constants formula for the adjoint equation $L_1^* x = f$.

Theorem 2.37. *(Variation of Constants). Suppose (2.15) is regressive. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$x^{\Delta} = -p(t)x^{\sigma} + f(t), \quad x(t_0) = x_0 \tag{2.20}$$

is given by

$$x(t) = e_{\ominus p}(t, t_0)\, x_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau) f(\tau) \Delta\tau$$

Proof. (See [1], Theorem 2.74 p. 77) □

Remark 2.38. *Because of Theorem 2.20 (v), an alternative form of the solution of the initial value problem (2.20) is given by the following*

$$x(t) = e_{\ominus p}(t, t_0) \left[x_0 + \int_{t_0}^{t} e_{\ominus p}(t_0, \tau)\, f(\tau) \Delta\tau \right].$$

Finally we can give the variation of constants formula for $L_1 y = f$.

Theorem 2.39. *(Variation of Constants). Suppose (2.15) is regressive. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^{\Delta} = p(t)y + f(t), \quad y(t_0) = y_0 \tag{2.21}$$

is given by

$$y(t) = e_p(t, t_0)\, y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau)) f(\tau) \Delta\tau$$

Proof. (See [1], Theorem 2.77 p.77) □

Let us use conclude by using the theorems in this section to solve some F.O.L.D.Es

We start by the simple F.O.L.D.E

$$y^{\Delta} = 2y + 3^t, y(0) = 0 \text{ where } \mathbb{T} = \mathbb{Z} \tag{2.22}$$

Solution Note that 2.22 is a non-homogenous and of course linear dynamic equation with the function p represented by 2 and f by 3^t. We apply Theorem 2.39 which says the answer is given by

$$y(t) = e_p(t, t_0)\, y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau)) f(\tau) \Delta\tau$$

Lets break down each part and join them in the above formula for brevity sake. The first term of the right hand of the equal sign is 0 because $y_0 = 0$. $e_p(t, \sigma(\tau))$ in our case is $e_2(t, \tau+1)$ which is equal, from Table 2.2, $(1+2)^{t-(\tau+1)}$. Lets put this in the general form of the solution, we get

$$\begin{aligned} y(t) &= \int_0^t 3^{t-1-\tau} 3^{\tau} \Delta\tau \\ &= 3^{t-1} \int_0^t 3^{-\tau} 3^{\tau} \Delta\tau \\ &= 3^{t-1} \int_0^t 1 \Delta\tau \\ &= 3^{t-1} \sum_0^{t-1} 1 \text{ (by Theorem 1.40)} \\ &= t3^{t-1} \end{aligned}$$

And now let us develop and solve an example that takes the well known logistic growth differential equation in literature and writes the corresponding time scales version of it. The example is left as an exercise in the paper by Sabrina Streipert, see [10] and we solve it below.

Example 2.40. *Let us consider a time scales analogue of the popular logistic growth model* $y' = ry\left(1 - \frac{y}{K}\right)$*, namely,*

$$y^{\Delta} = ry^{\sigma}\left(1 - \frac{y}{K}\right), \quad y(t_0) = y_0 \tag{2.23}$$

with growth rate $r > 0$*, and carrying capacity* $K > 0$*, and initial population size* $y(t_0) > 0$ *at time* $t_0 \in \mathbb{T}$*.*

Even though this is an example of a nonlinear dynamic equation of first order, we can apply the substitution $z = \frac{1}{y}$ for $y \neq 0$, to obtain the linear dynamic equation

$$\begin{aligned} z^{\Delta} &= \left(\frac{1}{y}\right)^{\Delta} \\ z^{\Delta} &= \frac{-y^{\Delta}}{yy^{\sigma}} \text{ (apply Theorem 1.12 (v))} \\ &= -rz + \frac{r}{K}, \text{ and the initial condition is } \quad z(t_0) = \frac{1}{y_0} \end{aligned}$$

Again note that our p is $-r$ and our f is $\frac{r}{K}$.

For $-r \in \mathcal{R}$, the solution is then given by Theorem 2.39. Using also Theorem 2.22 and re-substituting later yields

$$\begin{aligned} z(t) &= e_{-r}(t,t_0)\, y_0 + \int_{t_0}^{t} e_{-r}(t,\sigma(\tau))\, \frac{r}{K}\Delta\tau \text{ (by Theorem 2.39)} \\ &= e_{-r}(t,t_0)\, y_0 - \frac{1}{K}\int_{t_0}^{t} -re_{-r}(t,\sigma(\tau))\,\Delta\tau \\ &= e_{-r}(t,t_0)\, y_0 - \frac{1}{K}\Big[e_{-r}(t,t_0) - e_{-r}(t,t)\Big] \text{ (by applying Theorem 2.22)} \\ &= \frac{Ke_{-r}(t,t_0) - y_0\big(e_{-r}(t,t_0)-1\big)}{y_0 K} \\ &= \frac{e_{-r}(t,t_0)\,\big[K - y_0 + y_0\big]}{y_o K} \end{aligned}$$

And then by re-substituting to get the solution in terms of y,

$$y(t) = \frac{y_0 K}{e_{-r}(t,t_0)\,(K-y_0)+y_0}. \tag{2.24}$$

It can be easily checked that $y(t_0) = y_0$ and that y solves (2.23), Note that for $\mathbb{T} = \mathbb{R}$, ((2.23)) collapses to the Verhulst model $y' = ry\left(1 - \frac{y}{K}\right)$ and the solution (2.24) reads in this case as

$$y(t) = \frac{y_0 K}{e^{-r(t-t_0)}\,(K-y_0)+y_0},$$

which coincides with the classical solution.

We can visualize the Logistic Dynamic model on several time scales as show in 2.2 where we can see that the shape of the graph is the same for different time scales. The visualisation is due to Tom Cutcha who also runs a TimeScales wiki page.

The above equation has many uses that include modelling population growth under certain constraints, in medicine to model growth of tumors and even modelling pandemics for example the COVID-19 pandemic!

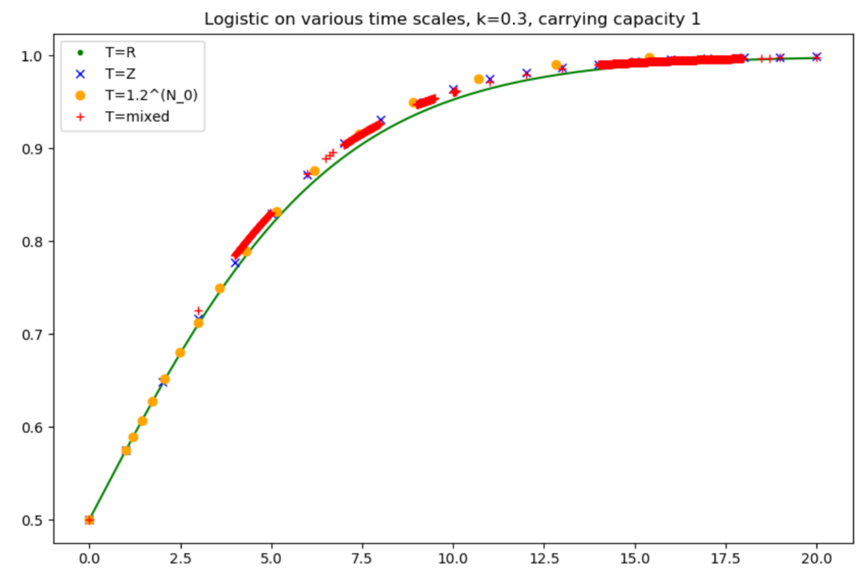

Figure 2.2. Visualization of Logistic dynamic equation on various time scales

CHAPTER 3

Conclusions

We have seen that Time Scale Calculus provides a rigorous framework that not only unifies differential and difference calculus together, but also extends them to different underlying time domains (dynamic equations). The major advantage of using time scales is to avoid proving results twice, once for continuous/ordinary calculus and once again for discrete calculus. So when one proves the result or theory on an arbitrary time scale (measure chain), they are basically done. Another advantage is that dynamic equations allows the modeling of hybrid systems that are a combination of discrete and continuous parts.

Together with my supervisor Prof. Seba we are now currently working on a research article that focuses on **oscillation criteria for solutions of dynamic equations on time scales**. The abstract of the latter has been accepted for presentation and participation at the INTERNATIONAL CONFERENCE ON NONLINEAR SCIENCE AND COMPLEXITY, July 2023 held in Turkey.

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.1 Appendix

In this thesis we have considered formulae, definitions and theorems involving the forward jump operator and the related delta (Δ) derivatives. This means we have considered time scales that contain left-scattered suprema and with interest in how functions change as time moves forward.
In this section we consider the definitions, theorems and differentiability formulae involving the backward jump operator. The former and the latter look very similar, that is why for brevity sake we proceeded using Delta derivatives. But care should be taken in that for time scales that contain right-scattered infinima, it is preferable to use a derivative that uses $\rho(t)$ in its definition rather $\sigma(t)$. In fact the whole thesis could be written using results for the so-called Nabla derivatives that use the backward jump operator ρ. The results in this section are due to Duke and Elizabeth, see [11].

Definition .1. *(Backward Graininess Function). The backwards graininess function,* $\nu : \mathbb{T} \to [0, \infty]$ *is defined by*

$$\nu(t) := t - \rho(t)$$

Definition .2. $(\mathbb{T}_k)$. *If* $\mathbb{T}$ *has a right-scattered minimum* a, *then* $\mathbb{T}_k = \mathbb{T} - \{a\}$. *Otherwise,* $\mathbb{T}_k = \mathbb{T}$

Further, $\mathbb{T}^k \cap \mathbb{T}_k$ is denoted $\mathbb{T}^k_k$. To describe a subset of an existing time scale with m left-scattered right endpoints removed and n right-scattered left endpoints removed, we use the notation $\mathbb{T}^{k^m}_{k_n}$, where $m, n = 0, 1, 2 \ldots$, e.g. $\mathbb{T}^{k^0} = \mathbb{T}, \mathbb{T}^{k^1} = \mathbb{T}^k$. Likewise, $\mathbb{T}^{k^2} = \mathbb{T}^k - \{\rho(b)\}$ if $\rho(b)$ is left-scattered and $\mathbb{T}^{k^2} = \mathbb{T}^k$ otherwise.

Definition .3. *(The Nabla Derivative). A function defined on* $\mathbb{T}$ *is nabla differentiable on* $\mathbb{T}_k$ *if for every* $\epsilon > 0$, *there exists a* $\delta > 0$ *with* $s \in (t - \delta, t + \delta) \cap \mathbb{T}$ *such that the inequality*

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| < \epsilon |\rho(t) - s|$$

holds. $f^{\nabla}(t)$ *is called the nabla derivative of* f *at* t.

And finally we give the theorem that gives differentiation formulae for nabla derivatives.

Theorem .4. *For* $f : \mathbb{T} \longrightarrow \mathbb{R}$ *and* $t \in \mathbb{T}_k$, *we have the following:*

(i) If f *is* ∇*-differentiable at* t, *then* f *is continuous at* t.

(ii) If f is d-continuous at a backward scattered point t, then it is ∇-differentiable at t and

$$f^{\nabla}(t) = \frac{f(\rho(t)) - f(t)}{\rho(t) - t}$$

(iii) If f is ∇-differentiable at a backward dense point t, then

$$f^{\nabla}(t) = \lim_{r \to t} \frac{f(t) - f(r)}{t - r}$$

(iv) If f is ∇-differentiable at t, then

$$f(\rho(t)) = f(t) - \nu(t) f^{\nabla}(t).$$

Proof. Same procedure as with forward jump operators. □

For more analogue theorems including nabla anti-derivatives, see [12].