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Model Reduction of MIMO Systems

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DEDICATIONS

To my father and mother, brothers and sisters;
To my big familly members and friends without exception;
To the girl I will never forget;
To all the teachers I had during my scolarship
especially Temam, Ait Ziane and Dr. Hariche;
To all people in the world struggling for Human Rights in their broadest sense, identity, dignity, peace, freedom, modernity and culture.

I DEDICATE MY MODEST WORK

Hakim Zabot

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Abstract

Over the past few decades, considerable improvements have been made in the approximation of high degree multivariable systems by low degree reduced models. The resulting reduced models are used to simplify the analysis and synthesis of complex systems as well as their modelling and implementation.

This problem has been the subject matter of several research studies and a variety of methods have been proposed [1,2,3,4,6,7,10,...etc]. The present thesis is an attempt to further investigate this problem using new tools.

The main contribution of this thesis may be summarized in the following:

(1) The elaboration of two model reduction procedures, the solvents based model reduction procedure and the spectral factors based model reduction procedure.

(2) Definition of a new truncation criterion, based on Hankel matrix difference, for the measure of the potential closeness of the reduced model to the original one.

(3) Application and promotion of the use of matrix polynomial theory in model reduction problems.

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Chapter 1

Introduction

The analysis and synthesis of complex physical or engineering systems always start by the building up of models which realistically describe their behaviour. The reason is that once a physical phenomenon has been adequately modelled so as to be a faithful representation of reality, all further analysis can be done on the model and experimentation on the process is no longer required. The advent of the digital computer has meant that relatively complex models can be built and manipulated. Often, this analysis, synthesis and modelling of a large scale multivariable system require the search for a reduced or approximated model to make an analog or digital simulation possible.

Model approximation and model reduction of dynamic systems refer to methods for simplification, approximation, and order reduction. Model approximation is of importance for the extraction of dominant features of a model, to reduce a high order time series model to a lower order structure motivated by physical considerations or, for evaluation of the relative importance of subsystems in some large scale systems. Model reduction and approximation arise in many important applications such as control and system engineering, electrical engineering, mechanical engineering, system identification, management and economics problems.

The term model reduction may cover several aspects such as model order reduction in a linear system, model approximation of non-linear differential equations by a linear system or, approximation of non-linear systems by ignoring higher order harmonics.

Because of the availability of the necessary mathematical tools in linear systems analysis, the first aspect (model order reduction) has received the most important consideration in control literature. It has been the focus of several research works and a variety of methods have been proposed. These methods have grown from either the state-space representation or the transfer function (frequency and time domain) representation.

A state-space model consists of a matrix triple (A,B,C) parametrizing, in discrete time case, a set of difference equations

$$\begin{cases} X(k+1) = AX(k) + BU(k) \\ Y(k) = CX(k) \end{cases} \tag{1.1}$$

or, in continuous-time case, a set of differential equations

$$\begin{cases} X(t) = AX(t) + BU(t), \\ Y(t) = CX(t) \end{cases} \tag{1.2}$$

where U,Y and X are, respectively, the input, output, and state vector of the system. The dimension of matrix A is said to be the order of the state-space model. The products CA^iB, $i = 0,1,2,\ldots$ are called the Markov parameters associated with the model (A,B,C).

The next class of methods are those developed from an input-output realisation. The input-output representation often takes the form of a transfer function (frequency domain characterisation) or an impulse response function (time-domain characterisation). For finite order systems, the transfer functions are rational: $H(s) = \frac{1}{a(s)} N(s)$ for continuous time case, or $H(z) = \frac{1}{a(z)} N(z)$ for discrete time case, where $a(.)$ and $N(.)$ are polynomials and polynomial matrices, respectively. Transfer functions are related to state-space representations by

$$H(s) = C(sI - A)^{-1}B \tag{1.3}$$

or,

$$H(z) = C(zI - A)^{-1}B \tag{1.4}$$

The impulse response of a system can be obtained by actually measuring its response to an impulse input or by estimating the correlation between the system output and a white noise input.

By denoting the impulse response of a discrete time system by $H(n)$ and that of a continuous-time system by $H(t)$, we can relate the impulse response to the state-space parameters (A,B,C) as

$$H(n) = CA^{n-1}B \tag{1.5}$$

and

$$H(t) = Ce^{At}B \tag{1.6}$$

The philosophy behind model reduction methods using the state-space representation, is to split the modes of the model into a dominant set (principal) and a dominated one (perturbational). Then by throwing away the least significant portion of the model, a reasonable lower order approximation is obtained. Among these methods, we have the balanced realisation method proposed by Moore [1], where he pointed out attention to the use of principal component analysis to model reduction problems and argued that the input-output behaviour of the model is not changed too much if the least controllable and least observable part is deleted. Latter, Pernebo and Silverman [2] further investigated this method in terms of stability, controllability and observability. Bamani [3], Rachid and Haskim [4] developed aggregation methods based on the Shur decomposition form while Kokotovic and Yackel [5] derived a singular perturbational analysis method using the state-space representation. Traditionally, the "relative dominance" is measured in terms of the eigenvalues of matrix A, or equivalently, the system modes: modes with slower decay are considered to be dominant.

However, recent progress in model reduction [64] shows that in order to get a reasonable approximation, it may not be appropriate to simply drop

the "fast" modes. Apparently the problem can be better answered if some error criterion that also takes into account the effect of matrices B and C on the system behaviour is introduced.

Clearly, the ultimate concern in model reduction is to approximate the input-output behaviour of a system. Knowledge about the internal structure (state-space model) of that system is only a convenience but not a necessity.

The next group of model reduction methods to be discussed are developed from an input-output description form. Successive techniques are derived from this representation. Among these, we mention the Padé approximation methods, the continued fraction methods and the moments-matching techniques. The moments of a system are defined to be the coefficients of the power series expansion or Taylor series expansions of the transfer function about certain points. A lower-order approximation, $\hat{H}(z)$, is constructed by matching up a desired number of its moments with those of $H(z)$.

This class of techniques developed from the input-output representation may be further divided into two major categories : the classical methods and the modified or stable methods.

The former approach is represented by the works of Chen and Shieh [6], Bosley, Kropholler and Lees [7], and Zakian [8], Qian and Meng [9]. But, they suffer from a serious drawback in that the approximations may be unstable even though the original system is. Hence, stable methods were proposed in order to remediate to this lack. The stable methods try to overcome this problem by insuring beforehand the presence of some desired modes in the reduced model. The characteristic equation is required to satisfy some criteria of stability such as the Routh stability criterion, Hurwitz polynomial, Mihailov criterion and the stability equation, while the parameters in the numerator are adjusted as in the classical reduction methods. For illustration purposes, we mention the method proposed by Wan [10] which uses the property of the Mihailov criterion to improve the Padé approximation method. The method developed by Hwang, Gou and Shieh

[11] uses the Routh approximation method to form the denominator and the minimisation of the integral squared error criterion for the derivation of the numerator. On the other hand, Xiheng [12] contributed to the performance of the FF-Padé method through some new concepts, while in [13] Shieh and Wei developed an elegant mixed model reduction method that combines dominant eigenvalue concept and the matrix continued fraction approach. However, the absolute stability of the stable methods is achieved only at the cost of a serious loss of accuracy, and compared to the classical approaches, fewer moments can be matched with approximates of the same order. Furthermore, they exhibit bad performance when approximating certain known classes of transfer functions [14].

Transfer functions based methods have intuitive and appealing frequency domain interpretations [15, 16]. However, the drawback with the majority of these techniques is the difficulty in extending them to multivariable cases [17].

Beside state-space models and transfer functions, impulse responses can also be a starting point for model reduction. The third group of model reduction methods that may be seen as an extension of the two previous groups, are the more recently developed Hankel approximation methods. The Hankel matrix associated with a linear system described by a transfer function $H(z)$ having a proper series expansion (in z^{-1}) $H(z) = \sum_{i=1}^{\infty} H_i z^{-i}$ is defined as

$$\Gamma\{H(z)\} = \begin{bmatrix} H_1 & H_2 & H_3 & \cdots \\ H_2 & H_3 & H_4 & \cdots \\ H_3 & H_4 & H_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad (1.7)$$

It has been a popular tool for system identification. The reason for its popularity is the well known Kronecker's theorem [18, 19, 20], a result that connect the degree (order) of a system with the rank of the corresponding block Hankel matrix.

Given a system transfer function $H(z)$, the rank of $\Gamma\{H(z)\}$ is equal to the degree (order) of the system $H(z)$; i.e the dimension of state vectors of minimal state space realisations.

Consider a state-space triple (A,B,C). It is noted that $H_i = CA^{i-1}B$. Hence

$$\Gamma\{H(z)\} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = O_b C_o \qquad (1.8)$$

where O_b and C_o, are respectively called the (extended) observability and Controllability matrices. Therefore, the rank of $\Gamma\{H(z)\}$is equal to that of O_b and C_o when O_b andC_o are of full rank (when the state-space realisation is observable and controllable, hence minimal).

Let's denote by "$\uparrow$" and "$\leftarrow$", respectively, the shift-up and the shift-left operators. Then it is seen that

$$O_b A = \begin{bmatrix} CA \\ CA^2 \\ \vdots \end{bmatrix} = O_b^{\uparrow} \qquad (1.9a)$$

$$AC_o = \begin{bmatrix} AB & A^2B & \cdots \end{bmatrix} = C_o^{\leftarrow} \qquad (1.9b)$$

and

$$O_b A C_c = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} A \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = \Gamma^{\uparrow} = \Gamma^{\leftarrow} \qquad (1.9c)$$

Hence, given a Hankel matrix Γ and a factorisation $O_b C_o$, a state feedback matrix A can be computed by solving any of the following three equations:

$$O_b A = O_b^{\uparrow} \qquad (1.10a)$$
$$AC_o = C_o^{\leftarrow} \qquad (1.10b)$$
$$O_b A C_o = \Gamma^{\uparrow} \qquad (1.10c)$$

where the matrices B and C are easily obtained from C_o and O_b.

If only noisy measurements of H_i are available, then the Hankel matrix formed from H_i may have unreasonably high rank which does not represent a

true system order. One should not insist on obtaining a system realisation which reproduces the noisy sequence of H_i; Rather, a realisation which produces some smoothed version of this sequence appears to be more desirable. Zeiger, Mc Ewen, Jaaksoo and Nurges [21, 22] developed a method in this order. Their method first applies singular value decomposition to the Hankel matrix, which decomposes the matrix into orthogonal components ordered according to the magnitude of the singular values. The components associated with small singular values are then treated as "perturbational" and removed by setting the corresponding singular values to zero. A least-square approximate solution for the state-feedback matrix A is then obtained from an equation similar to (1.10c). In the case where no "perturbation" is assumed, the procedure reduces to the Ho and Kalman's algorithm [19] for exact realisation.

Later and independently, Kung [23] arrived at a similar scheme for approximate realisation in which the state-feedback matrix is solved from an equation similar to (1.10a) or (1.10b). He also studied the stability of the realisation and reported some error analysis results. Connection between this method and the Moore's method of model reduction is mentioned in [23]. Further such connection is observed by Silverman [24].

The three closely related methods developed by Moore, Zeiger and Mc Ewen, and Kung are supported by good approximation results. Nevertheless they remain ad-hoc and heuristic approaches. Hence it is natural to question whether there is an optimal approximation in any sense, within this Hankel approximation framework. The answer is affirmative.

For the single input-output systems case, Adamjan, Arov and Krein [25], following some work of Nehari [26], developed a closed form optimal solution for model reduction problems with a Hankel norm error criterion, where the Hankel norm error is defined to be the spectral norm of the difference between the Hankel matrix associated with the original system and that associated with the approximating system. It has been shown [27] that the spectral norm of the Hankel matrix associated with a stable single

input-output system lies between the more conventional L_2 and L_∞ norms. Hence, the Hankel norm criterion can be viewed as a compromise between these two more popular error measures. Adamjan, Arov and Krein's work is restricted to scalar systems (single-input-output systems) except for a special multivariable extension problem [28] which in the context of linear systems approximation, may be called "zeroth order approximation problem". Their work is very mathematical and many investigators have worked on exploiting its engineering implications.

A general theory and algorithm for optimal Hankel norm approximation of multivariable systems was first proposed and developed by Kung and Lin [29, 30]. In [29, 30] an "Extension theorem" is derived which assures that with one block extension, any Hankel matrix can have certain desirable singular value/vector structure. Then a solution for optimal Hankel norm multivariable systems approximation problem is constructed.

In the present work two model reduction procedures are proposed. The first method belongs to the first group and it is developed from a block state-space description, while the second method is developed from a matrix transfer function and therefore may be categorised as belonging to the second group of methods. The truncation criterion is derived from the third class of methods through the use of the relative error between the singular values of the block-Hankel matrix of the original model and the singular values of the block-Hankel matrix of the neglected part of the model

The first procedure is based on the concept of dominance, between a complete set of solvents of the characteristic matrix polynomial of a multivariable system given in the form of a matrix fraction description. A block controller canonical form is formed from the coefficients of the matrix fraction description and then block diagonalised via the Vandermande similarity transformation. The obtained reduced model is further tuned at the eigenvalue level through the diagonalisation of the last block and the extraction of its dominant part.

Chapter 2

Review on Matrix Polynomials

A linear time invariant single-input single-output system is often described by a differential equation of the form

$$d(p)y(t) = n(p)u(t) \tag{2.1}$$

where $d(p)$ and $n(p)$ are scalar polynomials.

The Laplace transformation description of such system results in a transfer function with scalar polynomials in the numerator and denominator, written as

$$h(\lambda) = \frac{n(\lambda)}{d(\lambda)} \tag{2.2}$$

where λ is a complex variable. The multivariable analog of the scalar case is a transfer function in the form of a matrix fraction description (MFD)

$$H(\lambda) = N_R(\lambda)D_R^{-1}(\lambda) \tag{2.3}$$

where $N_R(\lambda)$ and $D_R(\lambda)$ are respectively $n \times m$ and $m \times m$ complex matrices known as matrix polynomials or lambda matrices ($\lambda - matrices$), λ is a complex variable and

$$H(\lambda) = L[H(t)] \tag{2.4}$$

$H(\lambda)$ is defined to be a RMFD (right matrix fraction description), the matrix $D_R(\lambda)$ in (2.3) is a right denominator λ-matrix [20].

An alternative factorisation of $H(\lambda)$ is the LMFD (left matrix fraction description) defined as:

$$H(\lambda) = D_L^{-1}(\lambda)N_L(\lambda) \tag{2.5}$$

where $D_L(\lambda)$ is a left denominator λ-matrix.

The stability and dynamic behaviour of multivariable systems formulated in (2.3) and (2.5) and related properties of λ-matrices have been investigated by Lancaster [31], Jury [32], Anderson and Bitmead [33], Papaconstantinou [34], Miller and Michel [35], Shieh and Sacheti [36], Shieh et al. [38], and Shieh and Tajvari [37].

The denominator matrix polynomial or $\lambda - matrix$ called also the characteristic matrix polynomial, characterises the properties of a system in terms of the spectrums of the different spectral factors.

By a n^{th} degree m^{th} order λ-matrix, we understand a matrix valued function of a complex variable of the form

$$A(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_{n-1}\lambda + A_n \tag{2.6}$$

where $A_i, i = 0,1,2,\ldots,n$, are $m \times m$ constant matrices and λ is a complex variable. In addition:

Definition 2.1: *Given $A(\lambda)$ as defined in (2.6) then:*

(i) $A(\lambda)$ *is monic if* $A_0 = I$.

(ii) $A(\lambda)$ *is comonic if* $A_n = I$.

(iii) $A(\lambda)$ *is regular if* $Det(A(\lambda)) \neq 0$.

(iv) $A(\lambda)$ *is nonsingular if* $Det(A(\lambda))$ *is not identically zero.*

(v) $A(\lambda)$ *is unimodular if* $Det(A(\lambda))$ *is nonzero constant.*

Other definitions for regularity and nonsingularity may be encountered in matrix polynomials literature. For example [39] defines a regular λ-matrix as one whose determinant is not identically zero and nonsingular λ-matrix as one whose determinant is a nonzero constant, thus making statements (iv) and (v) of definition 2.1 equivalent. Note that, if A_0 is nonsingular, one can always multiply by A_0^{-1} to get a monic matrix polynomial.

Another alternative formulation of the λ-matrix is:

$$A(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) & \cdots & a_{1n}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & \cdots & a_{2n}(\lambda) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(\lambda) & a_{n2}(\lambda) & \cdots & a_{nn}(\lambda) \end{pmatrix} \qquad (2.7)$$

where $a_{ij}(\lambda) = a_{ij0}\lambda^n + a_{ij1}\lambda^{n-1} + \cdots + a_{ijn}$,and $a_{ijk} = A_k(i,j)$ with a_{ijk} and λ belonging to a field $F(\lambda)$.

Clearly, the two forms (2.6) and (2.7) are equivalent. However, form (2.6) outlines the polynomial character of the matrix polynomial while form (2.7) stresses on the matrix one. In this thesis, we will be dealing only with monic matrix polynomials of the form (2.6) because it is the one that suits the present work.

There are also other forms such as the Smith normal form which is very useful in the study of λ-matrices. But, before introducing it, we have to define what is an elementary transformation on a $\lambda - matrix$.

2.1 Elementary Transformations on a λ-matrix

The reduction of a matrix of general form to a canonical form, will usually be achieved progressively by performing a sequence of simple transformations known as elementary transformations. We call an elementary transformation on a $\lambda - matrix$ $A(\lambda)$ over a field $F(\lambda)$, one of the following operations:

(1) Interchange of two rows.

(2) The multiplication of a row by a nonzero constant.

(3) The addition to one row another row multiplied by a polynomial $p(\lambda)$ of $F(\lambda)$.

The elementary column operations are defined in an entirely analogous fashion. A row transformation on $A(\lambda)$ is obtained by multiplying $A(\lambda)$ on the left by a convenient matrix H and a column transformation is obtained by multiplying $A(\lambda)$ on the right by a convenient matrix K. These

transformations can be seen as premultiplication and postmultiplication by elementary matrices of the following forms:

An interchange of rows (columns) i and j in $A(\lambda)$ is equivalent to multiplication on the left (right) by

$$\begin{array}{c} \\ \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & i & & j & & \end{array} \\ \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \end{array} \tag{2.8}$$

Adding to the i^{th} row of $A(\lambda)$ the j^{th} row multiplied by the polynomial $p(\lambda)$ is equivalent to multiplication on the left by

$$\begin{array}{c} \\ \\ \\ \\ i \rightarrow \\ \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccccc} & & & & & & j & & \end{array} \\ \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & p(\lambda) & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \end{array} \tag{2.9}$$

The same operation for columns is equivalent to multiplication on the right by the matrix

$$\begin{array}{c} \\ \\ \\ \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & p(\lambda) & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \tag{2.10}$$

Finally multiplication of the i^{th} row (column) in $A(\lambda)$ by a nonzero constant c is equivalent to the multiplication on the left (right) by

$$\begin{array}{c} \\ \\ \\ \\ i \rightarrow \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & c & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \qquad (2.11)$$

It is evident that the determinant of a matrix of type (2.8) is -1 and of type (2.9) and (2.10) is $+1$ while that of matrix of type (2.11) is the constant c. In addition, we state without proofs the following theorems [40].

Theorem 2.1: *Every elementary matrix in $F(\lambda)$ has an inverse, which is itself an elementary matrix in $F(\lambda)$.*

Theorem 2.2: *If $|A(\lambda)| = k \neq 0$ where $k \in F$, then $A(\lambda)$ is a product of elementary matrices.*

Theorem 2.3: *The rank of a λ-matrix is not changed by elementary transformations.*

2.2 Equivalent Transformations

Theorem 2.4: *Two λ-matrices $A(\lambda)$ and $D(\lambda)$ are equivalent if and only if there are unimodular matrices $P(\lambda) = H_s \cdots H_2 H_1$ and $Q(\lambda) = K_1 K_2 \cdots K_t$ such that*

$$D(\lambda) = P(\lambda) A(\lambda) Q(\lambda) \qquad (2.12)$$

Proof: It is evident from what precedes.

Theorem 2.5: *Every λ-matrix of rank r may be reduced by elementary transformations to the "Smith normal form" given by:*

$$D(\lambda) = \begin{bmatrix} d_1(\lambda) & & & & 0 \\ & \ddots & & & \\ & & d_r(\lambda) & & \\ & & & 0 & \\ & & & & \ddots \\ 0 & & & & 0 \end{bmatrix} \qquad (2.13)$$

Proof: See Gohberg et al. ref. [40].

Note that the Smith form $D(\lambda)$ is a diagonal matrix with monic scalar polynomials $d_i(\lambda)$ such that $d_i(\lambda)$ is divisible by $d_{i-1}(\lambda)$; $P(\lambda)$ and $Q(\lambda)$ are respectively matrix polynomials of sizes $m \times m$ and $n \times n$ with constant nonzero determinants.

Theorem 2.6: *Equivalent $m \times n$ $\lambda-$matrices are of the same rank.*

Proof: See Theorem 2.3.

2.3 Common Divisors

Theorem 2.7: *Let $A(\lambda)$ and $D(\lambda)$ be two equivalent matrices of rank r, then the greatest common divisor of all the minors of order s of $A(\lambda)$, $s \le r$, is also the greatest common divisor of all the minors of order s of $D(\lambda)$.*

Proof: See ref. [40].

When a $\lambda-$*matrix* of rank r is reduced to the form (2.13), the greatest common divisor of all the minors of order s of $A(\lambda)$, $s \le r$, is the greatest common divisor of all the minors of order s of $D(\lambda)$. From the above theorem 2.7, since in $D(\lambda)$ every $d_i(\lambda)$ divide $d_{i-1}(\lambda)$, the greatest common divisor of all the minors of order s of $D(\lambda)$ and also of $A(\lambda)$ is:

$$g_s(\lambda) = d_1(\lambda)d_2(\lambda)\cdots d_s(\lambda) \qquad (s = 1,2,\ldots,r)$$

To prove the uniqueness of the transformation, suppose that $A(\lambda)$ is reduced to

$$D_1(\lambda) = diag(d_1(\lambda), d_2(\lambda), \ldots, d_r(\lambda), 0, \ldots, 0)$$

and to

$$D_2(\lambda) = diag(h_1(\lambda), h_2(\lambda), \ldots, h_r(\lambda), 0, \ldots, 0)$$

from (2.13), we have

$$g_s(\lambda) = d_1(\lambda)d_2(\lambda)\cdots d_s(\lambda) = h_1(\lambda)h_2(\lambda)\cdots h_s(\lambda)$$

or $g_1(\lambda) = d_1(\lambda) = h_1(\lambda), \qquad g_2(\lambda) = d_1(\lambda)d_2(\lambda) = h_1(\lambda)h_2(\lambda)$

in such away that

$$d_2(\lambda) = h_2(\lambda), \ldots$$

In general, if we set $g_0(\lambda) = 1$, then

$$g_s(\lambda) \Big/ g_{s-1}(\lambda) = d_s(\lambda) = h_s(\lambda) \qquad s = (1,2,\ldots,r)$$

and we have:

Theorem 2.8: *The matrix $D(\lambda)$ of (2.13) obtained from $A(\lambda)$ through elementary transformations is unique.*

Hence, the normal smith matrices form a canonical set for the relation of equivalence on $F(\lambda)$.

2.4 Invariant Polynomials

The polynomials $d_1(\lambda), d_2(\lambda), \ldots, d_r(\lambda)$ in the diagonal of the smith form of $A(\lambda)$ are called invariant polynomials of $A(\lambda)$. The number r of invariant polynomials is defined as

$$r = \max_{\lambda \in C} \{rank(L(\lambda))\} \qquad (2.14)$$

This is evident from the fact that, $P(\lambda)$ and $Q(\lambda)$ are invertible matrices for every λ, we have rank ($A(\lambda)$) = rank ($D(\lambda)$) for every $\lambda \in F(\lambda)$. On the other hand, it is clear that rank $(D(\lambda)) = r$ if λ is not a zero of one of the invariant polynomials, otherwise rank $D(\lambda) < r$.

if $d_k(\lambda) = 1, \qquad k \le r$, then $d_1(\lambda) = d_2(\lambda) = \cdots = d_k(\lambda) = 1$.

As a consequence of theorem 2.8, we have:

Theorem 2.9: *Two square λ-matrices of order m on $F(\lambda)$ are equivalent if and only if they have the same invariant polynomials.*

2.5 Elementary Divisors

Representing each invariant polynomial as a product of linear factors

$$d_i(\lambda) = (\lambda - \lambda_{i1})^{\alpha_{i1}} \ldots (\lambda - \lambda_{ik_i})^{\alpha_{ik_i}}, \qquad i = 1,\ldots,r, \tag{2.15}$$

where $\lambda_{i1},\ldots,\lambda_{ik_i}$ are different complex numbers and $\alpha_{i1},\ldots,\alpha_{ik_i}$ are positive integers. The factors $(\lambda - \lambda_{ij})^{\alpha_{ij}}$, $j = 1,2,\ldots,k_i$, $i = 1,2,\ldots,r$, are called the elementary divisors of $A(\lambda)$. An elementary divisor is said to be linear or nonlinear depending on wether $\alpha_{ij} = 1$ or $\alpha_{ij} > 1$.

Theorem 2.10: *If $A(\lambda)$, $B(\lambda)$ are matrix polynomials over $F(\lambda)$, then the set of elementary divisors of the block diagonal matrix*

$$C(\lambda) = \begin{bmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{bmatrix} \tag{2.16}$$

is the union of the sets of elementary divisors of $A(\lambda)$ and $B(\lambda)$.

Proof: See Gohberg et al. ref. [40].

2.6 Latent Structure and Existence of Solvents of Matrix Polynomials

Consider an n^{th} degree m^{th} order λ-matrix described by

$$A(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_{n-1}\lambda + A_n \tag{2.17}$$

where $A_i \in C^{m\times m}$ and λ is a complex variable. The above equality remains unchanged if we replace λ by another scalar from $F(\lambda)$. For example, let $\lambda = k$, we obtain

$$A(k) = A_0k^n + A_1k^{n-1} + \cdots + A_{n-1}k + A_n \tag{2.18}$$

While, if we replace λ by a square matrix S, we may obtain different results since in general two square matrices do not commute. Then we define two matrix polynomials

$$A_R(S) = A_0S^n + A_1S^{n-1} + \cdots + A_{n-1}S + A_n \tag{2.19}$$

where S is an $m\times m$ matrix, as the right matrix polynomial associated with the λ-matrix $A(\lambda)$ And

$$A_L(S) = S^nA_0 + S^{n-1}A_1 + \cdots + SA_{n-1} + A_n \tag{2.20}$$

as the left matrix polynomial associated with the $\lambda - matrix$ $A(\lambda)$. A right solvent of $A(\lambda)$ is defined as a matrix R satisfying

$$A_R(R) = A_0R^n + A_1R^{n-1} + \cdots + A_{n-1}R + A_n = 0_m \tag{2.21}$$

and a left solvent as a matrix L satisfying

$$A_L(L) = L^nA_0 + L^{n-1}A_1 + \cdots + LA_{n-1} + A_n = 0_m \tag{2.22}$$

where 0_m is the $m \times m$ null matrix.

Let λ_j be a complex number such that

$$\det(A(\lambda_j)) = 0 \tag{2.23}$$

then, λ_j is a latent root of $A(\lambda)$. Also, let p_j be an $m \times 1$ vector such that

$$A(\lambda_j)p_j = 0_{m\times 1} \tag{2.24}$$

then, p_j is a right latent vector of $A(\lambda)$ associated with λ_j. Similarly, q_j is a left latent vector if

$$q_j^T A(\lambda_j) = 0_{1\times m} \tag{2.25}$$

where T in (2.25) designates transpose.

The relationship between latent roots, latent vectors, and the solvents can be stated as follows [41].

Theorem 2.11: *If $A(\lambda)$ has n linearly independent right latent vectors $p_1, p_2, \ldots, p_n$ (left latent vectors $q_1, q_2, \ldots, q_n$) corresponding to latent roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, then $P\Lambda P^{-1}$ $(Q^{-1}\Lambda Q)$ is a right (left) solvent, where $P = (p_1, p_2, \ldots, p_n)$ $[Q = (q_1, q_2, \ldots, q_n)^T]$ and $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$.*

From the above theorem, we can determine the relationship between a right solvent and the corresponding left solvent.

Theorem 2.12: *If $A(\lambda)$ has n latent roots $\lambda_1, \ldots, \lambda_n$, and the corresponding right latent vectors $p_1, \ldots, p_n$ as well as the left latent vectors $q_1, \ldots, q_n$ are both linearly independent, then the associated right solvent R and left solvent L are related by*

$$R = WLW^{-1} \tag{2.26}$$

where $W = PQ$, $P = (p_1,\ldots,p_n)$ *and* $Q = (q_1,\ldots,q_n)^T$.

Proof: the proof follows from theorem 2.11.

2.7 Jordan Structure

It is well known that a constant matrix with multiple eigenvalues is not necessarily diagonalisable. If it is not, the most compact form into which a matrix A may be reduced by means of a similarity transformation is the Jordan form. In fact, let $p_1, p_2, \ldots, p_k$ be a chain of eigenvectors of A, and if

$$Ap_1 = \lambda_0 p_1, \qquad Ap_2 = \lambda_0 p_2 + p_1, \qquad \ldots, \qquad Ap_k = \lambda_0 p_k + p_{k-1} \tag{2.27}$$

then $p_1, p_2, \ldots, p_k$ is said to be a Jordan chain of the matrix polynomial $(\lambda I - A)$. In the following, we wish to extend the notion of Jordan chain to matrix polynomials of higher orders.

For so doing, let p_i and q_i be respectively the right and left latent vectors of a $\lambda-$*matrix* $A(\lambda)$, corresponding to latent root λ_i of algebraic multiplicity m_i, satisfying:

$$\begin{aligned} A(\lambda_i)p_i^{(j)} &= 0 \qquad j = 1,2,\ldots,l_i \\ A^T(\lambda_i)q_i^{(j)} &= 0 \qquad j = 1,2,\ldots,l_i \end{aligned} \tag{2.28}$$

where l_i represents the geometric multiplicity of λ_i. The set of vectors defined above will be referred to as primary right [left] latent vectors. In case $l_i < m_i$, the set is to be completed by $m_i - l_i$ generalised latent vectors constructed from the so-called Jordan chain of $A(\lambda)$ associated with λ_i [42, 43]. Each of the Jordan chains has, as a leading vector, a primary latent vector while the remaining vectors of the chain are obtained according to the following definition:

Definition 2.2: *A set of vectors* $p_i^{(1)}, p_i^{(2)}, \ldots, p_i^{(k)}$ *is said to be a right Jordan chain of length k for* $A(\lambda)$, *corresponding to* λ_i, *if the vectors* $p_i^{(j)}$, $j = 1,2,\ldots,k$ *satisfy the following set:*

$$\begin{aligned} A(\lambda_i)p_i^{(1)} &= 0 \\ A(\lambda_i)p_i^{(2)} + \frac{1}{1!}\frac{dA(\lambda_i)}{d\lambda}p_i^{(1)} &= 0 \\ &\vdots \\ A(\lambda_i)p_i^{k} - \frac{1}{1!}\frac{dA(\lambda_i)}{d\lambda}p_i^{(k-1)} + \cdots + \frac{1}{(k-1)!}\frac{d^{k-1}A(\lambda_i)}{d\lambda^{k-1}}p_i^{(1)} &= 0 \end{aligned} \tag{2.29}$$

Putting this in matrix form we have:

$$\begin{pmatrix} A(\lambda_i) & 0 & \cdots & 0 \\ \frac{1}{1!}\frac{dA(\lambda_i)}{d\lambda} & A(\lambda_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(k-1)!}\frac{d^{k-1}A(\lambda_i)}{d\lambda^{k-1}} & \frac{1}{(k-2)!}\frac{d^{k-2}A(\lambda_i)}{d\lambda^{k-2}} & \cdots & A(\lambda_i) \end{pmatrix} \begin{pmatrix} p_i^{(1)} \\ p_i^{(2)} \\ \vdots \\ p_i^{(k)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.30}$$

For every primary latent vector $p_i^{(j)}$, $j = 1,2,\ldots,l_i$ will correspond a Jordan chain of length k_j where k_j is the size of the corresponding Jordan cell as shown by the following structure of a Jordan block:

$$J_i = Diag(J_i^{(k_1)} J_i^{(k_2)} \cdots J_i^{(k_{l_i})}) \tag{2.31}$$

where $J_i^{(k_j)}$ is the $k_j \times k_j$ matrix

$$J_i^{(k_j)} = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix} \tag{2.32}$$

In an analogous manner, a left Jordan chain of length k for $A(\lambda)$ is a set of k vectors $q_i^{(1)}, q_i^{(2)}, \ldots, q_i^{(k)}$ where the leading vector is a primary vector. The $q_i^{(j)}$ $j = 1,2,\ldots,k$ satisfies the following set of equations:

$$\begin{aligned} A^T(\lambda_i)q_i^{(1)} &= 0 \\ A^T(\lambda_i)q_i^{(2)} + \frac{1}{1!}\frac{dA^T(\lambda_i)}{d\lambda}q_i^{(1)} &= 0 \\ &\vdots \\ A^T(\lambda_i)q_i^{k} + \frac{1}{1!}\frac{dA^T(\lambda_i)}{d\lambda}q_i^{(k-1)} + \cdots + \frac{1}{(k-1)!}\frac{d^{k-1}A^T(\lambda_i)}{d\lambda^{k-1}}q_i^{(1)} &= 0 \end{aligned} \tag{2.33}$$

which can be also written in matrix form as:

$$\begin{pmatrix} A^T(\lambda_i) & 0 & \cdots & 0 \\ \frac{1}{1!}\frac{dA^T(\lambda_i)}{d\lambda} & A^T(\lambda_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(k-1)!}\frac{d^{k-1}A^T(\lambda_i)}{d\lambda^{k-1}} & \frac{1}{(k-2)!}\frac{d^{k-2}A^T(\lambda_i)}{d\lambda^{k-2}} & \cdots & A^T(\lambda_i) \end{pmatrix} \begin{pmatrix} q_i^{(1)} \\ q_i^{(2)} \\ \vdots \\ q_i^{(k)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (2.34)$$

Note that the vectors of Jordan chain of a matrix polynomial of degree greater than one need not be linearly independent, in opposition to the linear matrix polynomials of the type $(\lambda I - A)$.

2.8 Kronecker Product

In order to complete the derivations given in the rest of the thesis, we review here the Kronecker product of two matrices. The notion of Kronecker product arises naturally in matrix equations, and plays a prominent role in their resolution.

Definition 2.3: *Given two matrices $A \in F^{m\cdot m}$ and $B \in F^{n\cdot n}$, the Kronecker product of A and B, is defined to be a matrix $W = A \otimes B$ such that*

$$W = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix} \qquad (2.35)$$

A vector-valued function closely related to the Kronecker product and denoted by *Vec* is defined in the following as:

Definition 2.4: *Given a matrix $A \in F^{m\cdot n}$ whose columns are denoted by $\{A_1, A_2, \ldots, A_n\}$, the vector-valued function VecA is defined by*

$$VecA = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} \qquad (2.36)$$

Observe that the Vec-function is linear:

$$Vec(\alpha A - \beta B) = \alpha VecA - \beta VecB,$$

for any $A, B \in F^{m\cdot n}$ and $\alpha, \beta \in F$.

The following proposition [18] shows the close relationship between the Vec-function and the Kroneker product.

Proposition 2.1: *If* $A \in F^{m\times m}, B \in F^{n\times n}$, *and* $X \in F^{m\times n}$, *then*

$$Vec(AXB) = (B^T \otimes A)VecX$$

Proof: See Gantmacher ref. [18].

16.

Relying on the above proposition, we state in the following a theorem without [39] proof that will prove useful in our work.

Theorem 2.13: *Consider the general matrix equation*

$$A_1XB_1 + A_2XB_2 + \cdots + A_pXB_p = W \tag{2.37}$$

where $A_i \in C^{m\times m}$, $B \in C^{n\times n}$ *and* $X \in C^{m\times m}$, *then this equation has a solution* X *if and only if* X *satisfies*

$$GVecX = VecW \tag{2.38}$$

where $$G = \sum_{i=1}^{p} B_i^T \otimes A_i$$

The main importance of this theorem lies in the fact that the matrix equation $\sum_{i=1}^{p} A_iXB_i = W$ is transformed to the well-known matrix-vector equation of the form $Ax = b$.

We shall now determine criteria for the existence and uniqueness of solvents in light of the above theorem [43].

Let $$A(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_n.$$

By definition, if R is a right solvent of $A(\lambda)$, then

$$A_R(R) = A_0R^n + A_1R^{n-1} + \cdots + A_n = 0 \tag{2.39}$$

Let J be the Jordan form matrix of R and M its eigenvector matrix, i.e. the matrix whose columns are the eigenvectors and generalised eigenvectors of R.

Hence matrix R can be written as:

$$R = MJM^{-1} \tag{2.40}$$

using (2.38) we have

$$A_0 MJ^n M^{-1} + A_1 MJ^{n-1} M^{-1} + \cdots + A_{n-1} MJM^{-1} + A_n = 0 \tag{2.41}$$

multiplying equation (2.41) from the right by M, we get

$$A_0 MJ^n + A_1 MJ^{n-1} + \cdots + A_{n-1} MJ + A_n M = 0 \tag{2.42}$$

setting the mn vector $VecM$ as:

$$VecM = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_n \end{pmatrix} \tag{2.43}$$

where M_i is the i^{th} column eigenvector of M. And the matrix G as:

$$G = (J^T)^n \otimes A_0 + (J^T)^{n-1} \otimes A_1 + \cdots + J^T \otimes A_{n-1} + I \otimes A_n \tag{2.44}$$

then from theorem 2.13 it is clear that R is a right solvent of $A(\lambda)$ if an only if

$$GVecM = 0 \tag{2.45}$$

By denoting by $\aleph$, the null space of G, the former result is summarised in the following corollary.

Corollary 2.1: *Under the above notation, R is a right solvent of $A(\lambda)$ if and only if G is rank deficient or, equivalently, $\aleph$ is nonempty.*

Proof: The proof follows directly from equation $GVecM = 0$.

In a similar fashion a similar result [43] can be obtained for left solvents if we consider

$$L^n A_0 + L^{n-1} A_1 + \cdots + A_n = 0 \tag{2.46}$$

where L is a left solvent of $A(\lambda)$. Denoting by P the eigenvector matrix of L and by J its associated Jordan form, we can write $L = PJP^{-1}$. Hence,

$$PJ^n P^{-1} A_0 + PJ^{n-1} P^{-1} A_1 + \cdots + PJP^{-1} A_{n-1} + A_n = 0 \tag{2.47}$$

and after multiplication on the left by P^{-1}

$$J^n P^{-1} A_0 + J^{n-1} P^{-1} A_1 + \cdots + JP^{-1} A_{n-1} + P^{-1} A_n = 0 \tag{2.48}$$

Using theorem (2.13), and after denoting P^{-1} by Q, we get

$$\hat{A}VecQ = 0 \tag{2.49}$$

where $\hat{A} = \sum_{i=0}^{n} A_i^T \otimes J^{n-i}$

Corollary 2.2: *Under the above notation, L is a left solvent of $A(\lambda)$ if and only if $\hat{A}$ is rank deficient.*

It is important to mention that, by definition, M (resp. Q) must be nonsingular. Hence, only solutions $VecM$ (resp. $VecQ$) leading to nonsingular M (resp. Q) lead to solvents R (resp. L).

2.9 Complete Set of Solvents and Complete Factorisation

Solvents play an important role in the analysis of a λ-matrix [40]. A special set of solvents of a matrix polynomial is the complete set of solvents. The complete set of regular solvents $R_i, i = 1,2,\ldots,n$, is characterised by the following properties

$$(\rho(R_i)) \cap (\rho(R_j)) = \varnothing; \quad i \neq j \quad i,j = 1,2,\ldots,n \tag{2.50}$$

$$\bigcup_{i=1}^{m} (\rho(R_i)) = \rho(A(\lambda)) \tag{2.51}$$

where $\rho(R_i)$ is the spectrum of R_i and $\rho(A(\lambda))$ is the spectrum of $A(\lambda)$. The conditions for the existence and uniqueness of a complete set of solvents have been investigated by Lancaster [31], Dennis et al [42], Gohberg et al [44], and Markus and Mereuca [45].

The more general condition can be stated as follows [41].

Theorem 2.14: *If the elementary divisors of $A(\lambda)$ are linear, then $A(\lambda)$ has a complete set of right and left solvents.*

Theorem 2.15: *If the elementary divisors of $A(\lambda)$ are linear, then $A(\lambda)$ can be factored into the product of n-linear monic λ-matrices called a complete set of linear spectral factors.*

$$A(\lambda) = (\lambda I_m - Q_n)(\lambda I_m - Q_{n-1})\cdots(\lambda I_m - Q_1) \tag{2.52}$$

where $(\lambda I - Q_i), i = 1,2,\ldots,n$, are referred to as a complete set of linear spectral factors. The $m \times m$ complex matrices $Q_i, i = 1,2,\ldots,n$, are called the spectral factors of the $\lambda - matrix\ A(\lambda)$.

Note that, in order to apply theorems 2.14 and 2.15, the elementary divisors of $A(\lambda)$ should be linear, which can be tested by transforming $A(\lambda)$ into its Smith canonical form [40].

The right most spectral factor Q_1 is a right solvent of $A(\lambda)$ and the left most spectral factor Q_n is a left solvent of $A(\lambda)$, whereas the other spectral factors may or may not be solvents of $A(\lambda)$. The relationships between solvents and spectral factors are explored by Shieh and Tsay in [41], and various transformations have been developed.

2.10 Transformations of Solvents and Spectral Factors

Since the diagonal forms of a complete set of solvents and those of a complete set of spectral factors are identical, then they are related by similarity transformations.

2.10.1 Transformation of Right (Left) Solvents to Spectral Factors

Consider a complete set of right solvents $R_1, R_2, \ldots, R_n$ of a monic $\lambda - matrix\ A(\lambda)$, then $A(\lambda)$ can be factored as

$$A(\lambda) = N_n(\lambda) = (\lambda I_m - Q_n)(\lambda I_m - Q_{n-1})\cdots(\lambda I_m - Q_1) \tag{2.53}$$

by using the following recursive scheme

$$Q_k = N_{(k-1)R}(R_k) R_k N_{(k-1)R}(R_k)^{-1}, \qquad k = 1,2,\ldots,n \tag{2.54}$$

where

$$\begin{aligned} N_k(\lambda) &= (\lambda I_m - Q_k) N_{k-1}(\lambda) \\ &= N_{k-1}(\lambda)\lambda - Q_k N_{k-1}(\lambda), \quad k = 1,2,\ldots,n \end{aligned} \tag{2.55}$$

$$N_{kR}(R_j) = N_{(k-1)R}(R_j) R_j - Q_k N_{(k-1)R}(R_j), \qquad k = 1,2,\ldots,n \text{ and any } j$$

with $N_0(\lambda) = I_m \quad \text{rank}[N_{(k-1)R}(R_k)] = m, \quad k = 1,2,\ldots,n$

$$N_{0R}(R_j) = I_m, \quad \text{for any } j$$

Proof: See Shieh and Yih ref. [41].

In a similar manner, the spectral factors can be obtained from the known left solvents L_i of $A(\lambda)$, as follows:

$$Q_k = Q_{n-1-k} = M_{(k-1)L}(L_k)^{-1} L_k M_{(k-1)L}(L_k), \quad k = 1,2,\ldots,n \tag{2.56}$$

where $M_k(\lambda) = M_{(k-1)}(\lambda)(\lambda I_m - Q_k)$

$$= \lambda M_{(k-1)}(\lambda) - M_{(k-1)}(\lambda)Q_k , \quad k = 1,2,\ldots,n$$

$$M_{kL}(L_j) = L_j M_{(k-1)L}(L_j) - M_{(k-1)L}(L_j)Q_k , \ k = 1,2,\ldots,n \quad \text{and any } j$$

with

$M_0(\lambda) = I_m$, rank $[M_{(k-1)L}(L_k)] = m$, $k = 1,2,\ldots,n$

$M_{0L}(L_j) = I_m$ for any j

$M_{(k-1)L}(L)$ is a left matrix polynomial of $M_{(k-1)}(\lambda)$ having λ replaced by a left solvent L_j. The spectral factorisation of $A(\lambda)$ becomes

$$A(\lambda) = M_n(\lambda) = (\lambda I_m - Q_1)(\lambda I_m - Q_2)\cdots(\lambda I_m - Q_n) \tag{2.57}$$

2.10.2 Transformation of Spectral Factors to Right (Left) Solvents

Given a complete set of spectral factors of a $\lambda-matrix$ $A(\lambda)$, then a corresponding complete set of right (left) solvents can be obtained.

Before deriving the transformation we first state a lemma to show the existence of the transformations.

Lemma 2.1. *Let* $A(\lambda) = I_m\lambda^n + A_1\lambda^{n-1} + \cdots + A_{n-1}\lambda + A_n$ *and any* $m \times m$ *matrix* R *with eigenvalues* $\lambda_1,\ldots,\lambda_m$. *The eigenvalues of the matrix*

$$G_A(R) = (R^n)^T \otimes I_m + (R^{n-1})^T \otimes A_1 + \cdots + R^T \otimes A_{n-1} + I_m \otimes A_n \tag{2.58}$$

are the union of the eigenvalues of $A(\lambda_i)$, $i = 1,\ldots,m$, *where* $\otimes$ *designates the Kronecker product* [39].

Clearly, from the above lemma we have the following results.

Corollary 2.3: *The matrix function $G_A(R)$ defined in the above lemma is nonsingular if and only if $A(\lambda_i)$, $i = 1,2,\ldots,m$, are nonsingular, where the λ_i are the eigenvalues of the $m \times m$ matrix R.*

The Transformation of spectral factors to right (left) solvents of a λ-*matrix* can be derived as follows.

Theorem 2.13: *Given a monic λ-matrix with all elementary divisors being linear*

$$A(\lambda) = (\lambda I_m - Q_1)(\lambda I_m - Q_2)\cdots(\lambda I_m - Q_n) \tag{2.59}$$

where Q_i ($\triangleq Q_{n-1-i}$), $i = 1,2,\ldots,n$, are a complete set of spectral factors of $A(\lambda)$, and $Q_i \cap Q_j = \varnothing$.

Define λ-*matrices* $N_i(\lambda)$, $i = 1,2,\ldots,n$, as follows:

$$\begin{aligned} N_i(\lambda) &= (\lambda I - Q_i)^{-1} N_{i-1}(\lambda) \\ &= I_m \lambda^{n-i} + A_{1i}\lambda^{n-i-1} + \cdots + A_{(n-i-1)i}\lambda + A_{(n-i)i} \quad , i = 1,2,\ldots,n \end{aligned} \tag{2.60}$$

with

$$N_0(\lambda) = A(\lambda) \tag{2.61}$$

Then, the transformation matrix P_i which transforms the spectral factor Q_i ($\triangleq Q_{n-1-i}$) to the right solvent R_i ($\triangleq R_{n+1-i}$) of $A(\lambda)$ can be constructed from the new algorithm as follows

$$R_i = R_{n+1-i} = P_i Q_i P_i^{-1} \quad , \quad \text{if rank } P_i = m, \qquad i = 1,\ldots,m$$

where the $m \times m$ matrix P_i can be solved from the following matrix equation:

$$Vec(P_i) = G_{Ni}(Q_i)^{-1} Vec(I_m), \quad \text{rank } G_{Ni}(Q_i) = m^2, \quad i = 1,\ldots,m \tag{2.62}$$

where $G_{Ni}(Q_i)$ is defined by

$$G_{Ni}(Q_i) \triangleq (Q_i^{n-i})^T \otimes I_m + (Q_i^{n-i-1})^T \otimes A_{1i} + \cdots + Q_i \otimes A_{(n-i-1)i} + I_m \otimes A_{(n-i)i} \tag{2.63}$$

Proof: See Shieh and Yih ref. [41].

In the same fashion the complete set of spectral factors Q_i , $i = 1,2,\ldots,n$, can be converted into the left solvents L_i , $i = 1,2,\ldots,n$, using the following algorithm.

$$M_i(\lambda) = M_{i-1}(\lambda)(\lambda I_m - Q_i)^{-1}$$
$$= I_m\lambda^{n-i} + A_{1i}\lambda^{n-i-1} + \cdots + A_{(n-i-1)i}\lambda + A_{(n-i)i} \quad , \; i = 1,2,\ldots,n \tag{2.64}$$

$$H_{Mi}(Q_i) = I_m \otimes Q_i^{n-i} + A_{1i}^T \otimes Q_i^{n-i-1} + \cdots + A_{(n-i-1)i}^T \otimes Q_i + A_{(n-i)i}^T \otimes I_m \tag{2.65}$$

$$Vec(S_i) = H_{Mi}(Q_i)^{-1} Vec(I_m) \qquad i = 1,2,\ldots,n$$

$$L_i = S_i^{-1} Q_i S_i \tag{2.66}$$

2.11 Block Companion Form

In analogy with scalar polynomials a useful tool for the analysis of matrix polynomials is the block companion form matrix.

Given a $\lambda - matrix$

$$A(\lambda) = I\lambda^n + A_1\lambda^{n-1} + \cdots + A_n, \tag{2.67}$$

where $A_i \in C^{m\cdot m}$ and $\lambda \in C$, the associated bottom block companion form matrix is:

$$A_R = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_n & -A_{n-1} & -A_{n-2} & \cdots & -A_1 \end{pmatrix} \tag{2.68}$$

And the associated left block companion form matrix is:

$$A_L = \begin{pmatrix} 0 & 0 & \cdots & 0 & -A_n \\ I & 0 & \cdots & 0 & -A_{n-1} \\ 0 & I & \cdots & 0 & -A_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & -A_1 \end{pmatrix} \tag{2.69}$$

Note that A_L is the block transpose of A_R. If the matrix polynomial $A(\lambda)$ has a complete set of solvents, these companion matrices can be respectively block diagonalised via the right block Vandermande matrix defined by

$$V(R_1,R_2,\ldots,R_n)=\begin{pmatrix} I & I & \cdots & I \\ R_1 & R_2 & \cdots & R_n \\ \vdots & \vdots & \ddots & \vdots \\ R_1^{n-1} & R_2^{n-1} & \cdots & R_n^{n-1} \end{pmatrix} \tag{2.70}$$

and the left block Vandermande matrix defined by

$$V(L_1,L_2,\ldots,L_n)=\begin{pmatrix} I & L_1 & \cdots & L_1^{n-1} \\ I & L_2 & \cdots & L_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ I & L_n & \cdots & L_n^{n-1} \end{pmatrix} \tag{2.71}$$

where $R_1,R_2,\ldots,R_n$, $(L_1,L_2,\ldots,L_n)$ represent the complete set of right (left) solvents. Since the Vandermande matrices are nonsingular [42], we can write

$$[V(R_1,R_2,\ldots,R_n)]^{-1}A_R[V(R_1,R_2,\ldots,R_n)]=Diag(R_1,R_2,\ldots,R_n) \tag{2.72}$$

and

$$[V(L_1,L_2,\ldots,L_n)]A_L[V(L_1,L_2,\ldots,L_n)]^{-1}=Diag(L_1,L_2,\ldots,L_n) \tag{2.73}$$

These similarity transformations do a block decoupling of the spectrum of $A(\lambda)$ which is very useful in the analysis and design of large order systems.

2.12 Linearisation of λ-matrices

Given an n^{th} degree and m^{th} order $\lambda-matrix$

$$\overline{A}(\lambda)=\overline{A}_0\lambda^n+\overline{A}_1\lambda^{n-1}+\cdots+\overline{A}_n \tag{2.74}$$

with $\overline{A}_0\neq I_m$ and $Det(\overline{A}_0)\neq 0$. Hence, $\overline{A}(\lambda)$ is nonmonic.

Multiplying both sides of this equation by $\overline{A}_0^{-1}$, the above equation will look as follows:

$$\begin{aligned} A(\lambda)=\overline{A}_0^{-1}\overline{A}(\lambda) &= I\lambda^n+\overline{A}_0^{-1}\overline{A}_1\lambda^{n-1}+\cdots+\overline{A}_0^{-1}\overline{A}_n \\ &= I\lambda^n+A_1\lambda^{n-1}+\cdots+A_n \end{aligned} \tag{2.75}$$

The associated block companion form matrix is the $mn\times mn$ matrix A_C of the form:

$$A_C = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_n & -A_{n-1} & -A_{n-2} & \cdots & -A_1 \end{pmatrix} \tag{2.76}$$

The associated linear $\lambda - matrix$ is an $mn \times mn$ $\lambda - matrix$ defined by $A_C(\lambda) = \lambda I - A_C$.

Theorem 2.17: *Given* $\overline{A}(\lambda)$ *and* $A_c(\lambda)$ *as defined above, then the* $\lambda - matrix$ $A_c(\lambda)$ *and*

$$\begin{pmatrix} \overline{A}(\lambda) & 0 & \cdots & 0 \\ 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_m \end{pmatrix} \tag{2.77}$$

are equivalent.

Proof: first we define the $nm \times nm$ matrix polynomials $E(\lambda)$ and $F(\lambda)$

$$E(\lambda) = \begin{pmatrix} B_{n-1}(\lambda) & B_{n-2}(\lambda) & \cdots & B_1(\lambda) & B_0(\lambda) \\ -I_m & 0 & \cdots & 0 & 0 \\ 0 & -I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_m & 0 \end{pmatrix} \tag{2.78}$$

where $B_i(\lambda) = \lambda B_{i-1}(\lambda) + A_i$, $B_0(\lambda) = \overline{A}_0$ for $i = 1,2,\ldots,n-1$ and

$$F(\lambda) = \begin{pmatrix} I_m & 0 & 0 & \cdots & 0 \\ \lambda I_m & I_m & 0 & \cdots & 0 \\ \lambda^2 I_m & \lambda I_m & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda^{n-1} I_m & \lambda^{n-2} I_m & \lambda^{n-3} I_m & \cdots & I_m \end{pmatrix} \tag{2.79}$$

It can be verified that $Det(E(\lambda)) = \pm 1$ and $Det(F(\lambda)) = 1$, in other terms, $E(\lambda)$ and $F(\lambda)$ are unimodular, hence $F(\lambda)^{-1}$ exists and is also a matrix polynomial. It can also be verified that:

$$E(\lambda)A_C(\lambda)F(\lambda)=\begin{pmatrix}\bar{A}(\lambda) & 0 & \cdots & 0\\ 0 & I_m & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & I_m\end{pmatrix} \tag{2.80}$$

Particular Case: In case $\bar{A}(\lambda)$ is monic, then $\bar{A}_0 = I$ and $\bar{A}(\lambda)$ will reduce to $A(\lambda)$. The unimodular matrices $E(\lambda)$ and $F(\lambda)$ will remain the same except that $B_3(\lambda)$ is now equal to identity. The reader can verify that

$$E(\lambda)A_C(\lambda)F(\lambda)=\begin{pmatrix}A(\lambda) & 0\\ 0 & I\end{pmatrix} \tag{2.81}$$

The importance of the above result lies in the fact that it shows that $A_c(\lambda)$ and $A(\lambda)$ have exactly the same Jordan form matrix. Thus analysing $A(\lambda)$ is equivalent to analysing $A_c(\lambda)$ from the spectral point of view. Another linearisation defined when $A(\lambda)$ has a complete factorisation is:

$$A(\lambda) = (\lambda I_m - Q_1)(\lambda I_m - Q_2)\cdots(\lambda I_m - Q_n) \tag{2.82}$$

Let's define the $nm \times nm$ constant matrix U of the form:

$$U=\begin{pmatrix}Q_1 & 0 & 0 & \cdots & 0 & 0\\ I & Q_2 & 0 & \cdots & 0 & 0\\ 0 & I & Q_3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & Q_{n-1} & 0\\ 0 & 0 & 0 & \cdots & I & Q_n\end{pmatrix} \tag{2.83}$$

The associated linear $\lambda - matrix$ is an $nm \times nm$ $\lambda - matrix$ defined by $\lambda I - U$, then according to theorem (2.17) $\lambda I - U$ is a linearisation of $A(\lambda)$ if and only if there exists $P(\lambda)$ and $Q(\lambda)$ such that:

$$P^{-1}(\lambda)(\lambda I - U)Q(\lambda) \approx \begin{pmatrix}A(\lambda) & 0\\ 0 & I\end{pmatrix}. \tag{2.84}$$

This is satisfied by:

$$P(\lambda)=\begin{pmatrix} I & -(\lambda I-Q_1) & 0 & \cdots & 0 & 0 \\ 0 & I & -(\lambda I-Q_2) & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & -(\lambda I-Q_{n-1}) \\ 0 & 0 & 0 & \cdots & 0 & I \end{pmatrix} \tag{2.85}$$

and

$$Q(\lambda)=\begin{pmatrix} B_{n-1}(\lambda) & -I & 0 & \cdots & 0 & 0 \\ B_{n-2}(\lambda) & 0 & -I & \cdots & 0 & 0 \\ B_{n-3}(\lambda) & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_1(\lambda) & 0 & 0 & \cdots & 0 & -I \\ B_0(\lambda) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{2.86}$$

with $B_0 = I$ and $B_i(\lambda) = (\lambda I - Q_{n-i-1})B_{i-1}(\lambda) \quad i = 1,2,\ldots,n-1$

It can also be verified that $Det(Q(\lambda)) = \pm 1$ and $Det(P(\lambda)) = 1$. So $Q(\lambda)$ and $P(\lambda)$ are unimodular matrix polynomials.

2.13 Operations on Matrix Polynomials

Consider two matrix polynomials on $F(\lambda)$.

$$A(\lambda) = A_0\lambda^n + A_1\lambda^{n-1} + \cdots + A_{n-1}\lambda + A_n \tag{2.87}$$

and

$$B(\lambda) = B_0\lambda^k + B_1\lambda^{k-1} + \cdots + B_{k-1}\lambda + B_k \tag{2.88}$$

We will say that matrices (2.87) and (2.88) are equal, $A(\lambda) = B(\lambda)$, if $n = k$ and $A_i = B_i$, $i = 0,1,2,\ldots,n$.

The sum $A(\lambda) + B(\lambda)$ is a $\lambda - matrix$ $C(\lambda)$ obtained by adding the two corresponding elements of the two $\lambda - matrices$. The product, when meaningful, $A(\lambda)B(\lambda)$ is a $\lambda - matrix$ or matrix polynomial of degree at most equal to $n+k$. If $A(\lambda)$ or $B(\lambda)$ is nonsingular, the degree of $A(\lambda)B(\lambda)$ as well as that of $B(\lambda)A(\lambda)$ is exactly $n+k$.

If the leading coefficient B_0 is nonsingular, then there exists two unique couples of matrix polynomials $[Q(\lambda),\ R(\lambda)]$ and $[\bar{Q}(\lambda)),\ L(\lambda)]$ such that:

$$A(\lambda) = Q(\lambda)B(\lambda) + R(\lambda) \qquad \text{Right Division} \tag{2.89}$$

and

$$A(\lambda) = B(\lambda)\bar{Q}(\lambda) + L(\lambda) \qquad \text{Left Division} \tag{2.90}$$

where degree of $R(\lambda)$ ($L(\lambda)$) less than degree of $B(\lambda)$. In case, $R(\lambda) \equiv 0$ ($L(\lambda) \equiv 0$) then $B(\lambda)$ is said to be right (left) divisor of $A(\lambda)$.

Remark: A matrix polynomial of the form

$$B(\lambda) = b_0\lambda^k I_n + b_1\lambda^{k-1} I_n + \cdots + b_{k-1}\lambda I_n + b_k I_n \tag{2.91}$$

is called "a scalar polynomial".

A scalar matrix polynomial $B(\lambda) = b(\lambda)I_n$ commute with every matrix polynomial whose coefficients are square matrices of order n. If $B(\lambda) = b(\lambda)I_n$ in (2.89) and (2.90), then:

$$A(\lambda) = Q(\lambda)B(\lambda) + R(\lambda) = B(\lambda)\bar{Q}(\lambda) + L(\lambda) \tag{2.92}$$

Remainder Theorem. *Let $A(\lambda)$ be a λ-matrix and let $B = [b_{ij}]$ be a square matrix of order n on F. Since $(\lambda I - B)$ is linear, we can write:*

$$A(\lambda) = Q(\lambda)(\lambda I - B) + R \tag{2.93}$$

and

$$A(\lambda) = (\lambda I - B)\bar{Q}(\lambda) + L \tag{2.94}$$

where R and L are independent of λ.

Theorem 2.18: *If a matrix polynomial $A(\lambda)$ is divided by $(\lambda I - B)$, where $B = [b_{ij}]$ is square of order n, and if the obtained remainders are R and L, then*

$$R = A_R(B) = A_0 B^n + A_1 B^{n-1} + \cdots + A_{n-1}B + A_n \tag{2.95}$$

and

$$L = A_L(B) = B^n A_0 + B^{n-1} A_1 + \cdots + BA_{n-1} + A_n \tag{2.96}$$

Proof: The proof follows from equations (2.93) and (2.94) upon identification.

The above theorem can be used to prove the following corollary

Corollary 2.4: *A matrix B is a solvent of $A(\lambda)$ if and only if $(\lambda I - B)$ divides exactly $A(\lambda)$.*

Proof: See Hariche ref. [43].

In case $A(\lambda)$ is a scalar matrix polynomial, the remainders in (2.95) and (2.96) are identical.

Theorem 2.19: *If a scalar matrix polynomial $p(\lambda)I_n$ is divided by $(\lambda I - B)$ and if the obtained remainder is R, then $R = p(B)$.*

As a consequence, we have:

Theorem 2.20: *A scalar matrix polynomial $p(\lambda)I_n$ is divisible by $(\lambda I - B)$ if and only if $p(B) = 0$.*

2.14 Spectral Divisors

In this section, we consider a special case of division of matrix polynomial $A(\lambda)$, in which $R(\lambda)$ ($L(\lambda)$) is equal to the zero matrix.

Definition 2.5: *If $A(\lambda) = A_1(\lambda)A_2(\lambda)$ is a particular factorisation of the monic matrix polynomial $A(\lambda)$, with $\rho(A_1(\lambda)) \cap \rho(A_2(\lambda)) = \varnothing$, then the monic matrix polynomial $A_1(\lambda)$ and $A_2(\lambda)$ are called spectral divisors of $A(\lambda)$.*

The following result relates the latent roots and latent vectors of $A(\lambda)$ to those of its divisors.

Theorem 2.21: *Let $A(\lambda)$ be a λ-matrix as in (2.87) and $B(\lambda)$ be a right divisor of $A(\lambda)$, then*

- *Every latent root of $B(\lambda)$ is a latent root of $A(\lambda)$.*
- *Every Jordan chain of $B(\lambda)$ corresponding to latent root λ_i is also a Jordan chain of $A(\lambda)$ corresponding to the same latent root.*

Proof: See Hariche ref. [43].

By a set of divisors completely describing the spectral data of $A(\lambda)$, we mean a set of divisors $\{B_1(\lambda), B_2(\lambda), \ldots, B_s(\lambda)\}$ satisfying

$$\begin{aligned} &\bigcup_{i=1}^{s} \sigma(B_i(\lambda)) = \sigma(A(\lambda)) \\ &\sigma(B_i(\lambda)) \bigcap \sigma(B_i(\lambda)) = \varnothing \quad \text{for} \quad i \neq j \end{aligned} \tag{2.97}$$

2.15 Methods For Computation of a Complete Set of Solvents And Spectral Factors of Matrix Polynomials

Numerous methods [46, 47, 48] are available for computing the solvents and spectral factors of a matrix polynomial $A(\lambda)$. Simple approaches [46, 47] use the eigenvalues and the eigenvectors of a matrix polynomial to construct the solvents of the $\lambda-matrix$ $A(\lambda)$ using equation (2.26). However, it is often inefficient to explicitly determine the eigenvalues and eigenvectors of a matrix, which could be ill conditioned [49]. On the other hand, other methods with no prior knowledge of the eigenvalues and eigenvectors of the matrix are available [39, 46, 50, 51, 52]. For example in [50] the generalised Newton's method has been successfully used for determining the solvents of a $\lambda-matrix$. But, before dealing with this method, we have first to define what is a gradient of a matrix polynomial as well as the contraction operation.

2.15.1 Notion of Gradient of a Matrix Polynomial

The gradient of a matrix polynomial is defined [50] as:

$$\{\nabla A(X)\}_{i,j,k,l} = \frac{\partial}{\partial X_{k,l}} \{A(X)\}_{i,j} \quad : \quad i,j,k,l = 1,2,\ldots,m \tag{2.98}$$

where $X_{k,l}$ denotes the (k,l) element of X, $\{A(X)\}_{i,j}$ designates the (i,j) element of $A(X)$, and $\{\nabla A(X)\}_{i,j,k,l}$ denotes the (i,j,k,l) element of $\nabla A(X)$.

It is also shown in [50] that a contraction operation on $\nabla A_R(X)$ with respect to an arbitrary $m \times m$ square matrix Y is given by:

$$\nabla A_R(X)Y = \frac{d}{d\eta}\left(A_R(X+\eta Y)\right)\Big|_{\eta=0}$$
$$= \sum_{i=0}^{n} \frac{d}{d\eta}\left(A_i(X+\eta Y)\right)^{n-i}\Big|_{\eta=0} \tag{2.99}$$

And each term of the summation (2.99) can be computed as

$$\frac{d}{d\eta}\left(A_i(X+\eta Y)^r\right)\Big|_{\eta=0} = A_i \sum_{q=0}^{r-1} X^q Y X^{r-q-1} \tag{2.100}$$

Substituting (2.100) into (2.99) and rearranging indexes gives

$$\nabla A_R(X)Y = \sum_{i=0}^{n-1} A_i \sum_{q=0}^{n-i-1} X^q Y X^{n-i-q-1} \tag{2.101}$$

Performing index transformations or letting $k = i+q+1$ and $j = i$, (2.101) becomes

$$\nabla A_R(X)Y = \sum_{k=1}^{n} B_{kR}(X) Y X^{n-k} \tag{2.102}$$

where $B_{kR}(X)$ is the right matrix polynomial of the following $\lambda - matrix$ $B_k(\lambda)$:

$$B_k(\lambda) = \sum_{j=0}^{k-1} A_j \lambda^{k-j-1} \tag{2.103}$$

In a similar manner, for the left matrix polynomial $A_L(X)$, we have

$$\nabla A_L(X)Y = \sum_{k=1}^{n} X^{n-k} Y B_{kL}(X) \tag{2.104}$$

where $B_{kL}(X)$ is the left matrix polynomial of the $\lambda - matrix$ defined in (2.103).

2.15.2 Generalized Newton Method

Given a right matrix polynomial

$$A_R(X) = A_0 X^n + A_1 X^{n-1} + \cdots + A_n \tag{2.105}$$

It can be expanded around an $m \times m$ matrix X_0 as

$$A_R(X) = A_R(X_0) + \nabla A_R(X_0)(X - X_0) + O((X - X_0)) \tag{2.106}$$

where $O(X - X_0)$ is a matrix polynomial with high degree terms of $(X - X_0)$, and $\nabla A_R(X_0)(X - X_0)$ is a contracted gradient [50] of dimension $m \times m$.

The first degree approximation of (2.106) with $|\Delta X| < 1$ becomes

$$A_R(X) \cong A_R(X_0) + \nabla A_R(X_0)\Delta X \tag{2.107}$$

where $\Delta X \triangleq X - X_0$. We define a recursive formula

$$X_{i-1} = X_i + \Delta X_{i-1} \tag{2.108}$$

so, (2.107) becomes

$$A_R(X_{i-1}) \cong A_R(X_i) + \nabla A_R(X_i)\Delta X_{i-1} \tag{2.109}$$

If X_{i-1} is the right solvent of $A(\lambda)$, or $A_R(X_{i-1}) = 0_m$, then

$$A_R(X_i) + \nabla A_R(X_i)\Delta X_{i-1} = 0_m \tag{2.110}$$

Solving for ΔX_{i-1} in (2.110) and substituting it into (2.108) gives the recursive formula for solving the right solvent of $A(\lambda)$.

To solve for ΔX_{i-1}, we use the contracted gradient developed previously, as follows:

$$\nabla A_R(X_i)\Delta X_{i-1} = \sum_{k=1}^{n} B_{kR}(X_i)\Delta X_{i-1} X_i^{n-k} \tag{2.111}$$

Substituting (2.111) into (2.110) yields

$$\sum_{k=1}^{n} B_{kR}(X_i)\Delta X_{i-1} X_i^{n-k} = -A_R(X_i) \tag{2.112}$$

using theorem (2.13), we have:

$$Vec(\Delta X_{i-1}) = -G(X_i)^{-1} Vec\{A_R(X_i)\} \tag{2.113}$$

where

$$G(X_i) = \sum_{k=1}^{n} (X_i^{n-k})^T \otimes B_{kR}(X_i) \tag{2.114}$$

In a similar fashion, the recursive formula for solving the left solvents of $A(\lambda)$ is:

$$X_{i-1} = X_i + \Delta X_{i-1} \tag{2.115}$$

where ΔX_{i-1} is the solution of the following linear matrix equation:

$$\nabla A_L(X_i)\Delta X_{i-1} = \sum_{k=1}^{n} X_i^{n-k}\Delta X_{i-1} B_{kL}(X_i) = -A_L(X_i) \tag{2.116}$$

or

$$Vec\{\Delta X_{i-1}\} = -H(X_i)^{-1} Vec\{A_L(X_i)\} \tag{2.117}$$

where

$$H(X_i) = \sum_{k=1}^{n} B_{kL}(X_i)^T \otimes X_i^{n-k} \tag{2.118}$$

The convergence criterion is $\|\Delta X_{i-1}\| < \varepsilon$ where ε is an assigned small value. It should be also noted that the above procedure gives only one solvent at a time, hence this solvent must be simplified from the matrix polynomial $A(\lambda)$ through long division, then set the new obtained matrix polynomial as being the new $A(\lambda)$, and redo again the above process for the computation of the next solvent. But as it can be seen, this method depends largely on the initial guess, the reason for which most researchers [53] use it as a local method. In contrast other methods known as global methods are available among which we mention the Bernoulli's method and the *QD* algorithm.

2.15.3 Bernoulli's Method

The Bernoulli's iterative matrix method is a generalization of the scalar Bernoulli's method for the computation of the zero of a scalar polynomial with largest absolute value. Bernoulli's method is based on the solution of a difference equation [40]. From now on, a solvent S is said to be dominant solvent if every eigenvalue of S exceeds in modulus every eigenvalue of the quotient $A(\lambda)(I\lambda - S)^{-1}$ [40].

Theorem 2.22: *Let* $A(\lambda) = I\lambda^n + A_1\lambda^{n-1} + \cdots + A_n$ *be a monic matrix polynomial of degree n. Assume that* $A(\lambda)$ *has a dominant solvent* S, *and the transposed matrix polynomial* $A^T(\lambda)$ *also has a dominant solvent. Let* $\{U_r\}_{r=1}^{\infty}$ *be the solution of the system*

$$A_1 U_r + A_2 U_{r-1} + \cdots + A_{n-1} U_{r-n+1} + U_{r-n} = 0 \qquad r = 1,2,\ldots \tag{2.119}$$

where $\{U_r\}_{r=1}^{\infty}$ *is a sequence of* $m \times m$ *matrices to be found and is determined by the initial conditions*

$$U_0 = U_1 = \cdots = U_{n-1} = 0, \qquad U_n = I$$

then $U_{r-1}U_r^{-1}$ *exists for large* r *and*

$$U_{r-1}U_r^{-1} \to S \quad \text{as} \quad r \to \infty \tag{2.120}$$

Proof: See Gohberg et al. ref. [40].

For the case of a complete set of solvents the convergence is stated in terms of block Vandermande matrices [40].

Theorem 2.24: *If* $A(\lambda)$ *is a matrix polynomial of degree* n *such that:*

(i) *it has a complete set of solvents* $S_1, S_2, \ldots, S_n$.

(ii) S_1 *is a dominant solvent.*

(iii) $V(S_1, S_2, \ldots, S_n)$ *and* $V(S_2, S_3, \ldots, S_n)$ *are nonsingular, then*

$$\begin{aligned} \lim_{k\to\infty} U_{k-1}U_k^{-1} &= S_1 = R_1 \\ \lim_{k\to\infty} U_k^{-1}U_{k-1} &= S_1 = L_1 \end{aligned} \tag{2.121}$$

2.15.4 The QD Algorithm

The matrix quotient-difference algorithm is a generalisation of the *QD* algorithm for the determination of the zero's of a scalar polynomial [54].

Given a matrix polynomial with nonsingular coefficients as

$$A(\lambda) = I\lambda^n + A_1\lambda^{n-1} + \cdots + A_{n-1}\lambda + A_n \tag{2.122}$$

The objective is to find the spectral factors of $A(\lambda)$, that will allow us to write $A(\lambda)$ as a product of n linear spectral factors:

$$A(\lambda) = (\lambda I - Q_1)(\lambda I - Q_2)\cdots(\lambda I - Q_n) \tag{2.123}$$

Writing $A(\lambda)$ in block left companion form, we have

$$C_1 = \begin{bmatrix} -A_1 & I & 0 & \cdots & 0 \\ -A_2 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{n-1} & 0 & 0 & \cdots & I \\ -A_n & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.124}$$

The required transformation is a sequence of LR decompositions such that

$$C_3 = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.125}$$

where

$$C_{11} = \begin{bmatrix} -A_1 & I & 0 & \cdots & 0 \\ -A_2 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_{n-2} & 0 & 0 & \cdots & I \\ -A_{n-1} & 0 & 0 & \cdots & 0 \end{bmatrix} \quad C_{12} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \tag{2.126}$$

$$C_{21} = \begin{bmatrix} -A_n & 0 & \cdots & 0 & 0 \end{bmatrix} \quad C_{22} = \begin{bmatrix} 0 \end{bmatrix} \tag{2.127}$$

It is required to have $C = 0$, then let

$$X = \begin{bmatrix} X_1, X_2, X_3, \ldots, X_{n-1} \end{bmatrix} \tag{2.128}$$

we obtain the following set of equations:

$$\begin{gathered} -X_1 A_1 - X_2 A_2 - \cdots - X_{n-1} A_{n-1} = -A_n \\ X_1 = X_2 = X_3 = \cdots = X_{n-2} = 0 \\ X_{n-1} + D = 0 \end{gathered} \tag{2.129}$$

leading to the following decomposition of C_3:

$$C_3 = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & A_n A_{n-1}^{-1} & I \end{bmatrix} \begin{bmatrix} -A_1 & I & \cdots & 0 & 0 \\ -A_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n-1} & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & 0 & -A_n A_{n-1}^{-1} \end{bmatrix} \tag{2.130}$$

Hence, C_3 can be written as

$$C_3 = L_{-(n-2)} R_{-(n-2)} \tag{2.131}$$

Continuing this process of the block C_{11} up when C_3 is equivalent to a matrix R_0

$$C_3 = L_{-(n-2)} L_{-(n-3)} \cdots L_0 R_0 \tag{2.132}$$

where

$$R_0 = \begin{bmatrix} -A_1 & I & \cdots & 0 & 0 \\ 0 & -A_2 A_1^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -A_{n-1} A_{n-2}^{-1} & I \\ 0 & 0 & \cdots & 0 & -A_n A_{n-1}^{-1} \end{bmatrix} \tag{2.133}$$

$$L_0 = \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ A_2A_1^{-1} & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.134}$$

$$L_{-1} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & A_3A_2^{-1} & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I & 0 \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \quad \text{...etc.} \tag{2.135}$$

It is clear that if the matrices $L_0, L_{-1}, \ldots, L_{-(n-2)}$ are equal to identity matrices, then the block companion matrix C_1 will be similar to the following matrix

$$M = \begin{bmatrix} Q_1 & I & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & I & \cdots & 0 & 0 \\ 0 & 0 & Q_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Q_{n-1} & I \\ 0 & 0 & 0 & \cdots & 0 & Q_n \end{bmatrix} \tag{2.136}$$

The following theorem shows that under certain conditions, the sequence of $L_i, i = -(n-2), \ldots, 0$ converge to identities.

Theorem 2.25: Let $M = X\Lambda X^{-1}$ where

$$\Lambda = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{bmatrix} \tag{2.137}$$

If the following conditions are satisfied:

(a) There is a dominance relation between the R_i's: $R_1 > R_2 > \cdots > R_n > 0$.

(b) $X^{-1} = Y$ has a block LR factorisation $L_y R_y$.

(c) X has a block LR factorisation $L_x R_x$.

then the block LR algorithm just defined converges ($L_i \to I$).

Proof: See Dahimane ref. [53].

This means that we can start the QD algorithm by considering

$$E_1^{(0)} = A_2 A_1^{-1}; E_2^{(-1)} = A_3 A_2^{-1}; \ldots; E_{n-1}^{n-2} = A_n A_{n-1}^{-1} \tag{2.138}$$

and we deduce that

$$Q_1^{(0)} = -A_1; Q_2^{(-1)} = 0; \ldots; Q_n^{(n-1)} = 0 \tag{2.139}$$

Equations (2.138) and (2.139) provide us with the first two rows of the QD tableau. Hence, we can solve for the rhombus rules and get the row generation of the QD algorithm.

$$\begin{aligned} Q_i^{(j-1)} &= Q_i^{(j)} + E_i^{(j)} - E_{i-1}^{(j-1)} \\ E_i^{(j-1)} &= Q_{i+1}^{(j)} E_i^{(j)} [Q_i^{(j-1)}]^{-1} \end{aligned} \tag{2.140}$$

Writing this in tabular form yields:

$$\begin{array}{ccccccc} & -A_1 & & 0 & & 0 & \cdots \\ 0 & & A_2 A_1^{-1} & & A_3 A_2^{-1} & & A_4 A_3^{-1} & \cdots \\ & Q_1^{(1)} & & Q_2^{(0)} & & Q_3^{(-1)} & & \cdots \\ 0 & & E_1^{(1)} & & E_2^{(0)} & & E_3^{(-1)} & \cdots \\ & Q_1^{(2)} & & Q_2^{(1)} & & Q_3^{(2)} & & \cdots \\ 0 & & E_1^{(2)} & & E_2^{(1)} & & E_3^{(2)} & \cdots \\ & Q_1^{(3)} & & Q_2^{(2)} & & Q_3^{(3)} & & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{2.141}$$

where the $Q_i^{(j)}$ are the spectral factors of $A(\lambda)$. In addition, note that the QD algorithm gives all the spectral factors simultaneously and in dominance order. We have chosen, in the above, the row generation algorithm because it is more stable numerically. The interested reader may refer to Dahimane thesis [53] for further informations about the row generation algorithm and the column generation algorithm.

Chapter 3

Overview of the Existing Model Reduction Methods

In this chapter, an overview of the existing model reduction methods is presented. However, before doing so some familiar model approximation techniques that turn to be basic in system analysis are considered.

3.1 Basics

3.1.1 Linearization

Let $g(x,y,z,\ldots)$ be a function of many independent variables with continuous derivatives. Then the expansion of $g(x,y,z,\ldots)$ around a linearization point $(x_0,y_0,z_0,\ldots)$ using the Taylor series expansion, yields:

$$g(x,y,z,\ldots) = g(x_0,y_0,z_0,\ldots) + \frac{\partial g(x_0,y_0,z_0,\ldots)}{\partial x}(x-x_0) - \frac{\partial g(x_0,y_0,z_0,\ldots)}{\partial y}(y-y_0) - \frac{\partial g(x_0,y_0,z_0,\ldots)}{\partial z}(z-z_0)+\cdots \quad (3.1)$$

and by truncating the obtained expansion at some order k, a k^{th} order approximate equation will be obtained.

3.1.2 Discretization

When sample and hold devices are added to a system, it is said to be a discretized system. The discretization of system dynamics is a form of approximation where the original model and its approximate model are equal or close at the time instants as defined by the discretization or measurement

Piecewise-constant functions are often generated by a sample and a hold, called zero-order hold. These functions $u_i(t), i = 1,2,\ldots,p$, are described by

$$u_i(t) = u_i(kT) \qquad kT \le t \le (k-1)T \qquad (3.2)$$
$$k = 0,1,2,\ldots \qquad i = 1,2,\ldots,p$$

where T is a positive constant, called the sampling period

A linear system in the form

$$\begin{cases} x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \qquad (3.3)$$

with constant input $u(t)$ between two sampling instants can be descretized in time as follows:

$$\begin{cases} x(k+1) = \Phi x(k) + \Gamma u(k) \\ y(k) = Cx(k) \end{cases} \qquad (3.4)$$

or

$$\begin{cases} x_{k-1} = \Phi x_k + \Gamma u_k \\ y_k = Cx_k \end{cases} \qquad (3.5)$$

where $x(k) = x_k = x(t_k)$ at time instant t_k. For equidistant time instants $t_k = kT$, the matrices Φ and Γ are given by:

After application of the Laplace transform to equation (3.3), we get:

$$x(t) = L^{-1}\{(sI - A)^{-1}\}x(t_0) + L^{-1}\{(sI - A)^{-1}BU(s)\}$$
$$= e^{At}\Big|_{t_0}^{t} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Since the inputs are constant during the sampling period T, the vector $u(\tau)$ can be placed outside the integral sign, thus

$$x(t) = e^{At}\Big|_{t_0}^{t} x(t_0) + \left[\int_{t_0}^{t} e^{A(t-\tau)}Bd\tau\right]u(kT)$$

with $t_0 = kT$, $t = (k+1)T$ and by making a change of variable $s = (k-1)T - \tau$, the above equation reduces to:

$$x((k-1)T) = e^{AT}x(kT) - \left[\int_0^T e^{As}Bds\right]u(kT)$$

hence,

$$\Phi = e^{AT}$$
$$\Gamma = \int_0^T e^{As} B ds \tag{3.6}$$

3.1.3 Heuristic Model Reduction Methods

In this section, we are going to introduce some of the classical methods of model order reduction and point attention to their limitations. The most popular of them are the polynomial truncation, the method of dominating poles and pole-zero cancellations, which are all applicable to linear systems only. Hence, a nonlinear system requires to be linearized beforehand.

Let's start by considering a linear system with a transfer function

$$H(z) = \frac{0.35z^{-1}}{1 - 0.86z^{-1} - 0.0855z^{-2}} = \frac{0.35z^{-1}}{(1 - 0.95z^{-1})(1 + 0.09z^{-1})} \tag{3.7}$$

From the factored denominator of $H(z)$, it is clear that there is a large difference in the two time constants. A spontaneous but poor method of model reduction is simply to truncate the numerator and denominator polynomials with a static gain compensation (i.e. at z=1).

$$H(z) \cong H_1(z) = \frac{0.899z^{-1}}{1 - 0.86z^{-1}} \tag{3.8}$$

This is obtained by seeking $H_1(z)$ of the form

$$H_1(z) = \frac{kz^{-1}}{(1 - 0.86z^{-1})}$$

and for

$$H(1) = H_1(1) \Leftrightarrow \frac{0.35}{1 - 0.86 - 0.0855} = \frac{k}{1 - 0.86}$$

so

$$k = 0.899$$

It can be verified by impulse response and step response signals that even if the truncation is supported by static gain compensation, $H_1(z)$ is still

a poor approximation of $H(z)$. The reason is that the pole-zero location is very sensitive to the higher order coefficients.

It often makes more sense to keep the dominating poles and to eliminate the dominated ones while preserving the static gain. Applying this method to equation (3.7) yields

$$H(z) \approx H_2(z) = \frac{0.321z^{-1}}{(1-0.95z^{-1})} \tag{3.9}$$

this is also obtained seeking $H_2(z)$ of the form

$$H_2(z) = \frac{kz^{-1}}{(1-0.95z^{-1})}$$

and

$$H_2(1) = H(1) \Leftrightarrow \frac{k}{1-0.95} = \frac{0.35}{0.0545} \quad \text{so,}$$

$$k = 0.321$$

In a similar manner if we want to preserve the other pole, the computations yield:

$$H_3(z) = \frac{7z^{-1}}{1+0.09z^{-1}} \tag{3.10}$$

Clearly, it is easy to show serious shortcomings of the above heuristic methods according to criteria of preserved static gain, step responses, impulse responses, or least squares fitting. It is therefore desirable to derive methods for model approximation based on some sensitivity analysis of the input-output properties. According to the work of Kalman [56], a natural way to decrease the order of a model is to delete everything except the controllable and observable part. The reason is that this part of the model is structurally unstable, hence one should measure the sensitivities of the input output map in different directions of state space. The most controllable and observable part could then be used as a low order approximation for the model. This is exactly what is done in Moore [1], where the **controllability Gramian** and the **observability Gramian** are used to define measures of controllability and observability in certain directions of the state space.

In the following sections we present different methods which have been developed to derive reduced-order models starting from a higher-order ones In general these methods are either based on the state space description or on the frequency domain description

3.2 State Space Reduction Techniques

The class of methods that use the state space description is considered in this section, while methods depending on the transfer function description are treated in the next section.

3.2.1 Balanced Realisation and model reduction

For the sake of clarity, the development of the balanced realisation method is done in the discrete time domain, then the obtained results are extended to the continuous time case.

3.2.1.1 Balanced Realisation For Discrete-Time systems

Consider a linear time-invariant system in state space form

$$\begin{cases} x_{k+1} = \Phi x_k + \Gamma u_k \\ y_k = C x_k \end{cases} \qquad x_k \in R^n \tag{3.11}$$

The idea is to develop a quantitative measure on the observability and controllability of the states.

Direct calculation of the state x_N from an input sequence $\{u_k\}_{k=0}^{N-1}$ via equation (3.11), assuming $x_0 = 0$, yields

$$\begin{aligned} x_1 &= \Phi x_0 + \Gamma u_0 = \Gamma u_0 \\ x_2 &= \Phi x_1 + \Gamma u_1 = \Phi\Gamma u_0 + \Gamma u_1 \\ &\vdots \\ x_N &= \sum_{k=1}^{N} \Phi^{N-k}\Gamma u_{k-1} \\ &= \begin{bmatrix} \Phi^{N-1}\Gamma & \Phi^{N-2}\Gamma & \cdots & \Gamma \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = \Psi_N U_N \end{aligned} \tag{3.12}$$

In fact, there is an infinite number of control sequences U_N that result in the state x_N for $N > n$, however it is suitable to choose the one with smallest 2-norm. The minimum energy control sequence $\{u_k\}_{k=0}^{N-1}$ that results in the desired state x_N, is obtained by means of the pseudo-inverse of Ψ_N as:

$$U_N = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = \Psi_N^T (\Psi_N \Psi_N^T)^{-1} x_N \tag{3.13}$$

A special matrix called the controllability Gramian P_N is defined as:

$$P_N = \Psi_N \Psi_N^T = \sum_{k=0}^{N-1} \Phi^k \Gamma \Gamma^T (\Phi^T)^k \geq 0 \tag{3.14}$$

As $P_N \geq 0$ is positive semidefinite, it can be concluded that the above equation provides a quadratic form with a bound on the controllable states at time N. It is obvious from (3.14) that the controllability Gramian P_N satisfies the recursive equation

$$\begin{aligned} P_{N-1} &= \sum_{k=0}^{N} \Phi^k \Gamma \Gamma^T (\Phi^T)^k \\ &= \sum_{M=-1}^{N-1} \Phi^{M+1} \Gamma \Gamma^T (\Phi^T)^{M+1} \qquad M = k-1 \\ &= \Gamma \Gamma^T + \sum_{M=0}^{N-1} \Phi^{M+1} \Gamma \Gamma^T (\Phi^T)^{M+1} \\ &= \Gamma \Gamma^T + \Phi \Big(\sum_{M=0}^{N-1} \Phi^M \Gamma \Gamma^T (\Phi^T)^M \Big) \Phi^T \\ &= \Gamma \Gamma^T + \Phi P_N \Phi^T \end{aligned} \tag{3.15}$$

So

$$P_{N-1} = \Phi P_N \Phi^T + \Gamma \Gamma^T \tag{3.16}$$

And the solution P_N for a stable matrix Φ approaches the solution to the following Lyapunov equation

$$\Phi P \Phi^T + \Gamma \Gamma^T - P = 0. \tag{3.17}$$

The asymptotic controllability Gramian $P = \lim_{N \to \infty} P_N$ satisfies the above Lyapunov equation (3.17).

Likewise, the observability Gramian is defined as the infinite sum

$$Q_N = \sum_{k=0}^{\infty} (\Phi^T)^k C^T C \Phi^k \tag{3.18}$$

which satisfies similarly the Lyapunov equation

$$\Phi^T Q \Phi - Q + C^T C = 0 \tag{3.19}$$

and

$$\begin{aligned} Q_{k+1} &= \sum_{k=-1}^{\infty} (\Phi^T)^{k+1} C^T C \Phi^{k+1} \\ &= C^T C + \sum_{k=0}^{\infty} (\Phi^T)^{k+1} C^T C \Phi^{k+1} \\ &= C^T C + \Phi^T \sum_{k=0}^{\infty} (\Phi^T)^k C^T C \Phi^k \Phi \\ &= C^T C + \Phi^T Q_k \Phi \end{aligned} \tag{3.20}$$

Here also, the solution Q_k for a stable matrix Φ approaches the solution to the Lyapunov equation

$$\Phi^T Q \Phi - Q + C^T C = 0 \tag{3.21}$$

And the asymptotic observability Gramian $Q = \lim_{k \to \infty} Q_k$ satisfies the above Lyapunov equation (3.21).

It is clear that the controllability and observability Gramian matrices P and Q define quantitative measures of reachability and observability of the different subspaces of the state space. It is also well known that the Gramians are varying under state space transformations [1]. In order to clarify this, consider a state space transformation $z_k = Tx_k$ with its state space equations.

$$\begin{cases} z_{k+1} = \hat{\Phi} z_k + \hat{\Gamma} u_k = T \Phi T^{-1} z_k + T \Gamma u_k \\ y_k = \hat{C} z_k = C T^{-1} z_k \end{cases} \tag{3.22}$$

The corresponding Gramians for the system in equation (3.22) are

with $$P = \sum_{k=0}^{N-1} \Phi \Gamma \Gamma^T (\Phi^T)^k$$

$$
\begin{aligned}
P_z &= \sum_{k=0}^{N-1} (T\Phi T^{-1})^k (T\Gamma)(T\Gamma)^T ((T\Phi T^{-1})^T)^k \\
&= \sum_{k=0}^{N-1} T\Phi^k T^{-1} T\Gamma\Gamma^T T^T (T^T)^{-1} (\Phi^T)^k T^T \\
&= T \sum_{k=0}^{N-1} \Phi^k \Gamma\Gamma^T (\Phi^T)^k T^T \\
&= TPT^T
\end{aligned}
\tag{3.23}
$$

Similarly for the observability Gramian, we have

$$
Q = \sum_{k=0}^{\infty} (\Phi^T)^k C^T C \Phi^k
$$

then

$$
\begin{aligned}
Q_z &= \sum_{k=0}^{\infty} ((T\Phi T^{-1})^T)^k (CT^{-1})^T (CT^{-1})(T\Phi T^{-1})^k \\
&= \sum_{k=0}^{\infty} T^{-T} (\Phi^k)^T T^T T^{-T} C^T C T^{-1} T\Phi^k T^{-1} \\
&= T^{-T} \sum_{k=0}^{\infty} (\Phi^T)^k C^T C \Phi^k T^{-1} \\
&= T^{-T} Q T^{-1}
\end{aligned}
\tag{3.24}
$$

At this point, the question that rises is whether there is an transformation T such that $P_z = Q_z$. In fact, this property is achieved b choosing a state space representation z with equal and diagona controllability and observability Gramians such that

$$
P_z = Q_z = \Sigma = diag(\sigma_1, \sigma_2, \ldots), \quad \text{with} \quad \sigma_i = \sqrt{\lambda_i(PQ)} \tag{3.25}
$$

where $\lambda_i(PQ)$ denotes the i^{th} eigenvalue of the matrix PQ and Σ is diagonal matrix with elements σ_i. Hence the problem reduces to th derivation of the transformation T. One way is to use the Cholesky factor Q_1, U, Σ_1, of the matrices P, Q, Σ as intermediate results, and determines th state space transformation matrix T as follows:

$$
\begin{aligned}
&Q = Q_1^T Q_1 \\
&Q_1 P Q_1^T = U\Sigma^2 U^T \\
&U^T U = I \\
&\Sigma = \Sigma_1^T \Sigma_1 \\
&T = \Sigma_1^{-1} U^T Q_1
\end{aligned}
\tag{3.26}
$$

This transformation results in a state-space realization with similar ("balanced") properties of controllability and observability Gramians, and the magnitude of the elements σ_i of the Gramian Σ expresses the relative importance of each state $(z_k)_i$ for the input-output behaviour.

Let us now limit the discussion to asymptotically stable and balanced systems. Hence

$$P = Q = \Sigma^2 = Diag(\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2) \tag{3.27}$$

with large σ_i corresponding to essential states z_i while small σ_j correspond to states z_j that are less important for the input-output behaviour. We may view this as two interconnected systems and it seems reasonable to assume that the subsystem with small singular values does not affect the input-output behaviour of the system very much. Hence, its elimination is without significance.

Let $\hat{\Phi} = T\Phi T^{-1}$ and $\hat{\Gamma} = T\Gamma$ denote the transformed system matrices of equation (3.22) and let the state vector $z_k = Tx_k$ be decomposed as

$$z_k = \begin{bmatrix} z_k^1 \\ z_k^2 \end{bmatrix} \tag{3.28}$$

where z_k^2 is the vector of components with small singular values. The state space equation (3.22) can be written as:

$$\begin{aligned} z_{k+1} &= \begin{pmatrix} z_{k+1}^1 \\ z_{k+1}^2 \end{pmatrix} = \hat{\Phi} z_k + \hat{\Gamma} u_k = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} z_k^1 \\ z_k^2 \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} u_k \\ y_k &= \hat{C} z_k = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} z_k^1 \\ z_k^2 \end{pmatrix} \end{aligned} \tag{3.29}$$

By neglecting the dynamics of z_k^2 in front of those of z_k^1 and u_k, and consequently eliminating z_k^2 from the following system, we get:

$$\begin{aligned} z_{k+1}^1 &= \Phi_{11} z_k^1 + \Phi_{12} z_k^2 + \Gamma_1 u_k \\ z_{k+1}^2 &= \Phi_{21} z_k^1 + \Phi_{22} z_k^2 + \Gamma_2 u_k \end{aligned} \tag{3.30}$$

The reduced-order model will reduce to:

$$z_{k-1} = \Phi_1 z_k - \Gamma_1 u_k$$
$$y_k = C_1 z_k - \Gamma_0 u_k \qquad (3.31)$$

A model reduction guided by the magnitude of the singular values in the Gramian matrix Σ is called a balanced model reduction.

As an application of the above method, consider the following transfer function [57]

$$H(z) = \frac{0.22z^{-1}}{1 - 0.7z^{-1} - 0.08z^{-2}}$$

with the controllable canonical realization

$$x(k-1) = \begin{pmatrix} 0.7 & 0.08 \\ 1 & 0 \end{pmatrix} x(k) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} 0.22 & 0 \end{pmatrix} x(k)$$

Using the Cholesky factors, a balanced realization is produced as

$$x(k-1) = \begin{pmatrix} 0.7869 & 0.1079 \\ 0.1079 & -0.0869 \end{pmatrix} x(k) - \begin{pmatrix} 0.4579 \\ -0.1018 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} 0.4579 & -0.1018 \end{pmatrix} x(k)$$

with

$$\Sigma = \begin{pmatrix} 0.5510 & 0 \\ 0 & 0.0169 \end{pmatrix}$$

And the state space transformation matrix

$$T = \begin{pmatrix} 0.4579 & 0.0288 \\ -0.1018 & 0.1295 \end{pmatrix}$$

The elements of the diagonalized Gramian Σ are of different magnitudes, which indicates that a first order model would be sufficient. According to the previous development, the elimination of the second state vector component of the balanced system results in a balanced first order model

$$x(k-1) = 0.7869x(k) - 0.4579u(k)$$
$$y(k) = 0.4579x(k)$$

3.2.1.2 Balanced Model Reduction For Continuous-Time Systems

The continuous time analogs of the results presented in the previous section will be considered here. The system is assumed to be controllable, observable, asymptotically stable with a state space representation

$$\begin{aligned} x &= Ax - Bu \\ y &= Cx - Du \end{aligned} \tag{3.32}$$

The controllability and observability Gramians are defined as

$$\begin{aligned} W_c &= \int_0^\infty e^{At} BB^T e^{A^T t} dt \\ W_o &= \int_0^\infty e^{A^T t} C^T C e^{At} dt \end{aligned} \tag{3.33}$$

As in the discrete time case, it can be shown that it is possible to find an equivalence tranformation which makes the two gramians equal to some diagonal positive definite matrix Σ. The corresponding state space is said to be balanced and looks as follows:

$$\begin{aligned} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + Wu \end{aligned} \tag{3.34}$$

then by eliminating the least controllable and observable part, a reasonable approximation is obtained as:

$$\begin{aligned} z_1 &= \Phi_{11} z_1 + \Gamma_1 u \\ y &= C_1 z_1 + Wu \end{aligned} \tag{3.35}$$

For illustrative purposes, consider the following third order linear system [57]

$$G(s) = \frac{2}{(s-1)} \cdot \frac{229}{(s^2 - 30s - 229)}$$

The balanced state space model of this transfer function is

$$\begin{aligned} x &= \begin{pmatrix} -0.6683 & -1.6355 & -0.6166 \\ 1.6355 & -8.2111 & -7.4027 \\ -0.6165 & 7.4027 & -22.1206 \end{pmatrix} x + \begin{pmatrix} 1.2136 \\ -1.3380 \\ 0.5634 \end{pmatrix} u \\ y &= \begin{pmatrix} 1.2136 & 1.3380 & 0.5634 \end{pmatrix} x \end{aligned}$$

with Gramian and its singular values

$$\Sigma = \begin{pmatrix} 1.1018 & 0 & 0 \\ 0 & 0.1090 & 0 \\ 0 & 0 & 0.0072 \end{pmatrix}$$

and the transformation matrix T between x and $z = Tx$

$$T = \begin{pmatrix} 1.2136 & 36.6514 & 345.4191 \\ -1.3380 & -32.6780 & 26.8176 \\ 0.5634 & -5.6501 & 5.1795 \end{pmatrix}$$

The second-order reduced model is

$$\dot{x} = \begin{pmatrix} -0.6683 & -1.6355 \\ 1.6355 & -8.2111 \end{pmatrix} x - \begin{pmatrix} 1.2136 \\ -1.3380 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1.2136 & 1.3380 \end{pmatrix} x$$

and the corresponding transfer function is

$$G_2(s) = \frac{-0.3174s + 16.208}{s^2 - 8.27793s - 8.162338}$$

And finally a first order reduced model can be derived as:

$$\begin{cases} \dot{x} = -0.6683x - 1.2136u \\ y = 1.2136x \end{cases}$$

3.2.2 Model Reduction Based on Aggregation

Historically, the concepts and techniques of aggregation were developed at first in areas other than control systems. It has been the subject matter of extensive research studies in the economics literature. However, interaction between the economics and the control fields gave rise to a great interest in aggregation in this latter, because of the possibility that it offers in providing simplified models which could be more easily used in the analysis and synthesis of large scale controllers.

Consider a multivariable system described by the following state-space equations

$$\begin{cases} \dot{x}(t) = Ax(t) - Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3.36}$$

The triple (A,B,C) of dimensions respectively $n \times n$, $n \times m$ and $p \times n$ is assumed to be completely controllable and observable. The objective is to approximate the original model (3.36) with a satisfactory reduced model

$$\begin{aligned} \dot{z}(t) &= Fz(t) - Gu(t) \\ v(t) &= Hz(t) \end{aligned} \tag{3.37}$$

A reduced model is said to be "satisfactory" if the error between responses to a given class of inputs $\{u(t)\}$ of the original model output $y(t)$ and the aggregated model output $v(t)$ is negligible.

Note that the aggregated model order r is such that $m \le r \le n$. The relationship between the linear dynamic models (3.36) and (3.37) could be established by a linear transformation such as:

$$z(t) = Lx(t) \tag{3.38}$$

where L is an $r \times n$ constant aggregation matrix of rank r. Using equation (3.38) the equivalence between the models (3.36) and (3.37) is achieved provided that the conditions [58]:

$$\begin{aligned} FL &= LA \\ G &= LB \\ z(0) &= Lx(0) \end{aligned} \tag{3.39}$$

are satisfied. Since the $r \times n$ aggregation matrix L is assumed to be of full rank, it will possess a pseudo-inverse (an inverse in case of a square matrix) and therefore a least square solution for (3.39) of the form

$$F = LAL^{T}[LL^{T}]^{-1} \tag{3.40}$$

We shall emphasize here that the least square solution matrix F obtained above could be an exact or an approximate solution to (3.39), depending respectively on whether or not the aggregation matrix L is square.

Clearly, the basic step in models construction is the formation of the aggregation matrix L. Denoting the n-dimensional row vectors of L by $\{L_i^T\}, i = 1,2,\ldots,r$, we can write L as:

$$L = \left[L_1^T, \ \ldots \ , L_i^T, \ \ldots \ , L_r^T\right]^T \tag{3.41}$$

then

$$z(t) = \left[L_1^T, \ \cdots \ , L_i^T, \ \cdots \ , L_r^T\right]^T x(t) \qquad (3.42)$$

This means that $z_i(t)$ is a linear combination of some components of $x(t)$. With the help of the eigenspectrum of the original system, we select the elements of L such that L has at most one entry in each column, then n components of x can be grouped into at most r separate clusters. In this way, the vectors $\{L_i^T\}, i = 1,2,\ldots,r$, are mutually orthogonal which ensures the maximum rank of L. This procedure constitutes a method for determining the aggregation matrix.

An automatic method to compute the matrix L can be developed by considering the controllability matrices of systems (3.36) and (3.37).

Define

$$\begin{aligned} W_A &= \left[B \quad AB \quad \cdots \quad A^{n-1}B\right] \\ W_F &= \left[G \quad FG \quad \cdots \quad F^{n-1}G\right] \end{aligned} \qquad (3.43)$$

then from (3.39) we have:

$$LW_A = W_F \qquad (3.44)$$

Thus using the pseudo-inverse, L can be obtained as

$$\begin{aligned} L &= W_F W_A^- \\ L &= W_F W_A^T [W_A W_A^T]^{-1} \end{aligned} \qquad (3.45)$$

It is understood that W_A is of full rank n, since the original system is controllable. Thus, by specifying $F = Diag\{\lambda_1,\ldots,\lambda_r\}$ and choosing G so as to have a completely controllable reduced model, $rank(W_F) = r$, then L can be computed by equation (3.45).

In order to illustrate the aggregation procedure, let us consider the following fifth order system [55]:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0.01 & 0.05 & 0.25 \\ 0 & -4 & 0 & 0.45 & 0.1 \\ -0.088 & 0.2 & -10 & 0 & 0.22 \\ 1 & 0 & 0.075 & -4 & 0.05 \\ 0.11 & 0.2 & 0.999 & 0.44 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \\ 0.5 & 0.9 \\ 2 & 0.75 \\ 1 & 1 \end{bmatrix} u(t)$$

The eigenvalues of this system are {-10.03,-0.952,-0.2996,-4.073,-3.95}. A study of this eigenspectrum shows that an aggregate model of order three can be derived by retaining an average of the first and fourth modes, the second mode and an average of the third and fifth modes. This gives the following aggregation matrix L of the form

$$L = \begin{bmatrix} 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \end{bmatrix}$$

Accordingly, the matrices of the aggregated model are given by:

$$F = \begin{bmatrix} -1.975 & 0 & 0.1925 \\ 0.45 & -4 & 0.1 \\ 0.231 & 0.2 & -5.9 \end{bmatrix}$$

$$G = \begin{bmatrix} 1.5 & 0.626 \\ 0 & 1 \\ 0.75 & 0.95 \end{bmatrix}$$

and $H = CL^{-}$ for any given C.

3.2.3 Modal Analysis Approach

Modal analysis approach can be seen as a particular case of the aggregation method. Given a n^{th} order linear, time-invariant system in state space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) - Bu(t) \\ y(t) &= Cx(t) \end{aligned} \qquad (3.46)$$

the intent is to approximate the behaviour of this model by a reduced model of order r. The starting point for modal analysis approach, is the derivation of a modal matrix whose eigenvalues are the same as those of the system matrix. For so doing, we determine a transformation T that will order the eigenvalues of the original model in order of decreasing dominance such that they will be grouped into r and $(n-r)$ sets.

$$x(t) = Tz(t) \qquad (3.47)$$

hence,

$$\dot{z}(t) = (T^{-1}AT)z(t) + T^{-1}Bu(t)$$
$$= \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} u(t) \tag{3.48}$$

and

$$y(t) = CTz(t) \tag{3.49}$$

where $z_1(t)$ corresponds to the dominant components and $z_2(t)$ corresponds to the dominated components. Expanding the above system, yields

$$\begin{aligned} \dot{z}_1(t) &= J_1 z_1(t) + \Gamma_1 u(t) \\ \dot{z}_2(t) &= J_2 z_2(t) + \Gamma_2 u(t) \end{aligned} \tag{3.50}$$

The methods using the modal analysis approach seek a reduced order model described by:

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u(t) \\ y_r(t) &= C_r x_r(t) \end{aligned} \tag{3.51}$$

where $x_r(t) \in R^r$ is a subset of $x(t)$ containing the effective state variables. In the following three basic approaches will be considered. All of them retain the r dominant eigenvalues in the reduced order model.

Expanding equation (3.47), yields:

$$\begin{bmatrix} x_r(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} \tag{3.52}$$

The first method makes use of the fact that J_2 contains only non-dominant eigenvalues resulting in small time constants. Thus, $z_2(t)$ can approximately be replaced by linear combinations of the various inputs. Mathematically, the approximation is equivalent to putting $\dot{z}_2(t) = 0$ into (3.50) yielding:

$$J_2 z_2(t) + \Gamma_2 u(t) = 0 \quad \Leftrightarrow \quad z_2(t) = -J_2^{-1}\Gamma_2 u(t) \tag{3.53}$$

from (3.52), we have

$$x_r(t) = T_1 z_1(t) + T_2 z_2(t) \tag{3.54}$$

or equivalently:

$$z_1(t) = T_1^{-1}[x_r(t) - T_2 z_2(t)] \tag{3.55}$$

replacing (3.55) in (3.50) yields:

$$\dot{z}_2(t) = J_2\left\{T^{-1}[\dot{x}_r(t) - T_2\dot{z}_1(t)]\right\} - \Gamma_2 u(t) \tag{3.56}$$

replacing also (3.53) in (3.56) leads to:

$$\dot{z}_2(t) = J_2 T^{-1}[\dot{x}_r(t) - T_2 J_1^{-1}\Gamma_1 u(t)] - \Gamma_2 u(t) \tag{3.57}$$

By defining an r-dimensional auxiliary vector $w(t) = x_r(t) - T_2 J_1^{-1}\Gamma_1 u(t)$ and differentiating both sides of equation (3.52), we get:

$$\dot{x}_r(t) = T_1\dot{z}_2(t) + T_2\dot{z}_1(t) = T_1\dot{z}_2(t) \tag{3.58}$$

and after some minor manipulations

$$\dot{x}_r(t) = T_1 J_2 T_1^{-1}[x_r(t) - T_2 J_1^{-1}\Gamma_1 u(t)] + T_1\Gamma_2 u(t) \tag{3.59}$$

At this level, several remarks need to be mentioned:

(i) The retained (dominant) modes are excited in the same manner in both the original and the reduced systems.

(ii) The above method gives a correct steady-state error ($w(t) - x_r(t)$). However, this is not always the case for the transient response because of the second term $T_2 J_1^{-1}\Gamma_1 u(t)$. To ensure the correct initial state $w(0) = x_r(0)$ we let:

$$w(0) = x_r(0) - T_2 J_1^{-1}\Gamma_1 u(0) \tag{3.60}$$

Hence

$$w(t) \equiv x_r(t) \tag{3.61}$$

and equation (3.59) reduces to:

$$\dot{w}(t) = T_1 J_2 T_1^{-1} w(t) + T_1\Gamma_2 u(t) \tag{3.62}$$

Then the comparison of (3.51) with (3.62) gives

$$\begin{aligned} A_r &= T_1 J_2 T_1^{-1} \\ B_r &= T_1\Gamma_2 \end{aligned} \tag{3.63}$$

On the other hand, the expression for the unretained states can be derived from equation (3.52):

$$x_1(t) = T_3 z_2(t) + T_4 z_1(t) \tag{3.64}$$

replacing (3.53) and (3.55) in (3.64) yields:

$$\begin{aligned} x_2(t) &= T_3 T_1^{-1}\left[x_r(t) - T_2 z_2(t)\right] - T_4(-J_2^{-1}\Gamma_2 u(t)) \\ &= T_3 T_1^{-1} x_r(t) - (T_3 T_1^{-1} T_2 - T_4) J_2^{-1}\Gamma_2 u(t) \end{aligned} \tag{3.65}$$

A second development of the modal analysis method may be as follows:

Given a system in partitioned state space form

$$\begin{bmatrix} x_r(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_r(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \tag{3.66}$$

then

$$\begin{aligned} x_r(t) &= A_1 x_r(t) + A_2 x_2(t) + B_1 u(t) \\ x_2(t) &= A_3 x_r(t) + A_4 x_2(t) + B_2 u(t) \end{aligned} \tag{3.67}$$

and from (3.52) we get:

$$\begin{aligned} x_2(t) &= T_3 z_2(t) + T_4 z_2(t) \\ z_2(t) &= T_1^{-1} x_r(t) - T_1^{-1} T_2 z_2(t) \end{aligned} \tag{3.68}$$

By setting $z_1(t) = 0$, and replacing it in equation (3.53) yields

$$z_2(t) = -J_2^{-1}\Gamma_2 u(t) \tag{3.69}$$

The substitution of (3.69) and (3.68) into (3.67a) lead to:

$$\begin{aligned} x_r(t) &= A_1 x_r(t) + A_2\left[T_3 z_2(t) + T_4 z_2(t)\right] + B_1 u(t) \\ &= A_1 x_r(t) + A_2 T_3\left[T_1^{-1} x_r(t) - T_1^{-1} T_2 z_2(t)\right] + A_2 T_4 z_2(t) + B_1 u(t) \\ &= (A_1 + A_2 T_3 T_1^{-1}) x_r(t) + (A_2 T_4 - A_2 T_3 T_1^{-1} T_2) z_2(t) + B_1 u(t) \\ &= (A_1 + A_2 T_3 T_1^{-1}) x_r(t) - \left[B_1 - A_2(T_4 - T_3 T_1^{-1} T_2) J_2^{-1}\Gamma_2\right] u(t) \end{aligned} \tag{3.70}$$

In a similar manner, the comparison of (3.51) with (3.70) produces:

$$\begin{aligned} A_r &= A_1 + A_2 T_3 T_1^{-1} \\ B_r &= B_1 - A_2[T_4 - T_3 T_1^{-1} T_2] J_2^{-1}\Gamma_2 \end{aligned} \tag{3.71}$$

It can be shown that, by setting $x_r(0) = w(0)$ in the previous method, the present method gives the correct transient and steady state responses. The retained modes are excited in the same manner as in the original system except if $A_2 = 0$. The expression for the unretained modes is also given by equation (3.65).

A third choice is to completely neglect the effects of the nondominant variables $z_1(t)$ in the reduced model. In other terms the given system is seen as two independent subsystems described by

$$\begin{aligned} x_r(t) &= T_2 z_0(t) \\ x_1(t) &= T_3 z_0(t) \end{aligned} \tag{3.72}$$

substituting these equations in (3.48) with $z_0(t) = T_1^{-1} x_r(t)$, yields

$$\begin{aligned} x_r(t) &= T_1 \{J_0 T_1^{-1} x_r(t) + \Gamma_0 u(t)\} \\ &= T_1 J_0 T_1^{-1} x_r(t) + T_1 \Gamma_0 u(t) \end{aligned} \tag{3.73}$$

Hence by identification with equation (3.51), we have

$$\begin{aligned} A_r &= T_1 J_0 T_1^{-1} \\ B_r &= T_1 \Gamma_0 \end{aligned} \tag{3.74}$$

and the unretained variable $x_1(t)$ is expressed as:

$$x_1(t) = T_3 z_0(t) = T_3 T_1^{-1} x_r(t) \tag{3.75}$$

To demonstrate the use of the three reduced order methods, a fourth order single input single output system is chosen [55]:

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 & 6 & -2 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} x(t)$$

The system eigenvalues are $\{-1,-2,-3,-4\}$ with a unit step response of

$$y(t) = 1 - e^{-t} - e^{-2t} + 2e^{-3t} - e^{-4t}$$

A second order reduced model may be derived by retaining the state variables x_1 and x_2 using (3.62) we have:

$$\dot{w}(t) = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} w(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t)$$

$$x_r(t) = w(t) - \begin{pmatrix} 0.5833 \\ 0.4167 \end{pmatrix} u(t)$$

$$y_r(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} x_r(t)$$

Setting $w(0) = 0$, the unit step response of the reduced model is given by:

$$y_r(t) = 1 - e^{-t} - e^{-2t}$$

As it can be seen, the initial state response is not the same for both models. To overcome this, equation (3.60) is used with $x_r(0)=0$ and $u(0)=1$ to yield

$$w(0)=\begin{pmatrix}0.5833\\0.4167\end{pmatrix}$$

with an output step response of $y_r(t)=1.0833e^{-t}-0.1667e^{-2t}$

Application of the second method leads to:

$$\dot{x}_r(t)=\begin{pmatrix}-1 & -1\\0 & -2\end{pmatrix}x_r(t)+\begin{pmatrix}1\\0.1667\end{pmatrix}u(t)$$

$$y_r(t)=\begin{pmatrix}1 & 1\end{pmatrix}x_r(t)$$

with a unit step response ($x_r(0)=x_0(0)=0$)

$$y_r(t)=1-0.8333e^{-t}-0.1667e^{-2t}$$

which is identical to the result of the previous method.

Other methods using the state space description are available, such as subspace projection methods, optimal order reduction methods. etc... The interested reader may consult references[55, 57].

3.3 Frequency Domain Techniques

In the previous section we have seen model reduction techniques using the state space representation. In this section we will switch to those reduction techniques developed directly from a transfer function form.

The analysis and synthesis of large scale dynamical systems is usually done using models in the state space description form. However, the original model is often identified in the frequency domain and it is subsequently converted to the state space form. It is therefore natural to seek model simplification in the frequency domain itself.

With this goal, we start the analysis in this section by looking at the inter-relationship between state space and frequency domain concepts, then we will tackle the frequency domain techniques. For so doing, consider a linear time-invariant, asymptotically-stable dynamic system with n state variables, m input and q output variables described by the following state space equations:

$$\begin{cases} x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3.76}$$

The corresponding transfer function matrix of this system is

$$G(s) = C(sI - A)^{-1}B \tag{3.77}$$

where s is a complex variable. Expansion of the transfer function matrix (3.77) in a Maclaurin series about infinity ($s = \infty$) yields:

$$G(s) = \sum_{i=0}^{\infty} CA^{i}Bs^{-(i+1)} \tag{3.78}$$

where the quantities

$$M_i = CA^{i}B \qquad i = 0,1,\ldots \tag{3.79}$$

are termed the Markov parameters. On the other hand, expanding equation (3.77) in a Maclaurin series around the origin ($s = 0$) and assuming the system matrix to be nonsingular, yields

$$G(s) = \sum_{i=0}^{\infty} CA^{-(i+1)}Bs^{i} \tag{3.80}$$

where the coefficient matrices $CA^{-(i+1)}B$ are related to time moments H_i by:

$$\begin{aligned} H_i &= (-1)^i i! CA^{-(i+1)}B, \qquad i = 0,1,2,\ldots \\ H_i &= (-1)^i \left.\frac{d^i G(s)}{ds^i}\right|_{s=0} \\ &= (-1)^i i! CA^{-(i+1)}B \end{aligned} \tag{3.81}$$

For a single input single output system ($m = q = 1$), the transfer function is usually represented by

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_k + b_{k-1}s + \cdots + b_2 s^{k-2} + b_1 s^{k-1}}{a_k + a_{k-1}s + \cdots + a_1 s^{k-1} + s^k} \tag{3.82}$$

The mutivariable analog is a $n \times m$ matrix transfer function of the form

$$\begin{aligned} G(s) &= N(s)D^{-1}(s) = C(sI - A)^{-1}B \\ &= [B_k + B_{k-1}s + \cdots + B_2 s^{k-2} + B_1 s^{k-1}][A_k + A_{k-1}s + \cdots + A_1 s^{k-1} + s^k]^{-1} \\ &= \frac{Q_1 s^{l-1} + Q_2 s^{l-2} + \cdots + Q_{l-1}s + Q_l}{s^l + a_1 s^{l-1} + a_2 s^{l-2} + \cdots + a_{l-1}s + a_l} \end{aligned} \tag{3.83}$$

where $l = m \times k$ and $Q_i, i = 1,\ldots,l$, are constant matrices each of dimension $n \times m$. They are related to the triple $\{A,B,C\}$ as follows:

$$G(s) = C(sI - A)^{-1}B$$
$$= C\left[\frac{Adj[sI - A]}{\det(sI - A)}\right]B = \frac{Q_1 s^{l-1} + Q_2 s^{l-2} + \cdots + Q_{l-1}s + Q_l}{s^l + a_1 s^{l-1} + \cdots + a_{l-1}s + a_l} \qquad (3.84)$$

By identification of the coefficients of the same degree, we end up with the following relations:

$$\begin{aligned} Q_1 &= CB \\ Q_2 &= C[A + a_1 I]B \\ Q_3 &= C[A^2 + a_1 A + a_2 I]B \\ &\vdots \\ Q_n &= C[A^{l-1} + a_1 A^{l-2} + \cdots + a_{l-1} I]B \end{aligned} \qquad (3.85)$$

A number of simplification techniques have been developed to construct lower order transfer functions based on different ideas. Among these techniques, we discuss the continued fraction approximation, the time-moments matching method and the Pade-type approximation method.

3.3.1 Continued Fraction Approximation

The expansion of transfer function matrices into a matrix continued fraction and the inversion of a matrix fraction to a transfer function matrix represent two basic operations in multivariable system analysis and synthesis. Before we outline the simplification approach using continued fraction expansion, we will first present some introductory material on matrix continued fraction expansions.

3.3.1.1 The Three Cauer Forms

Consider an asymptotically stable system with a transfer function

$$G(s) = \frac{a_{2,n}s^{n-1} + \cdots + a_{2,3}s^2 + a_{2,2}s + a_{2,1}}{a_{1,n-1}s^n + \cdots + a_{1,3}s^2 + a_{1,2}s + a_{1,1}} \qquad (3.86)$$

where $a_{i,j}$ are constants. In principle, equation (3.86) can be expanded into several continued fraction forms; however, there are three basic forms of particular interest to systems engineering. These are called Cauer forms [61-63].

The first Cauer form obtained by long synthetic division of the above polynomials, or, alternatively by using the Routh's algorithm [13, 62] is:

$$G(s) = \cfrac{1}{h_1 s + \cfrac{1}{h_2 + \cfrac{1}{h_3 s + \cfrac{1}{h_4 + R(s)}}}} \tag{3.87}$$

where $R(s)$ is a residual transfer function.

We can use the same algorithm to derive the second Cauer form by arranging the polynomials of $G(s)$ in ascending order and then perform a long division.

$$G(s) = \cfrac{1}{k_1 + \cfrac{1}{\cfrac{k_2}{s} + \cfrac{1}{k_3 + \cfrac{1}{\cfrac{k_4}{s} + R(s)}}}} \tag{3.88}$$

A third Cauer form can also be developed using the same algorithm by combining the features of the first and second cauer forms, in such away that:

$$G(s) = \cfrac{1}{d_1 + f_1 s + \cfrac{1}{\cfrac{d_2}{s} + f_2 + \cfrac{1}{d_3 + f_3 s + \cfrac{1}{\cfrac{d_4}{s} + f_4 + R(s)}}}} \tag{3.89}$$

The multivarable analogs of the three Cauer forms are also available.

The first matrix Cauer form is:

$$G(s) = \left[H_1 s + \left[H_2 + \left[H_3 s + \left[H_4 + [\cdots]^{-1} \right]^{-1} \right]^{-1} \right]^{-1} \right]^{-1} \tag{3.90}$$

which has $2n$ matrices H_i and it represents a Maclaurin series expansion about $s = \infty$.

The second matrix Cauer form is:

$$G(s) = \left[K_1 + \left[K_2 \frac{1}{s} + \left[K_3 + \left[K_4 \frac{1}{s} + [\cdots]^{-1} \right]^{-1} \right]^{-1} \right]^{-1} \right]^{-1} \tag{3.91}$$

which has also $2n$ quotient matrices K_i and represents a Maclaurin series expansion about $s=0$. Whereas, the third Cauer form is:

$$G(s)=\left[D_1+F_1 s+\left[D_2\frac{1}{s}+F_2+\left[D_3+F_3 s+\left[D_4\frac{1}{s}+F_4+\left[\cdots\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1}\right]^{-1} \quad (3.92)$$

which has n matrices D_i and F_i and is equal to a Maclaurin series expansion about both $s=0$ and $s=\infty$.

Note that H_i, K_i, D_i and F_i are constant $m\times m$ quotient matrices, and any algorithm that computes the quotients D_i and F_i is capable of computing the H_i quotients (by setting all the D_i's equal to zero), whereas the computation of the K_i's quotients can be achieved by suppressing the quotients F_i throughout the implementation.

As the matrix quotients descend lower and lower in position, or equivalently the block develop to more and more inner loops, they have less significance as far as the overall system is concerned. Thus, a reduced order model can be obtained by retaining the first several dominant coefficients and discarding others.

A m^{th} order continued fraction approximation is obtained by means of the approximation $R_m(s)\cong 0$. In other words by truncating the sequence of coefficients after $2m$. The coefficients D_i and F_i of (3.92) are computed using the generalised Routh algorithm.

3.3.1.2 Generalized Routh Algorithm:

The computation of the matrix quotients in the third (mixed) matrix Cauer form is carried out using an algorithmic procedure based on a matrix Routh array. The structure of the Routh algorithm for a multivariable system is defined in such away that is similar to that of a single variable system.

Given

$$G(s)=\left[A_{2,n}s^{n-1}+A_{2,n-1}s^{n-2}+\cdots+A_{2,2}s+A_{2,1}\right]\left[A_{1,n+1}s^{n}+A_{1,n}s^{n-1}+\cdots+A_{1,2}s+A_{1,1}\right]^{-1} \quad (3.93)$$

$$A_{2,1} = \begin{bmatrix} -3 & 2 \\ -2 & -2 \end{bmatrix}, \quad A_{2,2} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A_{1,1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_{1,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{1,3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using the Routh algorithm, we obtain:

$$D_1 = \begin{bmatrix} -0.4 & -0.4 \\ 0.4 & -0.6 \end{bmatrix} \quad \left\langle \begin{matrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix} \right\rangle \quad F_1 = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 1.0 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} -1.152 & 0.364 \\ -0.667 & -0.667 \end{bmatrix} \quad \left\langle \begin{matrix} \begin{bmatrix} -3 & 2 \\ -2 & -2 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 2.7 & -0.6 \\ 0.3 & 3.6 \end{bmatrix} \end{matrix} \right\rangle \quad F_2 = \begin{bmatrix} 0.727 & 0.121 \\ 0.333 & 0.333 \end{bmatrix}$$

The most important features of the continued fraction expansion can be summarized in the following points:

1. It converges faster than other series expansions.
2. It contains most of the essential characteristics of the original model in the first few terms.
3. It does not require any knowledge of the model eigenspectrum.

The only disadvantage with the continued fraction is that the reduced model may be unstable even though the original model is stable [55].

Next, we present model approximation methods in which we identify a set of functions which are characteristic of the original full model and which can be calculated directly without computation of the time or frequency responses. We then match these functions for the simplified transfer function model by a suitable choice of parameters in the latter. The most popular methods of this type are the matching of time-moments of the impulse response [7, 8, 59] and the Pade approximation.

3.3.2 Time-Moments Matching

Consider the n^{th} order transfer function, given by

$$G(s) = \frac{b_n + b_{n-1}s + \cdots + b_2 s^{n-2} + b_1 s^{n-1}}{a_n + a_{n-1}s + \cdots + a_1 s^{n-1} + s^n} \tag{3.99}$$

Using long division, we can expand $G(s)$ in power series as follows:

$$G(s) = C_0 + C_1 s + C_2 s^2 + \cdots \tag{3.100}$$

where the constants C_i's are related to the time moments by the relation

$$C_i = (-1)^i \frac{1}{i!} H_i \tag{3.101}$$

Instead of computing the time moments, H_i's, it is better to calculate the coefficients C_i's using either of the following methods [55]:

(i) Set $C_0 = b_n / a_n$ and for $i > 0$ use

$$C_i = \frac{1}{a_n}\left[b_{n-i} - \sum_{j=1}^{i} a_{n-j} C_{i-j} \right] \tag{3.102}$$

(ii) Construct the array

$$\begin{array}{ccccc} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & \cdots \\ A_{2,1} & A_{2,2} & A_{2,3} & \cdots & \\ A_{3,1} & A_{3,2} & \cdots & & \\ A_{4,1} & \cdots & & & \\ \cdots & & & & \end{array} \tag{3.103}$$

where the subsequent elements are evaluated using the following recursive equation:

$$\begin{aligned} A_{1,j} &= a_{n-1-j}, \qquad A_{2,j} = b_{n-1-j} \\ A_{k,m} &= A_{k-1,1} A_{1,m-1} - A_{k-1,m-1} \\ k &= 3,4,\ldots,2n+1 \\ m &= 1,2,\ldots,n \end{aligned} \tag{3.104}$$

then use

$$\begin{aligned} C_0 &= A_{2,1} / A_{1,1} \\ C_j &= (-1)^j A_{j-2,1} \qquad j = 1,2,\ldots,n \end{aligned} \tag{3.105}$$

As it can be observed, the number of matched time moments determines the order of the reduced model. It can also be observed that the more matching of time moments, the more accurate will be the simplified transfer function.

However, this latter approach is not without defects: the simplified model order will be increased so the usefulness of simplification will be lost.

The inverse problem to the one just solved above, is the derivation of the simplified transfer function with its order and form fixed a priori. Given the simplified transfer function

$$G_r(s)=\frac{B_{21}+B_{22}s+B_{23}s^2+\cdots+B_{2,m-1}s^m}{1+B_{12}s+B_{13}s^2+\cdots+B_{1,r-1}s^r} \tag{3.106}$$

where $r>m$ and the $2r$ time-moments $H_0,\ldots,H_{2r-1}$, determine the unknown constant parameters $B_{12},\ldots,B_{1,r-1},B_{21},\ldots,B_{2,m-1}$.

The above procedure may be summarised in the following steps:

1. Calculate the coefficients C_i from the time moments H_i using (3.101).
2. Put every thing in matrix form:

$$\begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_m \\ \cdots \\ C_{m-1} \\ C_{m-2} \\ \vdots \\ C_{m-r} \end{bmatrix} = \left[\begin{array}{cccc|cccc} 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ -C_0 & 0 & & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ -C_{m-1} & -C_{m-2} & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ \hline & & & & & & & \\ -C_m & -C_{m-1} & \cdots & -C_0 & 0 & \cdots & \cdots & 0 \\ \vdots & & & & & & & \vdots \\ \vdots & & & & & -C_0 & 0 & 0 \\ -C_{m-r-1} & -C_{m-r-2} & \cdots & -C_m & & -C_1 & -C_0 & 0 \end{array}\right] \begin{bmatrix} B_{12} \\ B_{13} \\ \vdots \\ B_{1,r-1} \\ \hline \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{22} \\ \vdots \\ B_{2,m-1} \\ \hline \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

which can be written in compact form as:

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} f_1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_2 \\ 0 \end{bmatrix} \tag{3.107}$$

where

$$\begin{aligned} v_1 &= \begin{bmatrix} C_0 & C_1 & \cdots & C_m \end{bmatrix}^T \\ v_2 &= \begin{bmatrix} C_{m-1} & C_{m-2} & \cdots & C_{m-r} \end{bmatrix}^T \\ f_1 &= \begin{bmatrix} B_{12} & B_{13} & \cdots & B_{1,r-1} \end{bmatrix}^T \\ f_2 &= \begin{bmatrix} B_{21} & B_{22} & \cdots & B_{2,m-1} \end{bmatrix}^T \end{aligned} \tag{3.108}$$

$$W_{11} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ -C_0 & 0 & \cdots & \cdots & 0 \\ -C_1 & -C_0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ -C_{m-1} & -C_{m-2} & \cdots & -C_0 & 0 \end{bmatrix} (m+1) \times r \tag{3.109}$$

$$W_{21} = \begin{bmatrix} -C_m & -C_{m-1} & \cdots & -C_0 & \cdots & 0 \\ -C_{m+1} & -C_m & \cdots & -C_1 & \cdots & 0 \\ \vdots & \vdots & & & & \vdots \\ -C_{m+r-1} & -C_{m+r-2} & \cdots & \cdots & \cdots & -C_m \end{bmatrix} r \times r \tag{3.110}$$

$$W_{22} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ -C_0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & -C_0 & 0 & 0 \\ \cdots & \cdots & -C_1 & -C_0 & 0 \end{bmatrix} r \times (m+1) \tag{3.111}$$

$$W_{12} = [0](m+1) \times (m+1) \tag{3.112}$$

Then the unknown parameters are given by

$$f_1 = \begin{bmatrix} B_{12} \\ \vdots \\ B_{1,r-1} \end{bmatrix} = W_{21}^{-1} V_2 \tag{3.113}$$

$$f_2 = \begin{bmatrix} B_{21} \\ \vdots \\ B_{2,m-1} \end{bmatrix} = V_1 - W_{11} W_{21}^{-1} V_2 \tag{3.114}$$

It is assumed that the matrix W_{21} is nonsingular. However, if it happens to be singular then this means that the given set of moments can be matched with a simpler transfer function.

As an illustration, consider the fifth order transfer function [55]:

$$G(s) = \frac{2 + 3s + s^2 + 2s^3 + s^4}{1 + s + 2s^2 + s^3 + 2s^4 + 3s^5}$$

It is required to derive a simplified transfer function that matches the first six time-moments. This means that we are seeking a third order transfer function of the form:

$$G_3(s) = \frac{B_{21} + B_{22}s + B_{23}s^2}{1 + B_{12}s + B_{13}s^2 + B_{14}s^3}$$

with $r = 3$ and $m = 2$. Using the second method of calculating the time-moments from the array

1	1	2	1	2	3
2	3	1	2	1	0
−1	3	0	3	6	
−4	−2	−4	−8		
−2	−4	4			
2	−8				
10					

the coefficients C_{is} are obtained as:

$$C_0 = 2 \qquad C_1 = 1 \qquad C_2 = -4$$
$$C_3 = 2 \qquad C_4 = 2 \qquad C_5 = -10$$

The unknown parameters are computed from the following equations:

$$\begin{bmatrix} B_{12} \\ B_{13} \\ B_{14} \end{bmatrix} = \begin{bmatrix} 4 & -1 & -2 \\ -2 & 4 & -1 \\ -2 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \\ -10 \end{bmatrix} = \begin{bmatrix} -2.0909 \\ -1.6364 \\ -4.3636 \end{bmatrix}$$

$$\begin{bmatrix} B_{21} \\ B_{22} \\ B_{23} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 & -2 \\ -2 & 4 & -1 \\ -2 & -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2 \\ -10 \end{bmatrix} = \begin{bmatrix} 2 \\ -3.1818 \\ -9.3636 \end{bmatrix}$$

so that

$$G_3(s) = \frac{2 - 3.1818s - 9.3636s^2}{1 - 2.0909s - 1.6364s^2 - 5.818s^3}$$

3.3.3 Pade-Type Approximants

A Pade approximant [55], if it exists, is a unique rational function $[A_m(s)]/[B_n(s)]$, where $A_m(s)$ and $B_n(s)$ are polynomials in s of degrees m and n respectively. It is denoted symbolically by $P_{m,n}(G,s)$. The rational function $P_{m,n}(G,s)$ is said to be a Pade approximant of the function $G(s)$ if and only if the power series expansion of $P_{m,n}(G,s)$ is identical to that of

$G(s)$ up to and including terms of order s^{m-n}. To put this formal definition into a mathematical framework, let

$$G(s) = \frac{p_k + p_{k-1}s + \cdots + p_1 s^{k-1}}{d_k + d_{k-1}s + \cdots + d_0 s^k} \tag{3.115}$$

This transfer function can be expanded into the power series polynomial

$$G(s) = \sum_{j=0}^{\infty} C_j s^j \tag{3.116}$$

Let

$$\begin{aligned} P_{mn}(G,s) &= A_m(s) \Big/ B_n(s) \\ &= \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_m s^m}{b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n} \end{aligned} \tag{3.117}$$

be the $P_{m,n}(G,s)$ Pade approximant of $G(s)$. Then it follows from the above discussion that when the transfer function $G(s)$ and the $P_{m,n}(G,s)$ are expanded in their Maclaurin series,

$$G(s) - P_{mn}(G,s) = \sum_{j=m-n-1}^{\infty} e_j s^j \tag{3.118}$$

substitution of (3.117) in equation (3.118) yields:

$$G(s) - \frac{A_m(s)}{B_n(s)} = \sum_{j=m-n-1}^{\infty} e_j s^j \tag{3.119}$$

and rearranging the equations results in

$$B_n(s)G(s) - A_m(s) = \sum_{j=m-n-1}^{\infty} r_j s^j \tag{3.120}$$

Using (3.116) and (3.117) in (3.120) and carrying out the algebraic manipulations, we arrive at:

$$\begin{aligned} &b_0 C_0 + (b_0 C_1 + b_1 C_0)s + (b_0 C_2 + b_1 C_1 + b_2 C_0)s^2 + \cdots \\ &\quad \cdots - (a_0 + a_1 s + \cdots + a_m s^m) \\ &= r_{m-n-1} s^{m-n-1} + r_{m-n-2} s^{m-n-2} + \cdots \end{aligned} \tag{3.121}$$

Identification of terms of equal powers up to and including $(m+n)$ of both sides of the power series (3.121), leads to the following equations [59, 60].

$$\begin{aligned} b_0 &= 1 \quad , \qquad a_0 = C_0 \\ v &= Vy \\ Wx &= w \end{aligned} \tag{3.122}$$

where

$$v = \begin{bmatrix} a_1 - C_1 \\ a_2 - C_2 \\ \vdots \\ a_m - C_m \end{bmatrix} \tag{3.123}$$

$$V = \begin{bmatrix} C_0 & 0 & \cdots & \cdots & 0 \\ C_1 & C_0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ C_{m-1} & C_{m-2} & C_{m-3} & \cdots & C_0 \end{bmatrix} (m \times m) \tag{3.124}$$

$$y = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{3.125}$$

$$W = \begin{bmatrix} C_m & C_{m-1} & \cdots & C_0 & 0 & \cdots & 0 \\ C_{m-1} & C_m & \cdots & C_1 & C_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_{m-n-1} & C_{m-n-2} & \cdots & C_n & C_{n-1} & \cdots & C_m \end{bmatrix} (n \times n) \tag{3.126}$$

$$x = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad w = \begin{bmatrix} -C_{m-1} \\ -C_{m-2} \\ \vdots \\ -C_{m-n} \end{bmatrix} \tag{3.127}$$

The computation of $P_{m,n}(G,s)$ can now be done by the numerical solution of the linear equations (3.122), (3.123), (3.124), (3.125), (3.126), and (3.127).

Given

$$G(s) = \frac{b_n + b_{n-1}s + b_{n-2}s^2 + \cdots + b_1 s^{n-1}}{a_n + a_{n-1}s + a_{n-2}s^2 + \cdots + a_0 s^n} \tag{3.128}$$

where a_i and b_i are real numbers, we expand $G(s)$ into a Maclaurin series

$$G(s) = \sum_{i=0}^{\infty} \frac{G^{(i)}(0)}{i!} s^i = C_0 + C_1 s + C_2 s^2 + \cdots \qquad (3.129)$$

Multiplying both sides of equations (3.128) and (3.129) by $a_n + a_{n-1}s + a_{n-2}s^2 + \cdots$ and equating coefficients of like powers, leads to:

$$C_j = b_{n-j} - \sum_{k=0}^{j-1} C_k a_{n-j-k}, \qquad j = 1,2,\ldots \qquad (3.130)$$

with

$$a_n = 1, \qquad C_0 = b_n$$

Equations (3.130) constitute a simple means for the computation of the coefficients C_i's from the parameters of the transfer function. Next, consider the simplified transfer function $G_r(s)$ of the form:

$$G_r(s) = \frac{d_m + d_{m-1}s + d_{m-2}s^2 + \cdots + d_1 s^{m-1}}{e_r + e_{r-1}s + e_{r-2}s^2 + \cdots + e_0 s^r}; \qquad m \le r+1 \qquad (3.131)$$

which is the $P_{m,r}(G,s)$ Pade approximant.

To compute the unknown parameters, d_i's and l_i's, the set of equations developed previously may be used.

In the following we examine the important features of Pade-type approximants:

(i) The above procedure is simple and easily programmed on a digital computer.

(ii) The degrees of the numerator and denominator of the approximate model $P_{m,r}(G,s)$ are not restricted.

(iii) The Pade-type approximants are more accurate approximations than the time-moments-matching approximants since it fits the initial $(m+r)$ time moments of the original transfer function $G(s)$.

The weak point with the frequency domain model reduction methods presented above, is the fact that even if the original model is stable they produce unstable simplified models [55]. To overcome this problem many methodologies ensuring the stability of the simplified model beforehand have

been proposed and the interested reader may consult references [10, 11, 12, 13, ...etc].

The Extension of the single input single output systems Pade-type approximation results to the simplification of high order systems of equal inputs and outputs is relatively simple. To illustrate that, consider the following MIMO system described by:

$$\begin{aligned} G(s) &= C[sI_n - A]^{-1}B \\ &= \sum_{i=0}^{\infty} C_i s^i \end{aligned} \tag{3.132}$$

where the C_i's are constant $(p \times p)$ matrices.

Let the reduced order transfer function be of the form:

$$G_r(s) = D_r^{-1}(s)N_r(s) \tag{3.133}$$

where

$$\begin{aligned} D_r(s) &= \left[D_0 + D_1 s + \cdots + D_{r-1}s^{r-1} + I_r s^r\right] \\ N_r(s) &= \left[N_0 + N_1 s + \cdots + N_q s^q\right] \end{aligned} \tag{3.134}$$

and the D_i's and N_i's are constant $(r \times r)$ matrices. In order for $G_r(s)$ to be $P_{q,r}(G,s)$ matrix Pade type approximant, its power series expansion, about $s = 0$, must agree with that of $G(s)$ up to and including the term in s^{r-q}.

By equating coefficients of the same power, we have:

$$\begin{aligned} N_0 &= D_0 C_0 \\ N_1 &= D_0 C_1 + D_1 C_0 \\ &\cdots\cdots\cdots\cdots \\ N_q &= D_0 C_q + D_1 C_{q-1} + \cdots + D_q C_0 \\ 0 &= D_0 C_{q+1} + D_1 C_q + \cdots + D_{q+1} C_0 \\ &\cdots\cdots\cdots\cdots \\ 0 &= D_0 C_r + D_1 C_{r-1} + \cdots + D_{r-1} C_1 + C_0 \\ &\cdots\cdots\cdots\cdots \\ 0 &= D_0 C_{r-q} + D_1 C_{r-q-1} + \cdots + C_r \end{aligned} \tag{3.135}$$

The numerical solution of (3.135) gives the D_i's and N_i's uniquely.

Finally, we consider the following example to show the use of the Pade approximation in model simplification [55].

Let

$$G(s)=\begin{bmatrix} \dfrac{2s+10}{s^2+11s+10} & \dfrac{s+4}{s^2+7s+10} \\ \dfrac{s+10}{s^2+21s+20} & \dfrac{s+6}{s^2+5s+6} \end{bmatrix}$$

It is required to derive a Pade approximant that fits the first four time moments. Expanding $G(s)$ in power series about $s=0$, produces

$$G(s)=C_0+C_1s+C_2s^2+C_3s^3+\cdots$$

where the coefficient matrices, rounded to three decimals are:

$$C_0=\begin{bmatrix} 1 & 0.4 \\ 0.5 & 1 \end{bmatrix},\qquad C_1=\begin{bmatrix} -0.9 & -0.18 \\ -0.475 & -0.667 \end{bmatrix},$$

$$C_2=\begin{bmatrix} 0.89 & 0.086 \\ 0.474 & 0.389 \end{bmatrix},\qquad C_3=\begin{bmatrix} -0.889 & -0.042 \\ -0.474 & -0.213 \end{bmatrix}.$$

Since we need to match only four time-moments, the reduced model will look as follows:

$$G_2(s)=\left[D_0+D_1s+I_2s^2\right]^{-1}\left[N_0+N_1s\right]$$

and the application of (3.135) yields:

$$\begin{aligned} N_0 &= D_0C_0 \\ N_1 &= D_0C_1+D_1C_0 \\ 0 &= D_0C_2+D_1C_1+C_0 \\ 0 &= D_0C_3+D_1C_2+C_1 \end{aligned}$$

The resolution of the last two equations, lead to

$$D_0=\begin{bmatrix} 6.845 & 5.489 \\ -0.708 & 7.197 \end{bmatrix},\qquad D_1=\begin{bmatrix} 9.666 & 2.074 \\ 0.917 & 5.648 \end{bmatrix}$$

Using D_0, D_1, C_0 and C_1, the first two equations yield:

$$N_0=\begin{bmatrix} 9.59 & 8.227 \\ 2.893 & 6.915 \end{bmatrix},\qquad N_1=\begin{bmatrix} 1.935 & 1.049 \\ 0.958 & 1.345 \end{bmatrix}$$

Hence, the reduced model is:

$$G_2(s)=\frac{\begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix}}{\Delta_2(s)}$$

where

$$M_{11}(s) = 1.935s^3 + 18.533s^2 + 56.835s + 53.137$$
$$M_{12}(s) = 1.049s^3 + 11.363s^2 + 32.298s + 21.255$$
$$M_{21}(s) = 0.958s^3 + 10.379s^2 + 27.089s + 26.568$$
$$M_{22}(s) = 1.345s^3 + 18.948s^2 + 69.233s + 53.137$$

and

$$\Delta_2(s) = s^4 + 15.315s^3 + 67.723s^2 + 104.658s + 53.137$$

This ends up the presentation of the selected model reduction methods. Other methods and modified techniques are also available [10, 11, 12, 13, 21, 15, 16, ...etc]. Each method has its own features, merits and drawbacks.

In the next chapter we will tackle the heart of this thesis, represented by our contribution to model reduction of large scale systems.

Chapter 4

Proposed Methods

We have seen in the previous chapter some of the most popular model reduction procedures and pointed out some of their advantages and disadvantages. In the following, we try to bring attention to the potential of using matrix polynomials in the resolution of model reduction and approximation problems. To this end, this chapter is devoted to our contribution in model reduction and approximation. This contribution consists of the elaboration of two model reduction procedures.

The first procedure is developed from a block state-space description, based on the concept of dominance between a complete set of solvents of the characteristic matrix polynomial of a multivariable system in the form of a matrix fraction description. Whereas the second method, is based on the dominant spectral factors of this same characteristic matrix polynomial, and it is developed directly from the matrix transfer function description.

When dealing with high dimensional systems, the simplification problem becomes more complex and the coefficient matrices are much more difficult to handle. The usual SISO methods are no more applicable, but still a rule of thumb when searching for reduced models, is to take care of the following three tasks: selection of suitable coordinates, selection of the truncation criterion, and the model reduction approach.

4.1 Selection of Suitable Coordinates

The first task in any component truncation process is the selection of the coordinates in which coordinate truncation will be performed. Labelling these coordinates as $X_i, i = 1,2,\ldots,n$, the complete system can be described by a set of differential equations such as:

$$\begin{aligned} X_i &= A_i X_i + B_i U \\ Y_i &= C_i X_i + D_i U \qquad\qquad i = 1,2,\ldots,n \end{aligned} \tag{4.1}$$

where U is an m vector of control inputs, the vector X_i represents the block state of the i^{th} block component, and the output vector Y is defined as the p-vector whose norm is of interest in the modelling of control problems.

The system components X_i might be chosen such that the component model (4.1) describes a physical component or a mathematical component. The choice depends upon whether one wishes to analyse the effect of the deletion of a physical component as in failure mode analysis or the deletion of a mathematical component as in model reduction.

In problems where knowledge of the internal behaviour of the model is not needed, it is possible, as we have seen in the previous chapter, to develop model reduction methods directly from the input output relations. For SISO linear time-invariant systems, this corresponds to the well known transfer function representation, relating the Laplace transforms of the input and the output by an expression such as:

$$Y(s) = h(s)U(s) = \frac{n(s)}{d(s)}U(s) \tag{4.2}$$

where $n(s)$ and $d(s)$ are scalar polynomials.

In case of MIMO systems, this corresponds to the transfer matrix fraction description form (MFD)

$$\begin{aligned} Y(s) = H(s)U(s) &= N_R(s)D_R^{-1}(s)U(s) \\ &= D_L^{-1}(s)N_L(s)U(s) \end{aligned} \tag{4.3}$$

where $N_R(s), N_L(s), D_R(s)$, and $D_L(s)$ are matrix polynomials.

4.2 Selection of a Truncation Criterion

The second task of importance in model reduction and approximation is the truncation criterion. It is used to measure the potential closeness of the reduced model to the original one.

Up to now, no general results are yet available on the issue of 'best' component truncation. However, and in opposition to results from singular perturbation or classical dominant pole methods, it is well known that in order to get a satisfactory approximation of a model, it may not be appropriate to simply drop the fast modes. Robert E. Skelton [64], using cost decomposition analysis, has shown that retaining the 'fast' modes in the reduced model construction yields better models than retaining the 'slow' modes.

On the other hand, recent work has shown that the problem of component choice can be better solved with some criterion that also takes into account the effect of the state space description matrices B and C on the system behaviour. In this respect, Zeiger and Mc Ewen [21] developed a method that applies singular value decomposition to the Hankel matrix, which decomposes the matrix into orthogonal components ordered according to the singular values magnitude. The components associated with small singular values are then treated as 'perturbational' and removed by setting the corresponding singular values to zero.

However, the best achieved result is the one published by Moore [1], based on Kalman's canonical form [20]. Moore (see chapter 3 section 3.2.1) introduced the notion of 'principal component analysis' on the so called controllability and observability Gramians. It was shown that there exists an internally k^{th} dominant subsystem if and only if:

$$\left(\sum_{i=1}^{k}\sigma_i^4\right)^{1/2} >> \left(\sum_{j=k+1}^{n}\sigma_i^4\right)^{1/2} \qquad (4.4)$$

where $\sigma_i^2, i = 1,2,\ldots,n$, are the second order modes of the original model

In the present work the idea of second order modes dominance is kept, but expressed differently using Hankel matrices.

Let the original model transfer function be described by:

$$\begin{aligned} H(s) &= n_R(s)d_R^{-1}(s) \\ &= \left[n_1 s^{l-1} + n_2 s^{l-2} + \cdots + n_l\right]\left[d_0 s^l + d_1 s^{l-1} + \cdots + d_l\right]^{-1} \\ &= h(1)s^{-1} + h(2)s^{-2} + h(3)s^{-3} + h(4)s^{-4} + h(5)s^{-5} + h(6)s^{-6} + \cdots \end{aligned} \tag{4.5}$$

where $n_i, i = 1,2,\ldots,l$; and $d_i, i = 0,1,\ldots,l$, are constants, while the $h(i), i = 1,2,3,\ldots$ are the markov parameters of the transfer function $H(s)$. The corresponding Hankel matrix is given by

$$\Gamma\{H(s)\} = \begin{bmatrix} h(1) & h(2) & h(3) & \cdots & h(l) \\ h(2) & h(3) & h(4) & \cdots & h(l+1) \\ h(3) & h(4) & h(5) & \cdots & h(l+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h(l) & h(l+1) & h(l+2) & \cdots & h(2l-1) \end{bmatrix} \tag{4.6}$$

The singular value decomposition of the above Hankel matrix is given by

$$SVD\{\Gamma\{H(s)\}\} = U * Diag\{\sigma_i^2, i = 1,2,\ldots,l\} * V \tag{4.7}$$

where U and V are orthogonal matrices and $\sigma_i^2, i = 1,2,\ldots,l$ are the second order modes of the original model.

Assume that a reduced model $\hat{H}(s)$ of order k $(k < l)$ is required, and let it be of the form

$$\begin{aligned} \hat{H}(s) &= \left[\hat{n}_1 s^{k-1} + \hat{n}_2 s^{k-2} + \cdots + \hat{n}_k\right]\left[\hat{d}_0 s^k + \hat{d}_1 s^{k-1} + \cdots + \hat{d}_k\right]^{-1} \\ &= \hat{h}(1)s^{-1} + \hat{h}(2)s^{-2} + \hat{h}(3)s^{-3} + \hat{h}(4)s^{-4} + \hat{h}(5)s^{-5} + \hat{h}(6)s^{-6} + \cdots \end{aligned} \tag{4.8}$$

The corresponding Hankel matrix is then given by

$$\Gamma\{\hat{H}(s)\} = \begin{bmatrix} \hat{h}(1) & \hat{h}(2) & \hat{h}(3) & \cdots & \hat{h}(l) \\ \hat{h}(2) & \hat{h}(3) & \hat{h}(4) & \cdots & \hat{h}(l+1) \\ \hat{h}(3) & \hat{h}(4) & \hat{h}(5) & \cdots & \hat{h}(l+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{h}(l) & \hat{h}(l+1) & \hat{h}(l+2) & \cdots & \hat{h}(2l-1) \end{bmatrix} \tag{4.9}$$

A reduced model will be considered to be 'good', if the sum of the Hankel principal components of the neglected part of the system, is "small" compared to the sum of the Hankel principal components of the whole system.

The principal components of the neglected part of the system are the singular values of the Hankel matrix difference $\{\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}\}$ given by

$$SVD\{\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}\} = \overline{U} * Diag\{\overline{\sigma}_i^2, i = 1,2,\ldots,l\} * \overline{V} \qquad (4.10)$$

where $\overline{U}$ and $\overline{V}$ are orthogonal matrices.

Therefore, we shall say that a k^{th} order reduced model is the dominant subsystem of an l^{th} order system if and only if:

$$\left(\sum_{i=1}^{l} \sigma_i^4\right)^{1/2} >> \left(\sum_{j=1}^{l} \overline{\sigma}_j^4\right)^{1/2} \qquad (4.11)$$

or

$$\left(\frac{\left(\sum_{j=1}^{l} \overline{\sigma}_j^4\right)^{1/2}}{\left(\sum_{i=1}^{l} \sigma_i^4\right)^{1/2}}\right) < RE \qquad (4.12)$$

where $\sigma_i^2, i = 1,2,\ldots,l$, and $\overline{\sigma}_j^2, j = 1,2,\ldots,l$, are respectively the Hankel matrix singular values of the original model and the Hankel matrix singular values of the least significant subsystem with some zero's added, while RE is the allowed relative error.

4.3 Solvents Based Model Reduction Approach

In this section, we consider the first proposed model reduction procedure. The method retains the advantages of the concept of dominance between a complete set of solvents of the characteristic matrix polynomial, and the block decoupling of matrix polynomial using the Vandermonde similarity transformation.

4.3.1 Problem Formulation

Consider a stable, proper, and linear time-invariant multivariable system

$$H(s) = C(sI - A)^{-1}B \tag{4.13}$$

which can be represented in block state space form as:

$$\begin{aligned} \dot{X}(t) &= AX(t) + BU(t) \\ Y(t) &= CX(t) \end{aligned} \tag{4.14}$$

where $X(t)$ is the block state vector of the high order system,

$U(t)$ is the input vector,

$Y(t)$ is the output vector of the high order system.

The matrices (A,B,C) are constant matrices with appropriate dimensions. For the same class of inputs, we seek a reduced order model $\hat{H}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ whose state space representation is given by:

$$\begin{aligned} \dot{\hat{X}}(t) &= \hat{A}\hat{X}(t) + \hat{B}U(t) \\ \hat{Y}(t) &= \hat{C}\hat{X}(t) \end{aligned} \tag{4.15}$$

where $\hat{X}(t)$ is the block state vector of the reduced order system,

$\hat{Y}(t)$ is the output vector of the reduced order system.

The objective is to compute the triple $(\hat{A},\hat{B},\hat{C})$ such that the impulse response error $[H(t) - \hat{H}(t)]$ is minimum.

4.3.2 Theoretical Development

Consider a strictly proper $n \times m$ transfer function matrix in the form of a matrix fraction description

$$H(s) = D_l^{-1}(s)N_l(s) \quad \text{[LMFD: Left Matrix Fraction Description]} \tag{4.16}$$

or,

$$H(s) = N_R(s)D_R^{-1}(s) \quad \text{[RMFD: Right Matrix Fraction Description]} \tag{4.17}$$

$$= \left[N_1 s^{l-1} + N_2 s^{l-2} + \cdots + N_l\right]\left[D_0 s^l + D_1 s^{l-1} + \cdots + D_l\right]^{-1} \tag{4.18}$$

where $N_i, i = 1,2,\ldots,l$; and $D_i, i = 0,1,\ldots,l$, are respectively $n \times m$ and $m \times m$ constant matrices, with D_0 being the identity matrix.

Assume that $D_R(s)$ has a complete set of solvents $\{R_i, i = 1,2,\ldots,l\}$, meaning that they have disjoined spectra, while covering all the spectra of $D_R(s)$.

$$\begin{gathered} \left(\sigma(R_i)\right) \cap \left(\sigma(R_j)\right) = \varnothing \quad , \quad i \neq j \\ i = 1,2,\ldots,l \\ j = 1,2,\ldots,l \\ \bigcup \sigma(R_i) = \sigma\{D_R(s)\} \end{gathered} \tag{4.19}$$

Let us assume that the solvents $R_i, i = 1,2,\ldots,l$, are $m \times m$ real matrices satisfying

$$R_1 > R_2 > R_3 > \cdots > R_{l-1} > R_l \tag{4.20}$$

where > expresses dominance.

The following definition specifies the concept of dominance among matrices.

Definition 4.1: *A square matrix A is said to dominate another square matrix B if all the eigenvalues of A are greater, in modulus, than those of B.*

Writting the above system in block companion form yields:

$$A_c = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -D_l & -D_{l-1} & -D_{l-2} & \cdots & -D_1 \end{bmatrix} (lxm)x(lxm)$$

$$B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix} (lxm)xm \qquad C_c = \begin{bmatrix} N_l & N_{l-1} & N_{l-2} & \cdots & N_1 \end{bmatrix} nx(lxm) \tag{4.21}$$

The above system, may be written in compact form as:

$$\begin{aligned} X_c &= A_c X_c + B_c U \\ Y_c &= C_c X_c \end{aligned} \tag{4.22}$$

The idea, is to transform this system into a block diagonal form with the solvents in the main diagonal in order of decreasing dominance. One way of doing this is through the use of the Vandermande similarity transformation defined in chapter 2 as:

$$V_R = \begin{bmatrix} I & I & I & \cdots & I \\ R_1 & R_2 & R_3 & \cdots & R_l \\ R_1^2 & R_2^2 & R_3^2 & \cdots & R_l^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_1^{l-1} & R_2^{l-1} & R_3^{l-1} & \cdots & R_l^{l-1} \end{bmatrix}_{(mxl)x(mxl)} \tag{4.23}$$

This transformation changes the coordinates system as follows:

$$X_c = V_R X_R \Leftrightarrow X_R = V_R^{-1} X_c \tag{4.24}$$

Differentiating both sides of the above equation produces

$$X_R = V_R^{-1} X_c \tag{4.25}$$

and replacing (4.24) in (4.25) yields

$$\begin{aligned} X_R &= V_R^{-1}(A_c X_c + B_c U) \\ X_R &= (V_R^{-1} A_c V_R) X_R + (V_R^{-1} B_c) U \end{aligned} \tag{4.26}$$

and

$$Y = C_c X_c = (C_c V_R) X_R \tag{4.27}$$

Hence, the new coordinate system matrices are:

$$\begin{aligned} A_R &= V_R^{-1} A_c V_R \\ B_R &= V_R^{-1} B_c \\ C_R &= C_c V_R \end{aligned} \tag{4.28}$$

so that, the system may be written in expanded block form as:

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_l \end{bmatrix} = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_l \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_l \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_l \end{bmatrix} U \tag{4.29}$$

$$Y = \begin{bmatrix} C_1 & C_2 & \cdots & C_l \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_l \end{bmatrix}$$

As it can be seen, this is a block decoupled system. Thus, it can be decomposed into l independent subsystems

$$\begin{aligned} X_1 &= R_1 X_1 + B_1 U \Rightarrow X_1(s) = (sI - R_1)^{-1} B_1 U(s) \\ X_2 &= R_2 X_2 + B_2 U \Rightarrow X_2(s) = (sI - R_2)^{-1} B_2 U(s) \\ &\vdots \\ X_l &= R_l X_l + B_l U \Rightarrow X_l(s) = (sI - R_l)^{-1} B_l U(s) \end{aligned} \tag{4.30}$$

and the output can be written as a sum of l independent subsystems as follows:

$$\begin{aligned} Y(s) &= [C_1 X_1(s) + C_2 X_2(s) + \cdots + C_l X_l(s)] U(s) \\ &= [C_1 (sI - R_1)^{-1} B_1 + C_2 (sI - R_2)^{-1} B_2 \cdots + C_l (sI - R_l)^{-1} B_l] U(s) \end{aligned} \tag{4.31}$$

Clearly, the problem of approximating the l^{th} order model $H(s)$ with a reduced k^{th} order model $\hat{H}(s)$ satisfying the truncation criterion, reduces to truncate the system at the k^{th} subsystem as shown below:

$$\begin{aligned} \hat{Y}(s) &= [C_1 (sI - R_1)^{-1} B_1 + C_2 (sI - R_2)^{-1} B_2 + \cdots + C_k (sI - R_k)^{-1} B_k] U(s) \\ &= C_A (sI - A_A) B_A U(s) \end{aligned} \tag{4.32}$$

and the triple $\{A_A, B_A, C_A\}$ will have the following form:

$$A_A = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_k \end{bmatrix}_{(mxk)x(mxk)} \quad B_A = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix}_{(mxk)xm}$$

$$C_A = \begin{bmatrix} C_1 & C_2 & \cdots & C_k \end{bmatrix}_{nx(mxk)} \tag{4.33}$$

At this stage, we have obtained a reduced model at the subsystem level satisfying the following truncation criterion:

From the above analysis, the triple (A_R, B_R, C_R) can be split into two subsystems, a dominant one (A_A, B_A, C_A) and a dominated one (A_E, B_E, C_E).

$$A_R = \begin{bmatrix} A_A & 0 \\ 0 & A_E \end{bmatrix} \qquad B_R = \begin{bmatrix} B_A \\ B_E \end{bmatrix} \qquad C_R = \begin{bmatrix} C_A & C_E \end{bmatrix} \tag{4.34}$$

hence,

$$H(s) = C_R (sI - A_R)^{-1} B_R = H_1 s^{-1} + H_2 s^{-2} + H_3 s^{-3} + H_4 s^{-4} + \cdots \tag{4.35}$$

where

$$\begin{aligned} H_1 &= C_R B_R \\ H_2 &= C_R A_R B_R \\ H_3 &= C_R A_R^2 B_R \\ &\vdots \end{aligned} \tag{4.36}$$

and the corresponding Hankel matrix is given by

$$\Gamma\{H(s)\} = \begin{bmatrix} C_R B_R & C_R A_R B_R & \cdots & C_R A_R^{l-1} B_R \\ C_R A_R B_R & C_R A_R^2 B_R & \cdots & C_R A_R^{l} B_R \\ \vdots & \vdots & \ddots & \vdots \\ C_R A_R^{l-1} B_R & C_R A_R^{l} B_R & \cdots & C_R A_R^{2l-2} B_R \end{bmatrix}$$

$$\Gamma\{H(s)\} = \begin{bmatrix} C_A B_A & C_A A_A B_A & \cdots & C_A A_A^{l-1} B_A \\ C_A A_A B_A & C_A A_A^2 B_A & \cdots & C_A A_A^{l} B_A \\ \vdots & \vdots & \ddots & \vdots \\ C_A A_A^{l-1} B_A & C_A A_A^{l} B_A & \cdots & C_A A_A^{2l-2} B_A \end{bmatrix} + \begin{bmatrix} C_E B_E & C_E A_E B_E & \cdots & C_E A_E^{l-1} B \\ C_E A_E B_E & C_E A_E^2 B_E & \cdots & C_E A_E^{l} B \\ \vdots & \vdots & \ddots & \vdots \\ C_E A_E^{l-1} B_E & C_E A_E^{l} B_E & \cdots & C_E A_E^{2l-2} B \end{bmatrix}$$

$$\Gamma\{H(s)\} = \Gamma\{\hat{H}(s)\} + \Gamma\{H_\varepsilon(s)\}. \tag{4.37}$$

The relative error is measured by

$$\left(\frac{\left(\sum_{i=1}^{ml} \bar{\sigma}_i^4 \right)^{1/2}}{\left(\sum_{i=1}^{ml} \sigma_i^4 \right)^{1/2}} \right) < RE \tag{4.38}$$

where $\sigma_i^2, i = 1,2,\ldots,ml$ and $\bar{\sigma}_i^2, i = 1,2,\ldots,ml$ are respectively the singular values of $\Gamma\{H(s)\}$ and $\Gamma\{H_\varepsilon(s)\}$.

If it is required to tune the obtained approximation to the eigenvalue level, the diagonalisation of the last added subsystem $C_k(sI - R_k)^{-1}B_k$ seems to be well suited.

After diagonalisation, the triple $\{C_A, A_A, B_A\}$ will take the following form:

$$A_{AD} = \begin{bmatrix} R_1 & 0_m & \cdots & 0_m & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0_m & 0_m & \cdots & R_{k-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_{k1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_{km} \end{bmatrix} (m \times k)(m \times k)$$

$$C_{AD} = \begin{bmatrix} C_1 & C_2 & \cdots & C_{k-1} & c_{k1} & \cdots & c_{k2} \end{bmatrix} (n)(m \times k) \qquad (4.39)$$

$$B_{AD} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k-1} \\ b_{k1} \\ \vdots \\ b_{km} \end{bmatrix} (m \times k)(m)$$

where $\lambda_{ki}, i = 1,2,\ldots,m$, are the eigenvalues of the k^{th} solvent R_k and $b_{ki}, c_{ki}, i = 1,2,\ldots,m$, are respectively the transformed rows and columns of the matrices B_k and C_k where 0_m stands for the $m \times m$ zero matrix.

Again, at this stage, the relative error criterion satisfaction is tested for reduced models of order $((k-1)\times m + p) \times ((k-1)\times m + p)$, $p = 1,2,\ldots,m$. In case the relative error is satisfied for $p < m$, the system is truncated one more time at the $((k-1)\times m + p)$ component; Otherwise if, $p = m$, the subsystem $C_k(sI - R_k)^{-1}B_k$ is kept as it is.

The obtained tuned reduced model is then of the following form:

$$A_T = \begin{bmatrix} R_1 & 0_m & \cdots & 0_m & 0 & \cdots & 0 \\ 0_m & R_2 & \cdots & 0_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0_m & 0_m & \cdots & R_{k-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_{k1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_{kp} \end{bmatrix} ((k-1)\times m + p)((k-1)\times m - p)$$

$$C_T = \begin{bmatrix} C_1 & C_2 & \cdots & C_{k-1} & c_{k1} & \cdots & c_{kp} \end{bmatrix} (n)((k-1)\times m - p) \qquad (4.40)$$

$$B_T = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k-1} \\ b_{k1} \\ \vdots \\ b_{kp} \end{bmatrix} ((k-1)\times m + p)(m)$$

Finally, this triple $\{A_T, B_T, C_T\}$ is converted back to a transfer function description form using the Sauriau-Leverrier-Faddeev algorithm.

$$\hat{H}(s) = C_T (sI - A_T)^{-1} B_T \qquad (4.41)$$

The Sauriau-Leverrier-Faddeev algorithm give rise to an analytical expression for $(sI - A_T)^{-1}$, which is summarised in the following steps:

$$(sI - A_T)^{-1} = \frac{R(s)}{\Delta(s)}$$

with

$$R(s) = s^{n-1} I + s^{n-2} R_1 + \cdots + sR_{n-2} + R_{n-1}$$

and

$$\Delta(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

where the coefficients a_i and R_i are computed from the following relations:

$$
\begin{array}{lll}
A_1 = A_T & a_{n-1} = -trace(A_1) & R_1 = A_1 + a_{n-1}I \\
A_2 = A_T R_1 & a_{n-2} = -\frac{1}{2} trace(A_2) & R_2 = A_2 + a_{n-2}I \\
\vdots & \vdots & \vdots \\
A_{n-1} = A_T R_{n-2} & a_1 = -\frac{1}{(n-1)} trace(A_{n-1}) & R_{n-1} = A_{n-1} + a_1 I \\
A_n = A_T R_{n-1} & a_0 = -\frac{1}{n} trace(A_n) & R_n = 0
\end{array}
$$

Then

$$\hat{H}(s) = \frac{1}{\Delta(s)}[G(s)] = \frac{1}{\Delta(s)}\left[\sum_{i=0}^{n-1} C_T R_i B_T s^{n-1-i}\right] \tag{4.42}$$

In this correspondence, an efficient and simple algorithm is developed.

4.3.3 Algorithm

Step 1 : Choose the relative error RE.

Step 2 : Given $H(s) = N_R(s)D_R^{-1}(s)$, determine the solvents of $D_R(s)$ and classify them in order of decreasing dominance.

Step 3 : Construct the block controller form (C_C, A_C, B_C) and the corresponding Hankel matrix of the original model, then compute the singular values of this Hankel matrix.

$$SVD\{\Gamma\{H(s)\}\} = U * Diag\{\sigma_i^2, i = 1,2,\ldots,ml\} * V$$

with U, V being orthogonal matrices and $\sigma_i^2, i = 1,2,\ldots,lm$, are the singular values of $\Gamma\{H(s)\}$.

Step 4 : Form the Vandermande matrix V_R of the original model.

Step 5 : Transform the controller form to a block diagonal form using the block Vandermande similarity transformation.

Step 6 : With this decoupled system, start adding the block subsystems.

Step 7 : Form the Hankel matrix of the neglected subsystem as $\{\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}\}$ and determine its singular values.

$$SVD\{\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}\} = \bar{U} * \{\bar{\sigma}_i^2, i = 1,2,\ldots,ml\} * \bar{V}$$

where $\bar{U}$, $\bar{V}$ are orthogonal matrices and $\bar{\sigma}_i^2, i = 1,2,\ldots,lm$, are the singular values of the difference $\{\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}\}$.

Step 8 : Test for the relative error RE satisfaction

$$RE = \left(\left(\sum_{i=1}^{ml} \bar{\sigma}_i^4 \right)^{1/2} \Big/ \left(\sum_{i=1}^{ml} \sigma_i^4 \right)^{1/2} \right)$$

If the relative error is not satisfied, add another subsystem to the previous reduced model and go to step 7. Otherwise if the relative error is satisfied, the obtained approximation is tuned through the following steps to the eigenvalue level.

Step 9 : Diagonalise the last added subsystem and start truncating eigenvalues.

Step 10 : Again test for the relative error satisfaction. In case it is satisfied for $p < m$, reduce again the model by including only p eigenvalues of the k^{th} subsystem. Otherwise, if $p = m$ keep the subsystem as it is.

Step 11 : Use the Sauriau-Leverier-Faddeev algorithm in order to get a transfer function form.

$$\hat{H}(s) = \frac{1}{\Delta(s)}[G(s)]$$

4.3.4 Comments on the solvents based method

First of all note that the same development can be done for the left matrix fraction description form and similar results will be obtained.

As it will be seen in the coming chapter, the application of the above algorithm shows some very interesting results. The only drawback of this procedure is the numerical cost due to the inversion of the Vandermande matrix.

Procedures of bypassing this obstacle may be developed to increase the efficiency and stability of the algorithm. In this order a second procedure that avoids the inversion of huge Vandermande matrices is proposed in the next section.

4.4 Spectral Factors Based Model Reduction Approach

The second proposed model reduction method is developed directly from the matrix transfer function form. This method as well retains both the advantages of the concept of matrix polynomial division, and the dominant spectral factors of the characteristic matrix polynomial of the multi-input multi-output system written in the form of a matrix fraction description. In addition, if the reduced model is required to be tuned to the eigenvalue level, block decoupling is suggested.

4.4.1 Problem Formulation

Given a stable linear time-invariant multivariable system described by a matrix transfer function $H(s)$, the objective is to derive an approximation $\hat{H}(s)$ of $H(s)$ such that the impulse response error $[H(t) - \hat{H}(t)]$ is minimum. In other words, the problem reduces to eliminate any least significant subsystems which contribute little to the impulse response.

4.4.2 Theoretical Development

Consider a strictly proper $n \times m$ transfer function in the form of a matrix fraction description:

$$H(s) = D_l^{-1}(s)N_l(s) \qquad LMFD \qquad (4.43)$$

or,

$$\begin{aligned} H(s) &= N_R(s)D_R^{-1}(s) \qquad RMFD \\ &= \left[N_1 s^{l-1} + N_2 s^{l-2} + \cdots + N_l\right]\left[D_0 s^l + D_1 s^{l-1} + \cdots + D_l\right]^{-1} \end{aligned} \qquad (4.44)$$

where $N_i, i = 1,2,\ldots,l$; and $D_i, i = 0,1,2,\ldots,l$, are respectively $n \times m$ and $m \times m$ constant matrices, with D_0 being the identity.

Our main contribution lies in the derivation of an approximation $\hat{H}(s)$ from $H(s)$ based on the dominant spectral factors (that may be computed using any matrix polynomial method such as the ones seen in chapter 2) of

the characteristic matrix polynomial $D_R(s)$ such that the impulse response error between $H(t)$ and $\hat{H}(t)$ is minimum.

This problem is solved through the use of the notion of matrix polynomial division.

Let the reduced model be of the form:

$$\hat{H}(s) = \hat{N}_R(s)\hat{D}_R^{-1}(s) \tag{4.45}$$

Then the error function in the frequency domain may be written as follows:

$$\begin{aligned} H_\varepsilon(s) &= [H(s) - \hat{H}(s)] \\ &= N_R(s)D_R^{-1}(s) - \hat{N}_R(s)\hat{D}_R^{-1}(s) \end{aligned} \tag{4.46}$$

where $\hat{D}_R(s)$ is the dominant part of $D_R(s)$ containing the first k dominant spectral factors. Hence, it can be written as a product of k linear spectral factors

$$\hat{D}_R(s) = ((sI - Q_1)(sI - Q_2)\cdots(sI - Q_k)) \tag{4.47}$$

where $Q_1, Q_2, \ldots, Q_k$, represent a complete set of spectral factors of $\hat{D}_R(s)$ satisfying the following relation:

$$Q_1 > Q_2 > Q_3 > \cdots > Q_k \tag{4.48}$$

where $>$ specifies dominance between the set of spectral factors (see definition 4.1).

Note that $D_R(s)$ may or may not be able to be expressed as a product of l linear spectral factors, but it is evident that the dominant ones are within $\hat{D}_R(s)$. Hence, $D_R(s)$ can be written as a product of a dominant factorisable polynomial $\hat{D}_R(s)$ by a dominated polynomial $D(s)$ such that:

$$D_R(s) = \hat{D}_R(s)D(s) \tag{4.49}$$

where

$$D_R(s) = D_0 s^l + D_1 s^{l-1} + D_2 s^{l-2} + \cdots + D_{l-1}s + D_l \tag{4.50}$$

$$\hat{D}_R(s) = (sI - Q_1)(sI - Q_2)\cdots(sI - Q_r) \qquad 1 \le r \le l \tag{4.51}$$

and

$$D(s) = C_0 s^{l-r} + C_1 s^{l-r-1} + C_2 s^{l-r-2} + \cdots + C_{l-r-1} s + C_{l-r} \tag{4.52}$$

Justification:

Let

$$\begin{aligned} D(s) &= \sum_{i=0}^{\infty} C_i s^{l-r-i} \\ &= C_0 s^{l-r} + C_1 s^{l-r-1} + C_2 s^{l-r-2} + C_3 s^{l-r-3} + \cdots \end{aligned}$$

from (4.49), we have

$$D_R(s) = \hat{D}_R(s) D(s) \tag{4.53}$$

or equivalently:

$$\begin{aligned} & D_0 s^l + D_1 s^{l-1} + D_2 s^{l-2} + \cdots + D_{l-1} s + D_l \\ & \quad = [\hat{D}_0 s^r + \hat{D}_1 s^{r-1} + \hat{D}_2 s^{r-2} + \cdots + \hat{D}_{r-1} s + \hat{D}_r][C_0 s^{l-r} + C_1 s^{l-r-1} + C_2 s^{l-r-2} + \cdots] \end{aligned} \tag{4.54}$$

Combining terms of the same power yields

$$\begin{aligned} & D_0 s^l + D_1 s^{l-1} + D_2 s^{l-2} + D_3 s^{l-3} + \cdots + D_{l-1} s + D_l \\ & \quad = (\hat{D}_0 C_0) s^l + (\hat{D}_0 C_1 + \hat{D}_1 C_0) s^{l-1} + (\hat{D}_0 C_2 + \hat{D}_1 C_1 + \hat{D}_2 C_0) s^{l-2} + \cdots \end{aligned} \tag{4.55}$$

And identifying coefficients of the same power, leads to

$$\begin{aligned} D_0 &= \hat{D}_0 C_0 \\ D_1 &= \hat{D}_0 C_1 + \hat{D}_1 C_0 \\ D_2 &= \hat{D}_0 C_2 + \hat{D}_1 C_1 + \hat{D}_2 C_0 \\ D_3 &= \hat{D}_0 C_3 + \hat{D}_1 C_2 + \hat{D}_2 C_1 + \hat{D}_3 C_0 \\ &\vdots \\ D_{l-1} &= \hat{D}_0 C_{l-1} + \hat{D}_1 C_{l-2} + \hat{D}_2 C_{l-3} + \cdots + \hat{D}_{l-1} C_0 \\ D_l &= \hat{D}_0 C_l + \hat{D}_1 C_{l-1} + \hat{D}_2 C_{l-2} + \hat{D}_3 C_{l-3} + \cdots + \hat{D}_l C_0 \end{aligned} \tag{4.56}$$

Note that since $D_R(s)$ is 'divisible' by $\hat{D}_R(s)$ then $D(s)$ is a finite series. In other words, $D_i, i = (l+1), (l+2), \ldots$ are equal to zero matrices. Hence, the $C_i, i = 0,1,2,\ldots$ coefficients can be determined sequentially using

$$
\begin{aligned}
C_0 &= \hat{D}_0^{-1} D_0 \\
C_1 &= \hat{D}_0^{-1}[D_1 - \hat{D}_1 C_0] \\
C_2 &= \hat{D}_0^{-1}[D_2 - \hat{D}_1 C_1 - \hat{D}_2 C_0] \\
&\vdots \\
C_{l-1} &= \hat{D}_0^{-1}[D_{l-1} - \hat{D}_1 C_{l-2} - \hat{D}_2 C_{l-3} - \cdots - \hat{D}_{l-1} C_0] \\
C_l &= \hat{D}_0^{-1}[D_l - \hat{D}_1 C_{l-1} - \hat{D}_2 C_{l-2} - \hat{D}_3 C_{l-3} - \cdots - \hat{D}_l C_0]
\end{aligned}
\tag{4.57}
$$

Note also that (4.57) is a recursion relation that can always be solved since $\hat{D}_0^{-1}$ is nonsingular (it is in fact equal to identity since $\hat{D}_R(s)$ is monic).

Writting the above result in compact form yields:

$$
\begin{aligned}
&For \quad i = 0 \\
&C_0 = \hat{D}_0^{-1} D_0
\end{aligned}
\tag{4.58}
$$

and

$$
\begin{aligned}
&For \quad i = 1 \quad to \quad i = l \\
&C_i = \hat{D}_0^{-1}\left[D_i - \sum_{j=1}^{i} \hat{D}_j C_{i-j} \right]
\end{aligned}
\tag{4.59}
$$

Carrying out the above development and inverting both sides of equation (4.53), yields:

$$
D_R^{-1}(s) = D^{-1}(s)\hat{D}_R^{-1}(s) \tag{4.60}
$$

hence equation (4.46) can be rewritten as:

$$
\begin{aligned}
H_\varepsilon(s) &= N_R(s)D^{-1}(s)\hat{D}_R^{-1}(s) - \hat{N}_R(s)\hat{D}_R^{-1}(s) \\
&= \left[N_R(s)D^{-1}(s) - \hat{N}_R(s) \right]\hat{D}_R^{-1}(s)
\end{aligned}
\tag{4.61}
$$

where $\hat{D}_R(s)$ is the dominant factor of $D_R(s)$ and $D(s)$ is the dominated factor of $D_R(s)$.

By doing the 'long division' of $N_R(s)$ by $D(s)$, we get:

$$
N_R(s) = Q(s)D(s) + R(s) \tag{4.62}
$$

which represents a long division from the right.

Theorem 4.1:

Let

$$
N_R(s) = N_0 s^k + N_1 s^{k-1} + N_2 s^{k-2} + \cdots + N_{k-1} s + N_k \tag{4.63}
$$

$$
\begin{aligned}
C_0 &= \hat{D}_0^{-1} D_0 \\
C_1 &= \hat{D}_0^{-1}[D_1 - \hat{D}_1 C_0] \\
C_2 &= \hat{D}_0^{-1}[D_2 - \hat{D}_1 C_1 - \hat{D}_2 C_0] \\
&\vdots \\
C_{l-1} &= \hat{D}_0^{-1}[D_{l-1} - \hat{D}_1 C_{l-2} - \hat{D}_2 C_{l-3} - \cdots - \hat{D}_{l-1} C_0] \\
C_l &= \hat{D}_0^{-1}[D_l - \hat{D}_1 C_{l-1} - \hat{D}_2 C_{l-2} - \hat{D}_3 C_{l-3} - \cdots - \hat{D}_l C_0]
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$$
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&= \left[N_R(s)D^{-1}(s) - \hat{N}_R(s)\right]\hat{D}_R^{-1}(s)
\end{aligned}
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where $\hat{D}_R(s)$ is the dominant factor of $D_R(s)$ and $D(s)$ is the dominated factor of $D_R(s)$.

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which represents a long division from the right.

Theorem 4.1:

Let

$$
N_R(s) = N_0 s^k + N_1 s^{k-1} + N_2 s^{k-2} + \cdots + N_{k-1} s + N_k \tag{4.63}
$$

and

$$D(s) = D_0 s^p + D_1 s^{p-1} + D_2 s^{p-2} + \cdots + D_{p-1} s + D_p \quad \text{with} \quad k \geq p \tag{4.64}$$

Then there exists a unique matrix polynomial $Q(s)$ *of degree* $(k-p)$ *and a unique matrix polynomial* $R(s)$ *of degree not exceeding* $(p-1)$ *such that the equation*

$$N_R(s) = Q(s)D(s) + R(s) \tag{4.65}$$

is satisfied.

Proof:

Let

$$Q(s) = Q_0 s^{k-p} + Q_1 s^{k-p-1} + Q_2 s^{k-p-2} + \cdots + Q_{k-p-1} s + Q_{k-p} \tag{4.66}$$

equating coefficients of (4.65), the coefficients of $Q(s)$ and $R(s)$ can be successively and uniquely determined from the obtained k equations.

The coefficients matrices of $Q(s)$ are supplied recursively as:

$$\begin{aligned} Q_0 &= N_0 D_0^{-1} \\ Q_1 &= (N_1 - Q_0 D_1) D_0^{-1} \\ &\vdots \\ Q_{k-p} &= (N_{k-p} - Q_{k-2p} D_p - Q_{k-2p-1} D_{p-1} - \cdots - Q_{k-p-1} D_1) D_0^{-1} \end{aligned} \tag{4.67}$$

Replacing $(k-p)$ by i, the above equations will look as follows:

$$\begin{aligned} Q_0 &= N_0 D_0^{-1} \\ Q_1 &= (N_1 - Q_0 D_1) D_0^{-1} \\ &\vdots \\ Q_i &= (N_i - Q_{i-p} D_p - Q_{i-p-1} D_{p-1} - \cdots - Q_{i-2} D_2 - Q_{i-1} D_1) D_0^{-1} \end{aligned} \tag{4.68}$$

this can be written in a programmable form as:

$$\begin{aligned} &Q_0 = N_0 D_0^{-1} \\ &For \quad n = 1 \quad to \quad p \\ &Q_n = (N_n - \sum_{j=1}^{n} Q_{n-j} D_j) D_0^{-1} \\ &For \quad n = p+1 \quad to \quad (k-p) \\ &Q_n = N_n - \sum_{j=1}^{p} Q_{n-j} D_j \end{aligned} \tag{4.69}$$

and the remainder $R(s)$ is given by

$$R(s) = R_0 s^{p-1} + R_1 s^{p-2} + \cdots + R_{p-2} s + R_{p-1} \tag{4.70}$$

with the corresponding coefficients supplied by:

$$\begin{aligned} R_0 &= N_{k-p-1} - Q_{k-2p-1} D_p - \cdots - Q_{k-p} D_1 \\ R_1 &= N_{k-p-2} - Q_{k-2p-2} D_p - \cdots - Q_{k-p-1} D_3 - Q_{k-p} D_2 \\ &\vdots \\ R_{p-1} &= N_k - Q_{k-p} D_p \end{aligned} \tag{4.71}$$

here also replacing $(k-p)$ by i, the above equations reduce to

$$\begin{aligned} R_0 &= N_{i-1} - Q_{i-p-1} D_p - \cdots - Q_i D_1 \\ R_1 &= N_{i-2} - Q_{i-p-2} D_p - \cdots - Q_{i-1} D_3 - Q_i D_2 \\ &\vdots \\ R_{p-1} &= (N_k - Q_i D_p) \end{aligned} \tag{4.72}$$

writting this in a programmable form yields

$$R_j = N_{i-j-1} - \sum_{t=j-1}^{p} Q_{i-j-t-1} D_t \tag{4.73}$$

with $i = k-p$ and j varying between 0 and $(p-1)$, p being the order of $D(s)$ and k the order of $N_R(s)$ □□□.

Next, multiplying both sides of equation (4.62) by $D^{-1}(s)$ produces:

$$N_R(s) D^{-1}(s) = Q(s) + R(s) D^{-1}(s) \tag{4.74}$$

replacing equation (4.74) in equation (4.61), leads to:

$$\begin{aligned} H_\varepsilon(s) &= \left[Q(s) + R(s) D^{-1}(s) - \hat{N}_R(s) \right] \hat{D}_R^{-1}(s) \\ &= \left[Q(s) - \hat{N}_R(s) \right] \hat{D}_R^{-1}(s) + R(s) D^{-1}(s) \hat{D}_R^{-1}(s) \end{aligned} \tag{4.75}$$

The key idea is to take

$$\hat{N}_R(s) = Q(s) \tag{4.76}$$

then, equation (4.75) reduces to

$$\begin{aligned} H_\varepsilon(s) &= \{ H(s) - \hat{H}(s) \} \\ &= R(s) D^{-1}(s) \hat{D}_R^{-1}(s) \\ &= R(s) D_R^{-1}(s) \end{aligned} \tag{4.77}$$

which is also a transfer function in the form of a matrix fraction description (RMFD), required to be negligible in terms of impulse response with respect to the original model.

At this stage a reduced model at the spectral factor level satisfying the error criterion is obtained. In order to tune it at the eigenvalue level, the decoupling of the obtained reduced model and the diagonalisation of the last subsystem of the reduced model is done.

Let the reduced model be described by

$$\hat{H}(s) = \left[\hat{N}_1 s^{k-1} + \hat{N}_2 s^{k-2} + \cdots + \hat{N}_k\right]\left[\hat{D}_0 s^k + \hat{D}_1 s^{k-1} + \cdots + \hat{D}_k\right]^{-1} \tag{4.78}$$

The decoupling is achieved through the use of the Vandermande similarity transformation matrix. The Vandermande matrix is constructed using the dominant solvents derived from the corresponding dominant spectral factors using the transformation developed by Shieh and Tsay [41] (see also chapter 2). Hence

$$V(R_1, R_2, \ldots, R_k) = \begin{bmatrix} I & I & I & \cdots & I \\ R_1 & R_2 & R_3 & \cdots & R_k \\ R_1^2 & R_2^2 & R_3^2 & \cdots & R_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_1^{k-1} & R_2^{k-1} & R_3^{k-1} & \cdots & R_k^{k-1} \end{bmatrix} \tag{4.79}$$

Writing the obtained reduced model in companion from, yields:

$$A_c = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -\hat{D}_k & -\hat{D}_{k-1} & -\hat{D}_{k-2} & \cdots & -\hat{D}_1 \end{bmatrix}(k \times m)(k \times m) \qquad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}(k \times m)(m)$$

$$C_c = \begin{bmatrix} \hat{N}_k & \hat{N}_{k-1} & \hat{N}_{k-2} & \cdots & \hat{N}_1 \end{bmatrix}(n)(k \times m) \tag{4.80}$$

The above reduced system can be writen in compact form as:

$$\begin{aligned} X_c &= A_c X_c + B_c U \\ Y_c &= C_c X_c \end{aligned} \tag{4.81}$$

and making a change of coordinates using the Vandermande similarity transformation produces:

$$\begin{aligned} A_R &= V_R^{-1} A_c V_R \\ B_R &= V_R^{-1} B_c \\ C_R &= C_c V_R \end{aligned} \tag{4.82}$$

so that the above system may be written as a decoupled system

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_k \end{bmatrix} U$$

$$Y = \begin{bmatrix} C_1 & C_2 & \cdots & C_k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} \tag{4.83}$$

At this level, the diagonalisation of the last subsystem corresponding to the last added linear spectral factor is done in the same manner as for the previous procedure. The above system will then look as follows:

$$A_{AD} = \begin{bmatrix} R_1 & 0_m & \cdots & 0_m & 0 & \cdots & 0 \\ 0_m & R_2 & \cdots & 0_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & \cdots & R_{k-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_m \end{bmatrix} (m \times k)(m \times k) \quad B_{RD} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k-1} \\ b_{k1} \\ \vdots \\ b_{km} \end{bmatrix} (m \times k)(m)$$

$$C_{RD} = \begin{bmatrix} C_1 & C_2 & \cdots & C_{k-1} & c_{k1} & \cdots & c_{km} \end{bmatrix} (n)(m \times k) \tag{4.84}$$

And then we start truncating the eigenvalues up when the relative error criterion is again satisfied.

The relative error criterion satisfaction is tested for reduced models of order $((k-1) \times m + p) \times ((k-1) \times m + p)$, $p = 1,2,\ldots,m$. In case, the relative error is satisfied for $p < m$, the system is truncated one more time at the

$((k-1)\times m+p)$ component. Otherwise if, $p=m$, the subsystem $C_k(sI-R_k)^{-1}B_k$ is kept as a block without the diagonalisation.

The obtained tuned reduced model is then the following:

$$A_T=\begin{bmatrix} R_1 & 0_m & \cdots & 0_m & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0_m & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0_m & 0_m & \cdots & R_{k-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_{k1} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \lambda_{kp} \end{bmatrix}((k-1)\times m+p)((k-1)\times m+p)$$

$$C_T=\begin{bmatrix} C_1 & C_2 & \cdots & C_{k-1} & c_{k1} & \cdots & c_{kp} \end{bmatrix}(n)((k-1)\times m+p) \qquad (4.85)$$

$$B_T=\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k-1} \\ b_{k1} \\ \vdots \\ b_{kp} \end{bmatrix}((k-1)\times m+p)(m)$$

Finally, the above triple $\{A_T,B_T,C_T\}$ is converted back to a transfer function description form using the Sauriau-Leverrier-Faddeev algorithm which is very useful for the computation of the resolvent matrix.

$$\hat{H}(s)=C_T(sI-A_T)^{-1}B_T \qquad (4.86)$$

The corresponding resolvent matrix $(sI-A_T)^{-1}$ is denoted as follows:

$$(sI-A_T)^{-1}=\frac{adj(sI-A_T)}{\det(sI-A_T)}=\frac{R(s)}{\Delta(s)}$$

where

$$R(s)=Is^{n-1}+R_1s^{n-2}+\cdots+R_{n-2}s+R_{n-1}$$

and

$$\Delta(s)=s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0$$

The Sauriau-Leverrier-Fadeev indicates that the coefficients a_i and $R_i, i=0,1,\ldots,n-1$, can be obtained by the following recurrence formulae:

$$R_0 = I$$

$$a_{n-i} = -\frac{1}{i} tr(R_{i-1} A_T)$$

$$= -\frac{1}{i} tr(A_T R_{i-1})$$

$$R_i = R_{i-1} A_T + a_{n-i} I = A_T R_{i-1} + I a_{n-i}, \qquad i = (n-1), (n-2), \ldots, 1$$

$$a_0 = -\frac{1}{n} tr(A_T R_{n-1}) = -\frac{1}{n} tr(R_{n-1} A_T), \qquad or \qquad 0 = R_{n-1} A_T + a_0 I$$

where the tr stands for the trace of a matrix. The last equation can be used as a check. Then

$$\hat{H}(s) = \frac{1}{\Delta(s)}[G(s)] = \frac{1}{\Delta(s)}\left[\sum_{i=0}^{n-1} C_T R_i B_T s^{n-1-i}\right] \tag{4.87}$$

In this order an algorithm is developed and summarised in the following steps:

4.4.3 Algorithm

Step 1: Input the allowed relative error RE.

Step 2: Given $H(s) = N_R(s) D_R^{-1}(s)$, form its controller form and the corresponding Hankel matrix, then compute its singular values as

$$SVD\{\Gamma\{H(s)\}\} = U * Diag\{\sigma_i^2, i = 1,2,\ldots,ml\} * V$$

where U and V are orthogonal matrices and $\sigma_i^2, i = 1,2,\ldots,ml$ are the Hankel matrix singular values of the original model.

Step 3: Determine the first spectral factor of $D_R(s)$ and form the corresponding linear spectral factor to be $\hat{D}_R(s) = (sI - Q_1)$.

Step 4: Divide $D_R(s)$ by the $\hat{D}_R(s)$ such that

$$D(s) = \hat{D}_R^{-1}(s) D_R(s)$$

which represents a long division from the left.

Step 5: Divide the given $N_R(s)$ by $D(s)$ to yield

$$N_R(s) = \hat{N}_R(s) D(s) + R(s)$$

Step 6: Form the Hankel matrix difference $\{\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}\}$ and then compute its singular values.

$$SVD\{\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}\} = \bar{U} * Diag\{\bar{\sigma}_i^2, i = 1,2,\ldots,ml\} * \bar{V}$$

where $\bar{U}$ and $\bar{V}$ are orthogonal matrices and $\bar{\sigma}_i^2, i = 1,2,\ldots,ml$ are the Hankel matrix singular values of the neglected part of the model.

Step 7: Test for the relative error, RE, satisfaction using

$$RE = \frac{\left(\sum_{i=1}^{ml} \bar{\sigma}_i^4\right)^{1/2}}{\left(\sum_{i=1}^{ml} \sigma_i^4\right)^{1/2}}$$

Step 8: In case the relative error criterion is not satisfied, determine the next dominant spectral factor and form the new $\hat{D}_R(s)$ then go to step 4. Otherwise if it is satisfied, form the spectral factors level reduced model as:

$$\hat{H}(s) = \hat{N}_R(s)\hat{D}_R^{-1}(s)$$

Then tune it at the eigenvalue level through the following steps.

Step 9: Determine the solvents of $\hat{D}_R(s)$ from the corresponding spectral factors.

Step 10: Form the Vandermande matrix and transform the reduced model to a block diagonal form.

Step 11: Diagonalise the last block and form the new $\hat{H}(s)$ by truncating the eigenvalues.

Step 12: Determine the singular values of the Hankel matrix difference as

$$SVD\{\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}\} = \bar{U} * Diag\{\bar{\sigma}_i^2, i = 1,2,\ldots,ml\} * \bar{V}$$

Step 13: Test again for the relative error RE satisfaction. In case it is satisfied for $p < m$, the model is reduced to include only p eigenvalues of the last block subsystem. Otherwise if $p = m$, the spectral factor level reduced model is kept as it is.

Step 14: Use the Sauriau-Leverier-Fadeev algorithm in order to get a transfer function form.

4.4.4 Comments on the spectral factors based method

The major advantage offered by this approach over the previous one, is the fact that the inversion of huge Vandermande matrices is avoided, since

the original model is first reduced using the polynomial division then it is block decoupled using the Vandermande similarity transformation. In addition, it does not require the determination of the complete set of spectral factors.

The application of the later algorithm shows also very interesting results. The only weak point that may be mentioned about this procedure, is the need for a matrix polynomial method that produces a complete set of spectral factors in a specific order of dominance.

This completes the presentation of the two proposed model reduction methods. In the next chapter, we provide selected examples to illustrate the developed procedures.

Chapter 5

Numerical Results And Analysis

The two model reduction procedures, the solvents based and the spectral factors based procedures, developed in the previous chapter are implemented on computer using one of the most popular computer tools known as MATLAB.

MATLAB is a sophisticated mathematics and simulation environment that can be used to model and analyze dynamic systems. It handles continuous, discrete, linear, or nonlinear systems. As its name implies, it has extensive features for matrix manipulations. Matlab is an open environment for which many specialized toolboxes have been developed:

- Control Systems.
- Signal Processing.
- Optimization.
- Robust Control.
- μ Analysis and Synthesis (μ Tools).
- Spline.
- System Identification.
- Neural Networks and
- Chemometrics.

SIMULINK (formerly known as Simulab) is a graphical environment for modelling and simulating block diagrams and general nonlinear systems. Matlab supports some basic programming structures that allow looping and

conditioning commands along with relational and logical operators. The syntax and use of some of these structures are very similar to those found in other high level languages such as C, BASIC, and FORTRAN. The combination of these commands can lead to powerful programs or new functions that can be added to Matlab.

The main Matlab features used in the two algorithms range from the simplest matrix operations, and array operations, to matrix functions such as the Singular Value Decomposition, Impulse Response, Kronecker Product, ...etc.

5.1 Singular Value Decomposition

The singular values of the Hankel matrices $\Gamma\{H(s)\}$ and $\Gamma\{H(s)\} - \Gamma\{\hat{H}(s)\}$ are evaluated using a Matlab built-in function SVD. The notion of Singular Value Decomposition play a key role in the analysis of robust stability mainly for multivariable systems.

Consider a rectangular matrix H with rank ρ, then $SVD(H)$ produces a diagonal matrix $\hat{\Sigma}$ of the same dimension as H with non-negative elements, and unitary matrices U and V such that

$$H = U\hat{\Sigma}V^T \tag{5.1}$$

where H is $m \times n$, U is $m \times m$, and V is $n \times n$.

Note that V^T stands for the conjugate transpose of V. The matrix $\hat{\Sigma}$ is defined by

$$\hat{\Sigma} = \begin{cases} \left[\Sigma \mid 0\right] & \text{if} \quad n > m \\ \left[\begin{array}{c} \Sigma \\ \hline 0 \end{array}\right] & \text{if} \quad n < m \end{cases} \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_\rho \end{bmatrix} \tag{5.2}$$

and

$$\rho = \min\{m, n\}$$

The singular values are ordered as

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\rho \tag{5.3}$$

The smallest singular values are rarely exactly zeros, but if H is 'nearly singular' some of the singular values will be small. The ratio (σ_1 / σ_ρ) can be regarded as a condition number of the matrix H [66].

5.2 Impulse Response

The two algorithms presented in the previous chapter use the impulse response for the graphical comparison of the original and reduced models. The choice of the impulse response is not hazardous since its the one that characterizes completely the linear time invariant (LTI) system.

The Impulse response of continuous-time linear systems is also available in Matlab as a built in function under the notation

$$\mathrm{Im}\,pulse(A,B,C,D,iu) \tag{5.4}$$

which plots the time response of the linear system

$$\begin{cases} X = AX + BU \\ Y = CX + DU \end{cases} \tag{5.5}$$

to an impulse applied to the single input iu. The time vector is automatically determined. Impulse (NUM,DEN) can also be used for the scalar case and it plots the impulse response of polynomial transfer function G(s)=NUM(s)/DEN(s) where NUM and DEN contain the polynomial coefficients in descending powers of s.

$\mathrm{Im}\,pulse(A,B,C,D,iu,T)$ or $\mathrm{Im}\,pulse(NUM,DEN,T)$ uses the user-supplied time vector T which must be regularly spaced. When invoked with left hand arguments

$$\begin{aligned} [Y,X,T] &= \mathrm{Im}\,pulse(A,B,C,D,\ldots) \\ [Y,X,T] &= \mathrm{Im}\,pulse(NUM,DEN,\ldots), \end{aligned} \tag{5.6}$$

it returns the output and state time history in the matrices Y and X. Y has as many columns as there are outputs and length T rows, while X has as many columns as there are states. No plot is drawn on the screen.

5.3 Solvents Based Model Reduction Procedure

The solvents based method inputs two sets of matrices corresponding to the "numerator" coefficients and the coefficients of the characteristic matrix polynomial of the original transfer function. In addition the algorithm requires the supply of the allowed relative error RE to be used as a condition for the system truncation.

The algorithm starts by determining and classifying in order of decreasing dominance the solvents of the characteristic matrix polynomial using any matrix polynomial root finding method. It then computes the controller form of the given model and its corresponding Hankel matrix, and applies the Matlab command SVD to this Hankel matrix in order to compute its singular values. The block diagonalization of the original system is performed using the Vandermande matrix constructed from the determined solvents and its inverse computed using the Matlab inverse command (INV).

Following the block truncation, the last added subsystem is in turn tuned, after its diagonalization, using the Matlab command eigen (EIG). With this tuned reduced model, a Hankel matrix of the same dimension as the original model is formed and its corresponding singular values are also determined using the Matlab built-in function (SVD). Finally the sum of the singular values of the difference between the two models is divided by the sum of those of the original model and compared with the allowed relative error RE.

However, it is appropriate to mention that the lack of examples satisfying our assumptions in the current control literature constrained us to build our own examples for the testing of the developed algorithm.

To this end, the "numerator" matrix coefficients are chosen arbitrarily to avoid getting trapped by a particular case, while the coefficients of the characteristic matrix polynomial are constructed in such a way that it includes a complete set of solvents satisfying a dominance criterion.

This is achieved by first building the solvents using the relation defined in theorem 2.10 and given by

$$R = P\Lambda P^{-1} \tag{5.7}$$

where R is the solvent

Λ the Jordan form of R and

P is a matrix of eigenvectors of R.

Let

$$D_R(s) = D_0 s^l + D_1 s^{l-1} + D_2 s^{l-2} + \cdots + D_{l-1} s + D_l \tag{5.8}$$

where D_0 is the identity matrix. In addition, we assume that the constructed matrices $\{R_1, R_2, R_3, \ldots, R_l\}$ form a complete set of solvents of $D_R(s)$. Hence,

$$\begin{aligned} D_R(R_1) &= D_0 R_1^l + D_1 R_1^{l-1} + D_2 R_1^{l-2} + \cdots + D_{l-1} R_1 + D_l = 0 \\ D_R(R_2) &= D_0 R_2^l + D_1 R_2^{l-1} + D_2 R_2^{l-2} + \cdots + D_{l-1} R_2 + D_l = 0 \\ &\vdots \\ D_R(R_l) &= D_0 R_l^l + D_1 R_l^{l-1} + D_2 R_l^{l-2} + \cdots + D_{l-1} R_l + D_l = 0 \end{aligned} \tag{5.9}$$

This above system of equations can be expressed in matrix form as

$$\begin{bmatrix} D_l & D_{l-1} & D_{l-2} & \cdots & D_1 \end{bmatrix} \begin{bmatrix} I & I & I & \cdots & I \\ R_1 & R_2 & R_3 & \cdots & R_l \\ R_1^2 & R_2^2 & R_3^2 & \cdots & R_l^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_1^{l-1} & R_2^{l-1} & R_3^{l-1} & \cdots & R_l^{l-1} \end{bmatrix} = -\begin{bmatrix} R_1^l & R_2^l & R_3^l & \cdots & R_l^l \end{bmatrix} \tag{5.10}$$

and since we have assumed the existence of a complete set of solvents, the matrix

$$V_R = \begin{bmatrix} I & I & I & \cdots & I \\ R_1 & R_2 & R_3 & \cdots & R_l \\ R_1^2 & R_2^2 & R_3^2 & \cdots & R_l^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_1^{l-1} & R_2^{l-1} & R_3^{l-1} & \cdots & R_l^{l-1} \end{bmatrix} \tag{5.11}$$

is nonsingular [42]; Therefore we can write:

$$\begin{bmatrix} D_l & D_{l-1} & D_{l-2} & \cdots & D_1 \end{bmatrix} = -\begin{bmatrix} R_1^l & R_2^l & R_3^l & \cdots & R_l^l \end{bmatrix} \begin{bmatrix} V_R^{-1} \end{bmatrix} \tag{5.12}$$

To show the efficiency of the above algorithm, a large number of matrix transfer functions have been tested. Some of them are chosen as examples and discussed in the following.

Example 1:

The first chosen example is a 4^{th} order 3×3 matrix transfer function in the form of a right matrix fraction description (RMFD)

$$H(s) = N(s)D^{-1}(s)$$

where

$$N(s) = N_1 s^3 + N_2 s^2 + N_3 s + N_4$$
$$D(s) = D_0 s^4 + D_1 s^3 + D_2 s^2 + D_3 s + D_4$$

with

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad N_2 = \begin{pmatrix} 33 & 44 & 55 \\ -22 & -88 & 99 \\ 44 & 64 & 64 \end{pmatrix}; \quad N_3 = \begin{pmatrix} -25 & -51 & 22 \\ 11 & 22 & 10 \\ 55 & 39 & 74 \end{pmatrix}; \quad N_4 = \begin{pmatrix} 11 & -54 & 14 \\ 12 & 30 & 14 \\ 85 & 96 & 54 \end{pmatrix};$$

and

$$D_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 54.6829 & -3.4100 & -11.2305 \\ -4.8014 & 40.0253 & 3.6803 \\ 1.1092 & -2.2721 & 36.2918 \end{pmatrix};$$

$$D_2 = \begin{pmatrix} 1.0028e+3 & -0.1037e+3 & -0.4024e+3 \\ -0.1729e+3 & 0.4468e+3 & 0.1317e+3 \\ 0.0423e+3 & -0.0415e+3 & 0.3498e+3 \end{pmatrix}; \quad D_3 = \begin{pmatrix} 6.4121e+3 & -0.7832e+3 & -3.7313e+3 \\ -1.5983e+3 & 0.9637e+3 & 1.2178e+3 \\ 0.4022e+3 & -0.0089e+3 & 0.4137e+3 \end{pmatrix};$$

$$D_4 = \begin{pmatrix} 4.5100e+3 & -0.4791e+3 & -2.6615e+3 \\ -1.0450e+3 & 0.4596e+3 & 0.8352e+3 \\ 0.2140e+3 & 0.0769e+3 & 0.1931e+3 \end{pmatrix}.$$

The solvents of the above characteristic matrix polynomial are determined and classified according to step 2 of the algorithm. In order to be more informative, we have provided the corresponding spectrum of each solvent.

$$R_1 = \begin{pmatrix} -21.1725 & -0.5026 & -0.2009 \\ 0.5345 & -19.8128 & 0.0564 \\ -0.0639 & 1.1596 & -19.0147 \end{pmatrix}$$

whose spectrum is

{-21.0000, -20.0000, -19.0000}

$$R_2 = \begin{pmatrix} -18.1725 & -0.5026 & -0.2009 \\ 0.5344 & -16.8128 & 0.0564 \\ -0.0639 & 1.1597 & -19.0147 \end{pmatrix}$$

{-18.0000, -17.0000, -16.0000}

$$R_3 = \begin{pmatrix} -13.4865 & 1.8626 & 8.1876 \\ 1.4882 & -1.5966 & -1.1418 \\ 1.8800 & -0.4719 & -2.6170 \end{pmatrix} \qquad \{-15.0000,\ -1.5000,\ -1.2000\}$$

$$R_4 = \begin{pmatrix} -0.5107 & -0.2273 & -0.2632 \\ -0.9887 & -0.5086 & 0.4134 \\ 0.6688 & -0.3942 & -1.2807 \end{pmatrix} \qquad \{-1.0000,\ -0.8000,\ -0.4500\}$$

Then according to the eigenspectrum of the above solvents, a fair approximation is expected for a reduced model containing the first 8 large eigenvalues. This is obtained for a relative error of 2e-9, with a simplified scalar characteristic polynomial Δ(s) of order 8 in the form a rational matrix

$$\hat{H}(s) = \frac{1}{\Delta(s)}[G(s)]$$

where

$$\Delta(s) = a_0 s^8 + a_1 s^7 + a_2 s^6 + a_3 s^5 + a_4 s^4 + a_5 s^3 + a_6 s^2 + a_7 s + a_8$$

with

$$a_0 = 1;\ a_1 = 127.4434;\ a_2 = 6.9726e+3;\ a_3 = 2.1275e-5;\ a_4 = 3.9256e+6;$$
$$a_5 = 4.4203e+7;\ a_6 = 2.8824e+8;\ a_7 = 9.2905e-8;\ a_8 = 8.7591e+8$$

and

$$G(s) = G_0 s^7 + G_1 s^6 + G_2 s^5 + G_3 s^4 + G_4 s^3 + G_5 s^2 + G_6 s + G_7$$

where

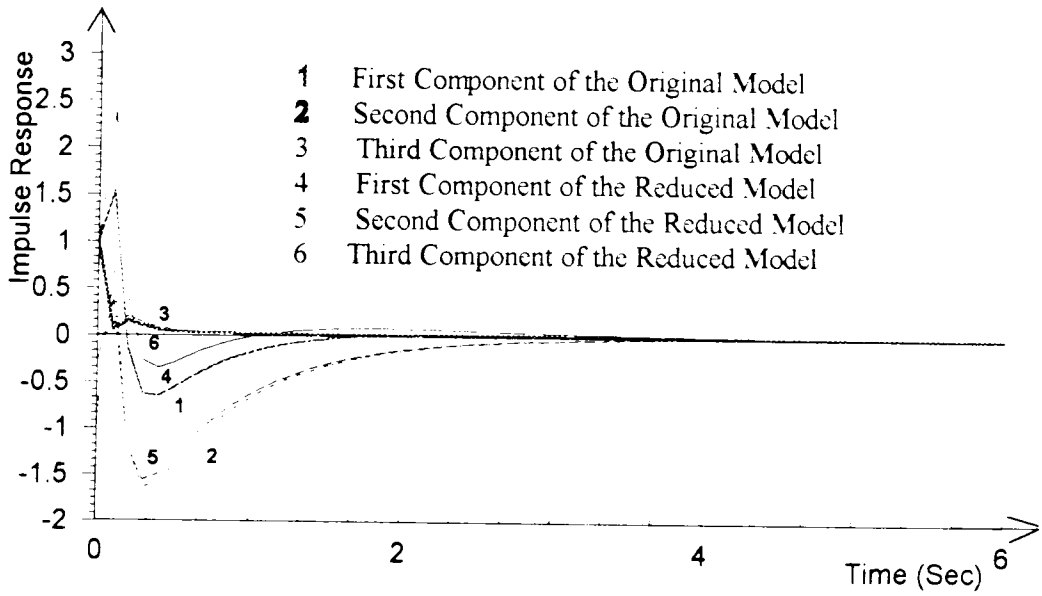
$$G_0 = \begin{pmatrix} 0.8712 & -0.5283 & -0.6162 \\ -0.1840 & 0.5691 & 0.6834 \\ -0.1788 & -0.7828 & 0.0830 \end{pmatrix};\quad G_1 = \begin{pmatrix} 89.7467 & -19.3896 & -11.9846 \\ -41.6038 & -54.5320 & 102.5558 \\ 21.4312 & -33.1551 & 38.2828 \end{pmatrix};$$

$$G_2 = \begin{pmatrix} 3664 & 660 & 1796 \\ -3157 & -7381 & 13597 \\ 2308 & 680 & 3097 \end{pmatrix};\qquad G_3 = \begin{pmatrix} 73.5e+3 & 43.2e+3 & 90.9e+3 \\ -112.9e-3 & -289.4e+3 & 468.7e-3 \\ 74.5e+3 & 59.4e+3 & 103.2e+3 \end{pmatrix};$$

$$G_4 = \begin{pmatrix} 68.9e+4 & 66.1e+4 & 160.4e+4 \\ -211.3e+4 & -539.3e+4 & 51.7e+4 \\ 104.8e+4 & 109.2e+4 & 156.1e+5 \end{pmatrix};\quad G_5 = \begin{pmatrix} 11.8e+5 & -2.2e+5 & 91.3e+5 \\ -204.5e+5 & -505.9e+5 & 813e+5 \\ 55.8e+5 & 43.6e+5 & 73.9e+5 \end{pmatrix};$$

$$G_6 = \begin{pmatrix} 24.8e+6 & -2e+6 & -42.4e+6 \\ -0.3e+6 & -209.8e+6 & 353.7e+6 \\ -7.1e+6 & -65.6e+6 & -55.6e+6 \end{pmatrix};\quad G_7 = \begin{pmatrix} 126.7e+6 & -519e+6 & -481.5e+6 \\ -89.3e+6 & -201.3e+6 & 390.7e+6 \\ -114.4e+6 & -527e+6 & -522.1e+6 \end{pmatrix}.$$

The impulse responses of the three components (y_1, y_2 and y_3) of the outputs of the original and reduced models are compared in the figure below. It is easy to see that the two responses are close to each other.



Note that a reduced model with a higher order will give a better approximation. This is an important requirement for any method of model reduction.

Example 2

As a second application for the solvents based model reduction procedure, we have chosen a 6^{th} order 2×3 matrix transfer function in the form of a matrix fraction description defined by:

$$H(s) = \left[N_1 s^5 + N_2 s^4 + N_3 s^3 + N_4 s^2 + N_5 s + N_6\right]\left[D_0 s^5 + D_1 s^5 + D_2 s^4 + D_3 s^3 + D_4 s^2 + D_5 s + D_6\right]^{-1}$$

where

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad N_2 = \begin{pmatrix} 3333 & 4444 & 5445 \\ -2212 & -2188 & 3199 \end{pmatrix}; \quad N_3 = \begin{pmatrix} -155 & -231 & 112 \\ 399 & 442 & 661 \end{pmatrix};$$

$$N_4 = \begin{pmatrix} 665 & -44 & 144 \\ 355 & 144 & 994 \end{pmatrix}; \quad N_5 = \begin{pmatrix} 774 & 885 & 221 \\ 121 & 477 & 999 \end{pmatrix}; \quad N_6 = \begin{pmatrix} 997 & 699 & 365 \\ 230 & 487 & 656 \end{pmatrix};$$

and

$$D_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 261.7789 & -37.744 & -41.9195 \\ -36.5803 & 196.1755 & 24.5753 \\ 43.6054 & -13.1646 & 157.6456 \end{pmatrix};$$

$$D_2 = \begin{pmatrix} 22728 & -6890 & -7576 \\ -6974 & 11780 & 4127 \\ 7369 & -3027 & 4565 \end{pmatrix}; \quad D_3 = \begin{pmatrix} 6.786e+5 & -3.2278e+5 & -3.5106e+5 \\ -3.3954e+5 & 2.0639e+5 & 1.8095e+5 \\ 3.2452e+5 & -1.6577e+5 & -1.3998e+5 \end{pmatrix};$$

$$D_4 = \begin{pmatrix} 1.9449e+6 & -1.0117e+6 & -1.0146e+6 \\ -1.0116e+6 & 0.5904e+6 & 0.5261e+6 \\ 0.9608e+6 & -0.5239e+6 & -0.4602e+6 \end{pmatrix}; \quad D_5 = \begin{pmatrix} 1.6780e+6 & -0.9554e+6 & -0.8537e+6 \\ -0.8875e+6 & 0.5454e+6 & 0.4454e+6 \\ 0.8402e+6 & -0.4945e+6 & -0.4059e+6 \end{pmatrix};$$

$$D_6 = \begin{pmatrix} 3.9672e+5 & -2.4898e+5 & -1.9125e+5 \\ -2.1205e+5 & 1.3691e+5 & 1.0063e+5 \\ 2.0022e+5 & -1.2824e+5 & -0.9373e+5 \end{pmatrix}.$$

The obtained solvents of the characteristic matrix polynomial of the original model with their associated spectrums are as follows:

$$R_1 = \begin{pmatrix} -91.6454 & 4.5195 & -6.3442 \\ -6.0781 & -90.2170 & -3.2531 \\ -2.9871 & -6.6767 & -93.1376 \end{pmatrix}$$ whose spectrum gives {-88.0000 ±5.0000i, -99.0000}

$$R_2 = \begin{pmatrix} -84.5276 & 1.5596 & 1.7489 \\ 5.8657 & -84.4691 & -2.1516 \\ -3.6509 & 2.5557 & -80.0034 \end{pmatrix}$$ {-85.0000, -83.0000, -81.0000}

$$R_3 = \begin{pmatrix} -80.2035 & 31.9386 & 41.0512 \\ 204.1226 & -85.7445 & -106.8212 \\ -160.5958 & 65.9966 & 81.9480 \end{pmatrix}$$ {-79.0000, -3.0000, -2.0000}

$$R_4 = \begin{pmatrix} -1.7291 & 0.2339 & 0.2623 \\ 0.8798 & -1.7202 & -0.3226 \\ -0.5476 & 0.3833 & -1.0506 \end{pmatrix}$$ {-1.7998, -1.5001, -1.2000}

$$R_5 = \begin{pmatrix} -0.8959 & 0.1870 & 0.1450 \\ 0.1284 & -0.7916 & 0.0317 \\ 0.0315 & 0.0934 & -0.7125 \end{pmatrix}$$ {-1.0000, -0.8000, -0.6000}

$$R_6 = \begin{pmatrix} -0.3192 & 0.1438 & 0.1029 \\ 0.1091 & -0.2275 & -0.0033 \\ 0.0089 & 0.0515 & -0.1534 \end{pmatrix}$$ {-0.4000, -0.2000, -0.1000}

The following reduced model is obtained for a relative error equal to 3e-22. Clearly, this value is very small and shows that the truncated subsystem is negligible in front of the original model in terms of singular values magnitude, hence a good approximation is expected. The reduced model is a rational matrix with a common denominator $\Delta(s)$ of order 11 such that:

$$\hat{H}(s) = \frac{1}{\Delta(s)}[G(s)]$$

where

$$\Delta(s) = a_0 s^{11} + a_1 s^{10} + a_2 s^9 + a_3 s^8 + a_4 s^7 + a_5 s^6 + a_6 s^5 + a_7 s^4 + a_8 s^3 + a_9 s^2 + a_{10} s + a_{11}$$

with

$$a_0 = 1;\ a_1 = 611.2816;\ a_2 = 1.6075e+5;\ a_3 = 2.3633e+7;\ a_4 = 2.108\text{-}e+9;$$
$$a_5 = 1.1542e+11;\ a_6 = 3.7008e+12;\ a_7 = 6.0725e+13;\ a_8 = 3.6194e+14;$$
$$a_9 = 9.6655e-14;\ a_{10} = 1.1945e+15;\ a_{11} = 5.579e+14.$$

and

$$G(s) = G_0 s^{10} + G_1 s^9 + G_2 s^8 + G_3 s^7 + G_4 s^6 + G_5 s^5 + G_6 s^4 + G_7 s^3 + G_8 s^2 + G_9 s + G_{10}$$

where

$$G_0 = \begin{pmatrix} 0.7971 & -3.1581 & -2.9157 \\ -0.1658 & 1.2233 & 0.5531 \end{pmatrix};\quad G_1 = \begin{pmatrix} 3558.7 & 2555.1 & 3708.2 \\ -2276.7 & -1636.7 & 3512 \end{pmatrix};$$

$$G_2 = \begin{pmatrix} 1.10e+6 & 1.55e+6 & 2.04e+6 \\ -1.01e+6 & -0.84e+6 & 1.49e+6 \end{pmatrix};\quad G_3 = \begin{pmatrix} 1.29e+8 & 3.02e+8 & 3.92e+8 \\ -1.82e+8 & -1.53e+8 & 2.63e+8 \end{pmatrix};$$

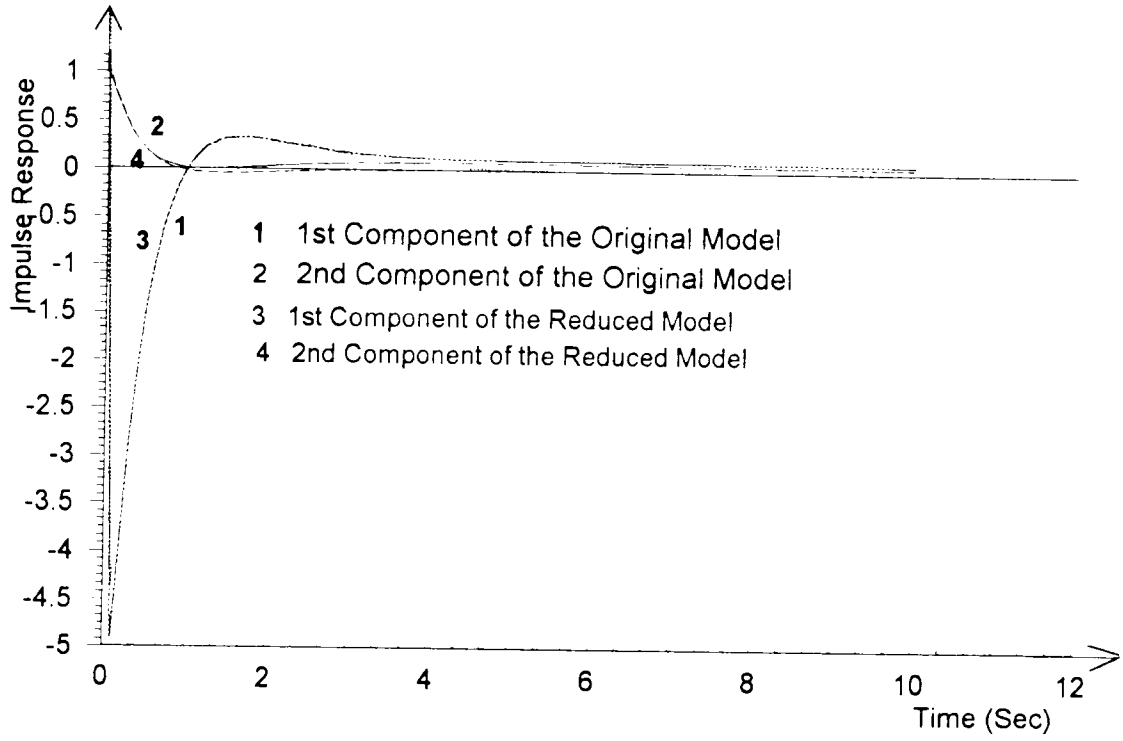
$$G_4 = \begin{pmatrix} 6.3e+9 & 28e+9 & 36.3e+9 \\ -16.9e+9 & -13.4e+9 & 23.6e+9 \end{pmatrix};\quad G_5 = \begin{pmatrix} 0.07e+12 & 1.27e+12 & 1.65e+12 \\ -0.8e+12 & -0.58e-12 & 1.09e+12 \end{pmatrix};$$

$$G_6 = \begin{pmatrix} -0.23e+13 & 2.17e+13 & 2.96e+13 \\ -1.65e+13 & -1.02e+13 & 2.21e+13 \end{pmatrix};\quad G_7 = \begin{pmatrix} -1.58e+13 & -5.5e+13 & -2.4e+13 \\ -4.88e+13 & -1.26e+13 & 8.25e+13 \end{pmatrix};$$

$$G_8 = \begin{pmatrix} -0.6e+14 & -8.35e-14 & -7.57e+14 \\ -0.64e+14 & 0.7e+14 & 1.79e+14 \end{pmatrix};\quad G_9 = \begin{pmatrix} -0.12e+15 & -1.96e+15 & -1.81e+15 \\ -0.11e+15 & 0.16e+15 & 0.38e+15 \end{pmatrix};$$

$$G_{10} = \begin{pmatrix} -0.09e+15 & -1.35e+15 & -1.23e+15 \\ -0.12e+15 & 0.08e+15 & 0.32e+15 \end{pmatrix}.$$

The impulse response curves of the full and reduced models are compared in the figure shown below.



Example 3

The next example is also a 6^{th} order matrix transfer function with solvents satisfying a dominance criterion, but very close to each other as will be apparent from the solvents spectrums.

$$H(s)=\left[N_1 s^5+N_2 s^4+N_3 s^3+N_4 s^2+N_5 s+N_6\right]\left[D_0 s^6+D_1 s^5+D_2 s^4+D_3 s^3+D_4 s^2+D_5 s+D_6\right]^{-1}$$

where the "numerator" coefficients are

$$N_1=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix};\quad N_2=\begin{pmatrix}3333e+4 & 1444e+4 & 5555e+4\\ -2222e+4 & -8888e+4 & 9889e+4\\ 5554e+4 & 9997e+4 & 4424e+4\end{pmatrix};$$

$$N_3=\begin{pmatrix}-5555e+4 & -5551e+4 & 2222e+4\\ 1521e+4 & 2552e+4 & 1221e+4\\ 3333e+4 & 8889e+4 & 4222e+4\end{pmatrix};\quad N_4=\begin{pmatrix}1111e+3 & -5444e+3 & 4114e+3\\ 2222e+3 & 3433e+3 & 5444e+3\\ 5475e+3 & 6869e+3 & 4112e+3\end{pmatrix};$$

$$N_5=\begin{pmatrix}6656e+4 & 4486e+4 & 8777e+4\\ 2555e+4 & 3666e+4 & 1411e+4\\ 2565e+4 & 3565e-4 & 6486e+4\end{pmatrix};\quad N_6=\begin{pmatrix}4799e+4 & 9818e+4 & 9777e+4\\ 7655e+4 & 3583e+4 & 8788e-4\\ 9876e+4 & 8598e+4 & 6989e-4\end{pmatrix}.$$

The characteristic matrix polynomial coefficients are:

$$D_0=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix};\quad D_1=\begin{pmatrix}480.5753 & -5.5798 & -0.9693\\ -7.8798 & 172.9928 & 9.8446\\ 5.0091 & 5.1369 & 465.4315\end{pmatrix};$$

$$D_2 = \begin{pmatrix} 9.6089e+4 & -0.2234e+4 & -0.0400e+4 \\ -0.3140e+4 & 9.3069e+4 & 0.3903e+4 \\ 0.1984e+4 & 0.2028e+4 & 9.0061e+4 \end{pmatrix}; \quad D_3 = \begin{pmatrix} 1.0231e+7 & -0.0357e+7 & -0.0066e+7 \\ -0.0500e+7 & 0.9751e+7 & 0.0618e+7 \\ 0.0314e+7 & 0.0320e+7 & 0.9273e+7 \end{pmatrix};$$

$$D_4 = \begin{pmatrix} 6.1181e+8 & -0.2854e+8 & 5.3573e+8 \\ -0.3973e+8 & 5.7366e+8 & 0.4881e+8 \\ 0.2477e+8 & 0.2514e+8 & 5.3573e+8 \end{pmatrix}; \quad D_5 = \begin{pmatrix} 1.9481e+10 & -0.1139e+10 & -0.0224e+10 \\ -0.1576e+10 & 1.7968e+10 & 0.1925e+10 \\ 0.0976e+10 & 0.0986e+10 & 1.6466e+10 \end{pmatrix};$$

$$D_6 = \begin{pmatrix} 2.5801e+11 & -0.1814e+11 & -0.0368e+11 \\ -0.2498e+11 & 2.3407e+11 & 0.3030e+11 \\ 0.1534e+11 & 0.1545e+11 & 2.1031e+11 \end{pmatrix}.$$

The computation of the solvents of the above characteristic matrix polynomial yields

$$R_1 = \begin{pmatrix} -89.3686 & 1.0758 & 0.4898 \\ 1.1535 & -88.3098 & -0.9243 \\ -0.4290 & -0.2753 & -87.3215 \end{pmatrix}$$ whose eigenvalues are {-90.000, -88.000, -87.000}

$$R_2 = \begin{pmatrix} -85.6565 & 0.5731 & 0.1861 \\ 0.7074 & -84.9929 & -0.7329 \\ -0.3611 & -0.3233 & -84.3505 \end{pmatrix}$$ {-86.0013, -84.9988, -83.9997 }

$$R_3 = \begin{pmatrix} -82.6565 & 0.5731 & 0.1861 \\ 0.7047 & -81.9929 & -0.7329 \\ -0.3611 & -0.3233 & -81.3505 \end{pmatrix}$$ {-83.0000, -82.0000, -80.9999}

$$R_4 = \begin{pmatrix} -79.6565 & 0.5731 & 0.1861 \\ 0.7047 & -78.9929 & -0.7329 \\ -0.3611 & -0.3233 & -78.3505 \end{pmatrix}$$ {-80.0000, -79.0000, -77.9999}

$$R_5 = \begin{pmatrix} -76.3787 & 0.9248 & -0.4019 \\ 1.9838 & -74.3733 & -3.4401 \\ -1.8271 & -2.1794 & -72.2480 \end{pmatrix}$$ {-76.9999, -76.0000, -70.0000}

$$R_6 = \begin{pmatrix} -66.8585 & 1.8599 & 0.3231 \\ 2.6257 & -64.331 & -3.2815 \\ -1.6697 & -1.7123 & -61.8105 \end{pmatrix}$$ {-68.0000, -65.0000, -60.0000}

As it may be seen, all the solvents show the same spectrum (the eigenvalues are almost of the same m tud); In other words there is no apparent dominance between the 6 solvents. This is the reason for which the only

plausible approximation given by our algorithm is the original model itself. Hence, we omit to give the reduced model transfer function and the impulse responses comparison of the two models.

Example 4

We finish with the solvents based method by selecting a 6^{th} order 3×3 matrix transfer function with "numerator" and "denominator" coefficients given as follows:

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad N_2 = \begin{pmatrix} 333 & 111 & 555 \\ -222 & -888 & 999 \\ 555 & 999 & 444 \end{pmatrix};$$

$$N_3 = \begin{pmatrix} -5e+3 & -55e+3 & 22e+3 \\ 15e+3 & 25e+3 & 12e+3 \\ 33e+3 & 88e+3 & 42e+3 \end{pmatrix}; \quad N_4 = \begin{pmatrix} 11e+4 & -54e+4 & 14e+4 \\ 22e+4 & 33e+4 & 44e+4 \\ 75e+4 & 69e+4 & 12e+4 \end{pmatrix};$$

$$N_5 = \begin{pmatrix} 56e+5 & 46e+5 & 7e+5 \\ 25e+5 & 36e+5 & 11e+5 \\ 25e+5 & 35e+5 & 66e+5 \end{pmatrix}; \quad N_6 = \begin{pmatrix} 47e+5 & 88e+5 & 77e+5 \\ 65e+5 & 35e+5 & 78e+5 \\ 76e+5 & 58e+5 & 69e+5 \end{pmatrix};$$

and

$$D_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 459.7188 & -192.8817 & 0.5155 \\ 46.1736 & 240.2222 & 14.8614 \\ 72.1022 & -93.6154 & 311.0590 \end{pmatrix};$$

$$D_2 = \begin{pmatrix} 7.9210e+4 & -5.6742e+4 & -0.3242e+4 \\ 1.4540e+4 & 1.3298e+4 & 0.1939e+4 \\ 2.2084e+4 & -2.7025e+4 & 3.1471e+4 \end{pmatrix}; \quad D_3 = \begin{pmatrix} 6.1847e+6 & -5.7046e+6 & -0.6664e+6 \\ 1.5608e+6 & -0.5751e+6 & -0.0531e+6 \\ 2.3035e+6 & -2.6648e+6 & 0.9912e+6 \end{pmatrix}$$

$$D_4 = \begin{pmatrix} 2.0018e+8 & -2.1178e+8 & -0.3650e+8 \\ 0.6126e+8 & -0.5465e+8 & -0.1090e+8 \\ 0.8801e+8 & -0.9708e+8 & -0.0554e+8 \end{pmatrix}; \quad D_5 = \begin{pmatrix} 1.4395e+9 & -1.6472e+9 & -0.2471e+9 \\ 0.4529e+9 & -0.4733e+9 & -0.0798e+9 \\ 0.6488e+9 & -0.7451e+9 & -0.0827e+9 \end{pmatrix}$$

$$D_6 = \begin{pmatrix} 2.5707e+9 & -3.1333e+9 & -0.4029e+9 \\ 0.8049e+9 & -0.9201e+9 & -0.1335e+9 \\ 1.1564e+9 & -1.3889e+9 & -0.1659e+9 \end{pmatrix}.$$

The corresponding solvents are computed and given below with their corresponding spectrums.

$$R_1 = \begin{pmatrix} -120.0000 & 30.0000 & -30.0000 \\ -1.2500 & -95.0000 & -22.5000 \\ -2.5000 & 15.0000 & -130.0000 \end{pmatrix}$$

whose eigenvalues are

{-120.0000, -115.0000, -110.0000}

$$R_2 = \begin{pmatrix} -93.0765 & -27.1166 & 24.4240 \\ 5.7696 & -121.3471 & 18.2699 \\ 1.9232 & -12.1160 & -85.5764 \end{pmatrix} \quad \{-105.0000, -100.0000, -95.0000\}$$

$$R_3 = \begin{pmatrix} -90.0000 & 30.0000 & -30.0000 \\ -1.2500 & -65.0000 & -22.5000 \\ -2.5000 & 15.0000 & -100.0000 \end{pmatrix} \quad \{-90.0000, -85.0000, -80.0000\}$$

$$R_4 = \begin{pmatrix} -154.8688 & 36.0344 & 162.3556 \\ -41.1307 & 3.7307 & 43.4907 \\ -62.8111 & 15.6643 & 61.1300 \end{pmatrix} \quad \{-75.0000, -8.0000, -7.0000\}$$

$$R_5 = \begin{pmatrix} -2.2628 & -3.8425 & 0.0527 \\ 2.2635 & -8.1582 & -0.0526 \\ 6.8951 & -9.7372 & -4.5790 \end{pmatrix} \quad \{-6.0000, -5.0000, -4.0000\}$$

$$R_6 = \begin{pmatrix} 0.0010 & -3.6677 & 0.3332 \\ 1.7508 & -4.9175 & 0.0832 \\ 4.2517 & -8.0853 & -1.0835 \end{pmatrix} \quad \{-3.0000, -2.0000, -1.0000\}$$

A convincing approximation of order 10 is derived for a relative error equal to $1e-11$, with scalar characteristic polynomial coefficients given by:

$a_0 = 1$; $a_1 = 975.7036$; $a_2 = 4.2738e+5$; $a_3 = 1.1067e+8$; $a_4 = 1.8762e+10$; $a_5 = 2.1758e+12$;
$a_6 = 1.7480e+14$; $a_7 = 9.6056e+15$; $a_8 = 3.4555e+17$; $a_9 = 7.3481e+18$; $a_{10} = 7.0139e+19$.

The numerator matrix polynomial coefficients are:

$$G_0 = \begin{pmatrix} 0.8933 & -0.2295 & 0.1722 \\ 0.1799 & 1.3326 & -0.6697 \\ -0.7605 & 1.6838 & 1.2935 \end{pmatrix}; \quad G_1 = \begin{pmatrix} 1.0939e+3 & -0.3291e+3 & 0.6374e+3 \\ -0.2723e+3 & 0.3664e+3 & 0.3814e+3 \\ -0.1700e+3 & 2.6249e+3 & 1.3671e+3 \end{pmatrix};$$

$$G_2 = \begin{pmatrix} 0.4677e+6 & -0.1872e+6 & 0.3926e+6 \\ -0.2278e+6 & -0.0488e+6 & 0.4374e+6 \\ -0.0490e+6 & 1.6484e+6 & 0.6107e+6 \end{pmatrix}; \quad G_3 = \begin{pmatrix} 1.0883e+8 & -0.7364e+8 & 1.1289e+8 \\ -0.7408e+8 & -0.1880e+8 & 1.4262e+8 \\ -0.4240e+8 & 5.4917e+8 & 1.5415e+8 \end{pmatrix};$$

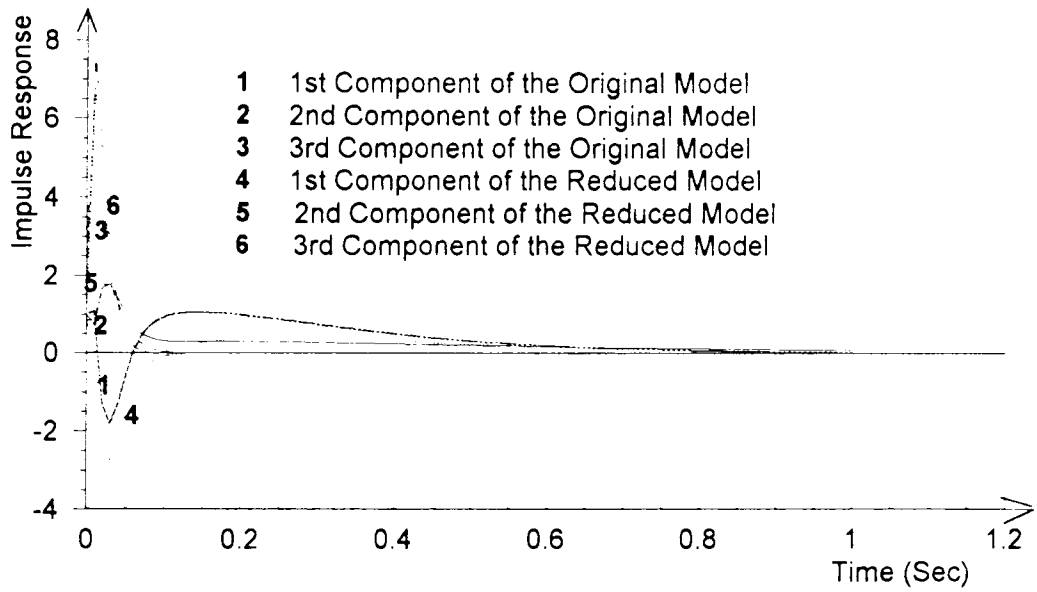
$$G_4 = \begin{pmatrix} 0.1695e+11 & -0.2011e+11 & 0.1879e+11 \\ -0.1406e+11 & 0.0247e+11 & 0.2363e+11 \\ -0.1509e+11 & 1.0922e+11 & 0.2447e+11 \end{pmatrix}; G_5 = \begin{pmatrix} 0.1982e+13 & -0.3542e+13 & 0.1957e+13 \\ -0.1757e+13 & 0.1459e+13 & 0.2216e+13 \\ -0.2674e+13 & 1.3676e+13 & 0.2549e+13 \end{pmatrix};$$

$$G_6 = \begin{pmatrix} 0.1755e+15 & -0.3888e+15 & 0.1300e+15 \\ -0.1489e+15 & 0.2250e+15 & 0.1148e+15 \\ -0.2678e+15 & 1.0912e+15 & 0.1749e+15 \end{pmatrix}; G_7 = \begin{pmatrix} 1.0744e+16 & -2.5641e+16 & 0.5368e+16 \\ -0.8269e+16 & 1.7130e+16 & 0.2616e+16 \\ -1.5533e+16 & 5.3943e+16 & 0.7636e+16 \end{pmatrix}$$

$$G_8 = \begin{pmatrix} 0.3895e+18 & -0.9304e+18 & 0.1255e+18 \\ -0.2705e+18 & 0.6641e+18 & -0.0118e+18 \\ -0.4883e+18 & 1.5082e+18 & 0.1928e+18 \end{pmatrix};$$

$$G_9 = \begin{pmatrix} 0.6166e+19 & -1.4286e+19 & 0.1270e+19 \\ -0.3926e+19 & 1.0507e+19 & -0.1171e+19 \\ -0.6457e+19 & 1.8246e+19 & 0.2144e+19 \end{pmatrix}.$$

The comparison between the original and reduced models curves is shown in the following figure.



5.5 Spectral Factors Based Model Reduction Procedure

The spectral factors-based method also inputs two sets of matrices: the matrix coefficients of the "numerator" and the coefficients of the characteristic matrix polynomial of the transfer function given in the form of a matrix fraction description. As for the solvents-based method, we need to supply the allowed relative error *RE*.

This algorithm also searches for the roots of the characteristic matrix polynomial, but this time it searches for the spectral factors and classifies them in order of decreasing dominance. After the formation of the original model Hankel matrix and the determination of its singular values, the characteristic matrix polynomial of the reduced model is built using matrix polynomial multiplication while the new "numerator" is determined using matrix polynomial division.

At this level, the Hankel matrix difference between the original and the reduced models is formed and its singular values are computed using the Matlab command SVD. Then, when the relative error is satisfied, the reduced model is block-decoupled using the Vandermande matrix whose solvents are obtained from the corresponding spectral factors of the reduced model using a similarity transformation with the help of the Matlab built in commands Kronecker (KRON) and (VEC). Afterward, the last sub system is diagonalized using the Matlab command eigen (EIG) allowing extraction of the dominant part of the last added subsytem. Finally, the two models are compared using the function impulse (IMPULSE) supplied also with Matlab.

The problem of lack of examples satisfying our assumptions arises also for this method. In order to provide a reliable test for the algorithm, we have constructed our own examples and, in the following, we outline the procedure.

The choice of the "numerator" matrix coefficients is random, while the coefficients of the characteristic matrix polynomial are derived as follows:

First the spectral factors are formed using the relation

$$Q = MDM^{-1} \tag{5.13}$$

where Q is the spectral factor,

D is a Jordan matrix containing the eigenvalues of Q and

M is a matrix of eigenvectors of Q.

Then, assuming that $D_R(s)$ has a complete set of spectral factors $\{Q_1, Q_2, \ldots, Q_l\}$, it can be writen as a product of l linear spectral factors

$$\begin{aligned} D_R(s) &= (sI - Q_1)(sI - Q_2)\cdots(sI - Q_l) \\ &= D_0 s^l + D_1 s^{l-1} + D_2 s^{l-2} + \cdots + D_{l-1} s + D_l \end{aligned} \tag{5.14}$$

To illustrate the efficiency of the spectral factors based model reduction procedure we have chosen a set of examples. These examples are generated in such a way that the spectral factors of the characteristic matrix polynomial satisfy a dominance criterion.

Example 1

This example is a 6^{th} order 2×3 transfer function in the form of a right matrix fraction description described by:

$$H(s) = \left[N_1 s^5 + N_2 s^4 + N_3 s^3 + N_4 s^2 + N_5 s + N_6\right]\left[D_0 s^6 + D_1 s^5 + D_2 s^4 + D_3 s^3 + D_4 s^2 + D_5 s + D_6\right]^{-1}$$

where

$$N_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad N_1 = \begin{pmatrix} 3333 & 4444 & 5445 \\ -2212 & -2188 & 3199 \end{pmatrix}; \quad N_2 = \begin{pmatrix} -155 & -231 & 122 \\ 399 & 442 & 661 \end{pmatrix};$$

$$N_3 = \begin{pmatrix} 665 & -44 & 144 \\ 355 & 144 & 994 \end{pmatrix}; \quad N_4 = \begin{pmatrix} 774 & 885 & 221 \\ 121 & 477 & 999 \end{pmatrix}; \quad N_5 = \begin{pmatrix} 997 & 699 & 365 \\ 230 & 487 & 656 \end{pmatrix}.$$

And the matrix coefficients of the characteristic matrix polynomial are

$$D_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 259.3207 & -38.5824 & -36.9661 \\ -205.0275 & 263.1699 & 112.5201 \\ 167.7410 & -62.4038 & 93.1095 \end{pmatrix};$$

$$D_2 = \begin{pmatrix} 2.4686e+4 & -0.7223e+4 & -0.8133e+4 \\ -3.5612e+4 & 2.2989e+4 & 1.9048e+4 \\ 2.8659e+4 & -1.1277e+4 & -0.6488e+4 \end{pmatrix}; \quad D_3 = \begin{pmatrix} 0.8648e+6 & -0.3464e+6 & -0.4352e+6 \\ -1.5995e+6 & 0.6940e+6 & 0.8363e+6 \\ 1.2690e+6 & -0.5253e+6 & -0.6300e+6 \end{pmatrix};$$

$$D_4 = \begin{pmatrix} 2.5399e+6 & -1.1337e+6 & -1.3022e+6 \\ -4.7714e+6 & 2.1993e+6 & 2.4794e+6 \\ 3.7826e+6 & -1.7149e+6 & -1.9243e+6 \end{pmatrix}; \quad D_5 = \begin{pmatrix} 2.1970e+6 & -1.1315e+6 & -1.1204e+6 \\ -4.1498e+6 & 2.1691e+6 & 2.1264e+6 \\ 3.2885e+6 & -1.7085e+6 & -1.6677e+6 \end{pmatrix}$$

$$D_6 = \begin{pmatrix} 5.1561e+5 & -3.0925e+5 & -2.5262e+5 \\ -9.7671e+5 & 5.8909e+5 & 4.7866e+5 \\ 7.7368e+5 & -4.6607e+5 & -3.7761e+5 \end{pmatrix}.$$

As it has been stated, the algorithm starts by determining the spectral factors of the above characteristic matrix polynomial, leading to:

$$Q_1 = \begin{pmatrix} -91.6454 & 4.5195 & -6.3442 \\ -6.0781 & -90.2170 & -3.2531 \\ -2.9871 & -6.6767 & -93.1376 \end{pmatrix}$$ whose spectrum gives {-88.0000 ±5.0000i, -99.0000}

$$Q_2 = \begin{pmatrix} -84.5276 & 1.5596 & 1.7489 \\ 5.8657 & -84.4691 & -2.1516 \\ -3.6509 & 2.5557 & -80.0034 \end{pmatrix}$$ {-85.0000, -83.0000, -81.0000}

$$Q_3 = \begin{pmatrix} -80.2035 & 31.9386 & 41.0512 \\ 204.1226 & -85.7445 & -106.8212 \\ -160.5958 & 65.9966 & 81.9480 \end{pmatrix}$$ {-79.0000, -3.0000, -2.0000}

$$Q_4 = \begin{pmatrix} -1.7291 & 0.2339 & 0.2623 \\ 0.8798 & -1.7202 & -0.3226 \\ -0.5476 & 0.3833 & -1.0506 \end{pmatrix}$$ {-1.8000, -1.5000, -1.2000}

$$Q_5 = \begin{pmatrix} -0.8959 & 0.1870 & 0.1450 \\ 0.1284 & -0.7916 & 0.0317 \\ 0.0315 & 0.0934 & -0.7125 \end{pmatrix}$$ {-1.0000, -0.6000, -0.8000}

$$Q_6 = \begin{pmatrix} -0.3192 & 0.1438 & 0.1029 \\ 0.1091 & -0.2275 & -0.0033 \\ 0.0089 & 0.0515 & -0.1534 \end{pmatrix}$$ {-0.4000, -0.2000, -0.1000}

From the eigenspectrum of the above spectral factors, it is apparent that a good approximation can be obtained for a reduced model containing the first 7 large eigenvalues. This is achieved for a relative error equal to $8e-4$, giving a reduced model under the from

$$\hat{H}(s) = \frac{1}{\Delta(s)}[G(s)]$$

where the coefficients of the scalar characteristic polynomial $\Delta(s)$ are:

$a_0 = 1$; $a_1 = 603.0001$; $a_2 = 1.5573e+5$; $a_3 = 2.2328e+7$; $a_4 = 1.9196e+9$;
$a_5 = 9.8958e+12$ $a_6 = 2.8324e+12$; $a_7 = 3.4722e+13$.

and the coefficients of the matrix polynomial $G(s)$ are as follows:

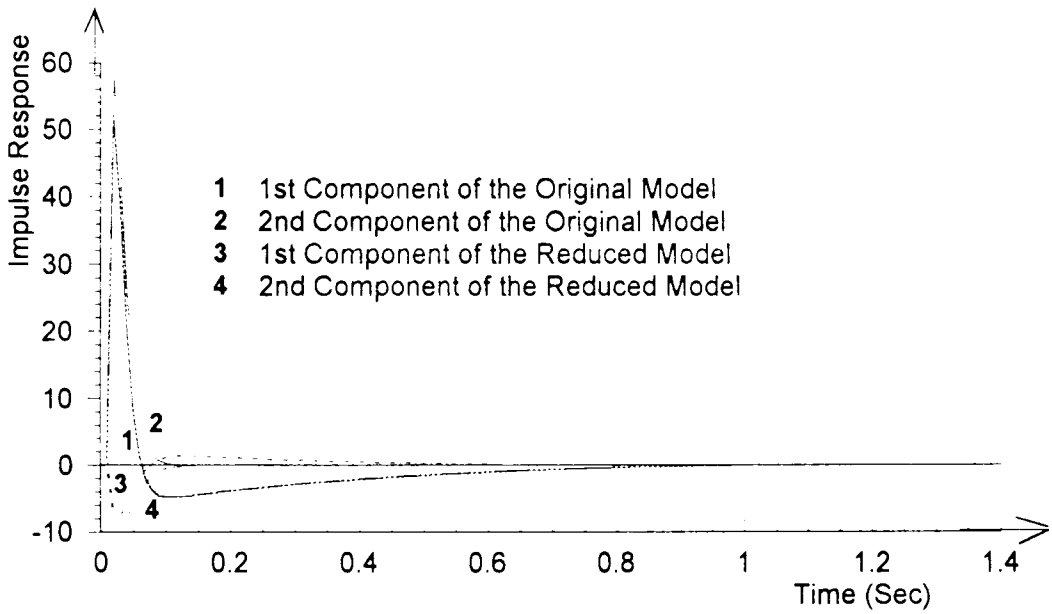
$$G_0 = \begin{pmatrix} 0.2120 & 2.9378 & 4.2110 \\ -9.9602 & 1.5097 & 7.2005 \end{pmatrix}; \quad G_1 = \begin{pmatrix} 3.2056e+3 & 6.2480e+3 & 8.0109e+3 \\ -7.9832e+3 & -1.5426e+3 & 7.4066e+3 \end{pmatrix};$$

$$G_2 = \begin{pmatrix} 1.0665e+6 & 2.4448e+6 & 3.0623e+6 \\ -3.2073e+6 & -0.5063e+6 & 2.8652e+6 \end{pmatrix}; \quad G_3 = \begin{pmatrix} 1.3094e+8 & 4.1461e+8 & 5.1615e+8 \\ -6.4565e+8 & -0.4599e+8 & 5.3436e+8 \end{pmatrix};$$

$$G_4 = \begin{pmatrix} 0.6696e+10 & 3.5793e+10 & 4.4771e+10 \\ -6.6770e+10 & -0.0503e+10 & 5.1921e+10 \end{pmatrix}; \quad G_5 = \begin{pmatrix} 0.0890e+12 & 1.5514e+12 & 1.9672e+12 \\ -3.4166e+12 & 0.1110e+12 & 2.5409e+12 \end{pmatrix};$$

$$G_6 = \begin{pmatrix} -0.1931e+13 & 2.6884e+13 & 3.4826e+13 \\ -6.8753e+13 & 0.3837e+13 & 4.9592e+13 \end{pmatrix}.$$

It is interesting to observe that the impulse response error between the original and reduced models is almost negligible. This is illustrated by the following figure.



Note also here that a better approximation can be obtained for higher order reduced models. This is done by choosing a smaller relative error RE.

Example 2

The next example is a 6^{th} order 3×3 matrix transfer function with "numerator" coefficients given by:

$$N_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad N_1 = \begin{pmatrix} 333 & 111 & 555 \\ -222 & -888 & 999 \\ 555 & 999 & 444 \end{pmatrix}; \quad N_2 = \begin{pmatrix} -5e+3 & -55e+3 & 22e+3 \\ 15e+3 & 25e+3 & 12e+3 \\ 33e+3 & 88e+3 & 42e+3 \end{pmatrix};$$

$$N_3 = \begin{pmatrix} 11e+4 & -54e+4 & 14e+4 \\ 22e+4 & 33e+4 & 44e+4 \\ 75e+4 & 69e+4 & 12e+4 \end{pmatrix}; N_4 = \begin{pmatrix} 56e+5 & 46e+5 & 7e+5 \\ 25e+5 & 36e+5 & 11e+5 \\ 25e+5 & 35e+5 & 66e+5 \end{pmatrix}; N_5 = \begin{pmatrix} 47e+5 & 88e+5 & 77e+5 \\ 65e+5 & 35e+5 & 78e+5 \\ 76e+5 & 58e+5 & 69e+5 \end{pmatrix}$$

and "denominator" coefficients given by

$$D_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 191.1518 & 301.9950 & -76.6902 \\ -189.0205 & 644.5408 & -31.0642 \\ -168.7804 & 248.8579 & 354.3072 \end{pmatrix};$$

$$D_2 = \begin{pmatrix} -0.0092e+5 & 0.7609e+5 & -0.0943e+5 \\ -0.5788e+5 & 1.2436e+5 & 0.0288e+5 \\ -0.5622e+5 & 0.7805e+5 & 0.5150e+5 \end{pmatrix}; \quad D_3 = \begin{pmatrix} -0.2700e+7 & 0.7401e+7 & 0.0369e+7 \\ -0.6943e+7 & 1.1081e+7 & 0.1521e+7 \\ -0.7098e+7 & 0.9459e+7 & 0.3750e+7 \end{pmatrix};$$

$$D_4 = \begin{pmatrix} -2.4688e+8 & 3.9742e+8 & 0.5700e+8 \\ -4.0845e+8 & 5.5766e+8 & 0.9961e+8 \\ -4.1008e+8 & 5.4636e+8 & 1.2396e+8 \end{pmatrix}; \quad D_5 = \begin{pmatrix} -0.7552e+10 & 1.0796e+10 & 0.0832e+10 \\ -1.0347e+10 & 1.4324e+10 & 0.1224e+10 \\ -0.9460e+10 & 1.3085e+10 & 0.1222e+10 \end{pmatrix};$$

$$D_6 = \begin{pmatrix} -1.9495e+10 & 2.7531e+10 & 0.2226e+10 \\ -2.6104e+10 & 3.6236e+10 & 0.3098e+10 \\ -2.3659e+10 & 3.2932e+10 & 0.2865e+10 \end{pmatrix}.$$

The computation of the spectral factors of the above characteristic matrix polynomial yields:

$$Q_1 = \begin{pmatrix} 10 & -246.6667 & 93.3333 \\ 120 & -336.6667 & 73.3333 \\ 60 & -103.3333 & -93.3333 \end{pmatrix}$$

whose eigenvalues are

{-150.0000, -140.0000, -130.0000}

$$Q_2 = \begin{pmatrix} -96.1538 & -54.2308 & 48.8462 \\ 11.5385 & -152.6923 & 36.5385 \\ 3.8462 & -24.2308 & -81.1538 \end{pmatrix}$$

{-120.0000, -110.0000, -100.0000}

$$Q_3 = \begin{pmatrix} -90 & 60 & -60 \\ -2.5 & -40 & -45 \\ -5 & 30 & -110 \end{pmatrix}$$

{-90.0000, -80.0000, -70.0000}

$$Q_4 = \begin{pmatrix} -64.6296 & 18.1481 & -1.2963 \\ 4.0741 & -30.3704 & -29.2593 \\ -5 & 10 & -55 \end{pmatrix} \qquad \{-60.0000,\ -50.0000,\ -40.0000\}$$

$$Q_5 = \begin{pmatrix} 49.6316 & -75.5789 & -4.5263 \\ 54.1579 & -79.8947 & -4.6316 \\ 110.6842 & -153.2105 & -13.7368 \end{pmatrix} \qquad \{-35.0000,\ -5.0000,\ -4.0000\}$$

$$Q_6 = \begin{pmatrix} 0.0000 & -3.6667 & 0.3333 \\ 1.7500 & -4.9167 & 0.0833 \\ 4.2500 & -8.0833 & -1.0833 \end{pmatrix} \qquad \{-3.0000,\ -2.0000,\ -1.0000\}$$

From the eigenspectrum of the above spectral factors, it is suggested that a 13^{th} order reduced model (scalar characteristic polynomial) will be adequate in approximating the behaviour of the 18^{th} order original model. This is fulfilled for a relative error equal to 0.008, giving

$$\hat{H}(s) = \frac{1}{\Delta(s)}[G(s)]$$

where the scalar coefficients of $\Delta(s)$ are:

$a_0 = 1$; $a_1 = 1.1750e+3$; $a_2 = 6.2840e+5$; $a_3 = 2.0243e+8$; $a_4 = 4.3796e+10$;
$a_5 = -6.7165e+12$ $a_6 = 7.5067e+14$; $a_7 = 6.1869e+16$; $a_8 = 3.7576e+18$;
$a_9 = 1.6599e+20$; $a_{10} = 5.1806e+21$; $a_{11} = 1.0809e+23$;
$a_{12} = -1.3506e+24$; $a_{13} = -7.6281e+24$.

and the coefficients of the matrix polynomial $G(s)$ are as follows:

$$G_0 = \begin{pmatrix} 0.6234 & 0.5375 & -0.2842 \\ -0.1033 & 1.0942 & -0.0174 \\ -0.0356 & 0.4963 & 0.4658 \end{pmatrix}; \quad G_1 = \begin{pmatrix} 0.8762e+3 & 0.4380e+3 & 0.2990e+3 \\ -0.1538e+3 & -0.2473e+3 & 1.0097e+3 \\ 0.6821e+3 & 1.3313e+3 & 0.6393e+3 \end{pmatrix};$$

$$G_2 = \begin{pmatrix} 5.9994e+5 & -0.9763e+5 & 3.7763e+5 \\ -1.3967e+5 & -4.7165e+5 & 7.9352e+5 \\ 9.3608e+5 & 4.7648e+5 & 4.3816e+5 \end{pmatrix}; \quad G_3 = \begin{pmatrix} 2.1163e+8 & -1.3264e+8 & 1.5203e+8 \\ -0.4818e+8 & -1.8722e+8 & 2.7775e+8 \\ 4.8205e+8 & -0.1441e+8 & 1.6529e+8 \end{pmatrix};$$

$$G_4 = \begin{pmatrix} 0.4107e+11 & -0.4062e+11 & 0.3261e+11 \\ -0.0626e+11 & -0.4189e+11 & 0.5737e+11 \\ 1.3385e+11 & -0.4258e+11 & 0.3739e+11 \end{pmatrix}; \quad G_5 = \begin{pmatrix} 0.4335e+13 & -0.6174e+13 & 0.4299e+13 \\ 0.0229e+13 & -0.6366e+13 & 0.7765e+13 \\ 2.2864e+13 & -1.1429e+13 & 0.5418e+13 \end{pmatrix};$$

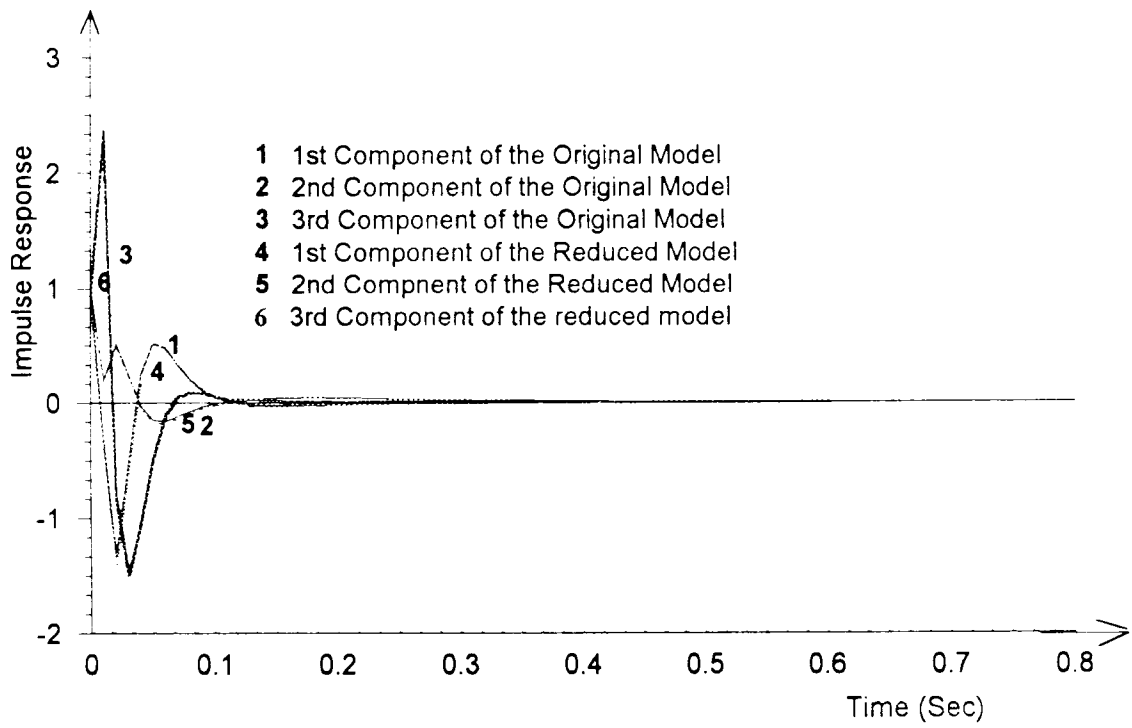
$$G_6 = \begin{pmatrix} 0.1854e+15 & -0.5095e+15 & 0.3694e+15 \\ 0.1757e+15 & -0.6957e+15 & 0.7213e+15 \\ 2.5443e+15 & -1.5967e-15 & 0.5256e-15 \end{pmatrix};\quad G_7 = \begin{pmatrix} -0.0912e+17 & -0.1934e+17 & 0.2077e+17 \\ 0.2444e+17 & -0.5479e+17 & 0.4692e+17 \\ 1.8794e+17 & -1.3633e+17 & 0.3509e+17 \end{pmatrix};$$

$$G_8 = \begin{pmatrix} -1.6144e+18 & 0.2019e-18 & 0.6913e+18 \\ 1.7682e+18 & -3.0326e+18 & 2.1521e+18 \\ 9.1295e-18 & -7.3500e-18 & 1.6190e+18 \end{pmatrix};\quad G_9 = \begin{pmatrix} -0.8777e-20 & 0.5194e-20 & 0.0642e+20 \\ 0.7344e-20 & -1.1319e-20 & 0.6978e+20 \\ 2.8039e-20 & -2.4322e-20 & 0.4880e-20 \end{pmatrix};$$

$$G_{10} = \begin{pmatrix} -2.2847e-21 & 2.0901e+21 & -0.4568e+21 \\ 1.6841e+21 & -2.6699e+21 & 1.5921e+21 \\ 5.0253e-21 & -4.4443e+21 & 0.7299e+21 \end{pmatrix};$$

$$G_{12} = \begin{pmatrix} -0.8555e-23 & 2.1086e+23 & -1.6309e+23 \\ 0.3865e+23 & -1.9567e+23 & 1.8655e+23 \\ 1.7930e+23 & 0.6200e+23 & -2.3751e+23 \end{pmatrix}.$$

The original and reduced models impulse response curves comparison is shown below



Example 3

The next example is a 6^{th} order 3×3 matrix transfer function with "numerator" coefficients given by:

$$N_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};\quad N_1=\begin{pmatrix}3457&4424&5845\\-2013&-2981&3187\\5641&3201&8924\end{pmatrix};\quad N_2=\begin{pmatrix}210&-511&-322\\477&632&761\\654&921&320\end{pmatrix};$$

$$N_3=\begin{pmatrix}761&-144&532\\565&241&879\\147&954&632\end{pmatrix};\quad N_4=\begin{pmatrix}844&655&521\\321&445&986\\568&788&399\end{pmatrix};\quad N_5=\begin{pmatrix}1897&499&815\\430&768&647\\485&696&832\end{pmatrix}.$$

And "denominator" coefficients given by:

$$D_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};\quad D_1=\begin{pmatrix}334.4496&-122.3903&24.0180\\35.0898&173.6470&114.3352\\11.5207&-41.8154&343.9035\end{pmatrix};$$

$$D_2=\begin{pmatrix}3.9658e+4&-2.2531e+4&0.0851e+4\\0.6610e+4&0.8856e+4&2.1482e+4\\0.2087e+4&-0.7735e+4&4.1101e+4\end{pmatrix};\quad D_3=\begin{pmatrix}2.1162e+6&-1.6019e+6&0.0994e+6\\0.4296e+6&0.0221e+6&1.4270e+6\\0.1227e+6&-1.6019e+6&2.0951e+6\end{pmatrix};$$

$$D_4=\begin{pmatrix}4.8484e+7&-4.8274e+7&1.3041e+7\\1.0228e+7&-0.6737e+7&3.8110e+7\\0.2369e+7&-0.9542e+7&4.3270e+7\end{pmatrix};\quad D_5=\begin{pmatrix}3.3308e+8&-4.5065e+8&2.9776e-8\\0.4043e+8&-0.5169e+8&3.3418e-8\\-0.0503e+8&-0.2887e+8&2.6839e-8\end{pmatrix};$$

$$D_5=\begin{pmatrix}6.9559e+8&-9.9582e+8&3.8161e+8\\0.7988e+8&-1.6861e+8&5.7096e+8\\-0.1389e+8&-0.4581e+8&4.0346e+8\end{pmatrix}.$$

The computed spectral factors with their corresponding spectrums are the following:

$$Q_1=\begin{pmatrix}-100.7031&-25.4688&186.0156\\-4.2969&-104.5313&78.9844\\-0.5469&-2.0312&-79.7656\end{pmatrix}$$

whose spectrum gives

{-100.0000, -95.0000, -90.0000}

$$Q_2=\begin{pmatrix}-110&86&-134\\-30&5&130\\-15&42&-143\end{pmatrix}$$

{-85.0000, -83.0000, -80.0000}

$$Q_3=\begin{pmatrix}-55.2386&4.7045&-31.2614\\8.0341&-58.3864&-32.5341\\7.8750&-5.2500&-73.3750\end{pmatrix}$$

{-70.0000, -62.0000, -55.0000}

$$Q_4 = \begin{pmatrix} -64.5556 & 57.1111 & -26.2222 \\ -14.2222 & -2.5556 & -20.8889 \\ -6.2222 & 10.4444 & -37.8889 \end{pmatrix} \qquad \{-45.0000,\ -35.0000,\ -25.0000\}$$

$$Q_5 = \begin{pmatrix} -0.3333 & -4.00 & -16.6667 \\ 5.6667 & -13.00 & -7.6667 \\ 2.6667 & -4.00 & -7.6667 \end{pmatrix} \qquad \{-9.0000,\ -7.0000,\ -5.0000\}$$

$$Q_6 = \begin{pmatrix} -3.6190 & 4.0435 & -1.8833 \\ -0.2715 & -0.1737 & -2.2299 \\ -0.2933 & 0.6522 & -2.2073 \end{pmatrix} \qquad \{-3.0000,\ -2.0000,\ -1.0000\}$$

According to the spectral factors eigenspectrums, a fair approximation will be obtained for a reduced model of order 12 under the form

$$\hat{H}(s) = \frac{1}{\Delta(s)}[G(s)]$$

where the characteristic scalar polynomial coefficients are:

$a_0 = 1;\quad a_1 = 825.0001;\quad a_2 = 3.0867e+5;\quad a_3 = 6.9209e+7;\quad a_4 = 1.0349e+10$

$a_5 = 1.0864e+12;\quad a_6 = 8.2017e+5;\quad a_7 = 4.4817e+15;\quad a_8 = 1.7571e+18;$

$a_9 = 4.8138e+18;\quad a_{10} = 8.7334e+19;\quad a_{11} = 9.4040e+20;\quad a_{12} = 4.5355e+21.$

and the coefficient matrices of the matrix polynomial $G(s)$ are as follows:

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\quad G_1 = \begin{pmatrix} 3.9476e+3 & 4.5464e+3 & 5.8210e+3 \\ -2.0481e+3 & -2.3296e+3 & 3.0727e+3 \\ 5.6295e+3 & 3.2428e+3 & 9.4051e+3 \end{pmatrix};$$

$$G_2 = \begin{pmatrix} 1.5744e+6 & 3.6089e+6 & 2.2045e+6 \\ -0.9355e+6 & -1.8765e+6 & 1.8674e+6 \\ 2.5475e+6 & 3.1686e+6 & 3.8898e+6 \end{pmatrix};\quad G_3 = \begin{pmatrix} 0.2570e+9 & 1.1186e+9 & 0.2550e+9 \\ -0.1740e+9 & -0.5627e+9 & 0.5033e+9 \\ 0.4693e+9 & 1.0844e+9 & 0.5998e+9 \end{pmatrix};$$

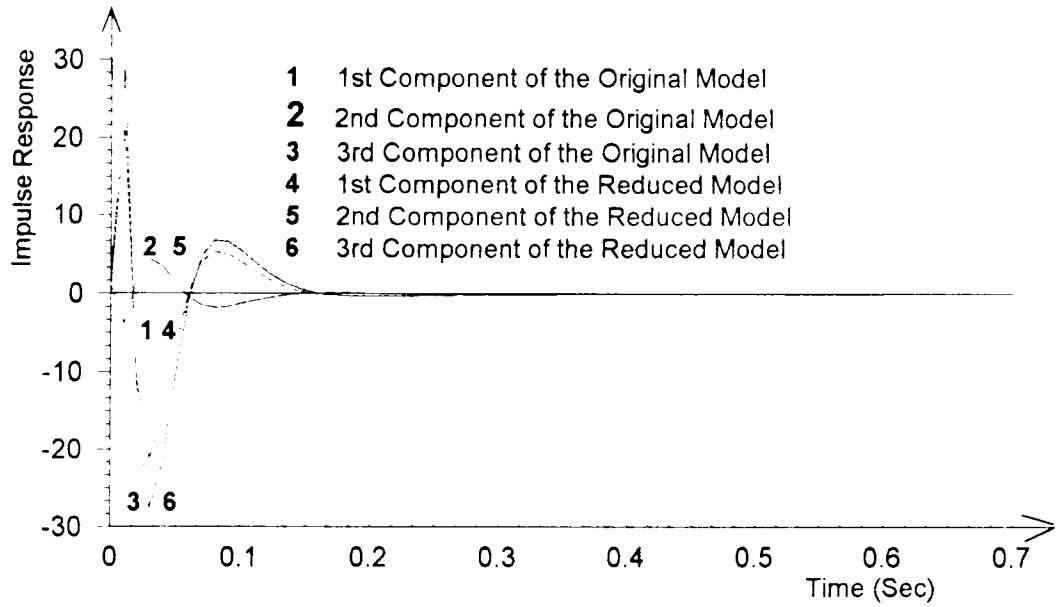
$$G_4 = \begin{pmatrix} 0.1955e+11 & 1.8344e+11 & -0.0589e+11 \\ -0.1648e+11 & -0.9012e+11 & 0.7766e+11 \\ 0.4412e+11 & 1.8881e+11 & 0.3152e+11 \end{pmatrix};\quad G_5 = \begin{pmatrix} 0.0437e+13 & 1.7656e+13 & -0.4074e+13 \\ -0.0776e+13 & -0.8568e+13 & 0.7419e+13 \\ 0.2079e+13 & 1.8951e+13 & -0.1885e+13 \end{pmatrix};$$

$$G_6 = \begin{pmatrix} -0.0341e+15 & 1.0239e+15 & -0.4078e+15 \\ -0.0097e+15 & -0.4958e+15 & 0.4437e+15 \\ 0.0287e+15 & 1.1357e+15 & -0.3513e+15 \end{pmatrix};\quad G_7 = \begin{pmatrix} -0.2658e+16 & 3.4631e+16 & -1.9222e+16 \\ 0.0637e+16 & -1.6880e+16 & 1.5969e+16 \\ -0.1450e+16 & 3.9547e+16 & -1.9620e+16 \end{pmatrix};$$

$$G_8 = \begin{pmatrix} -0.7140e+17 & 6.100e+17 & -4.4089e+17 \\ 0.2753e+17 & -3.0062e+17 & 3.0631e+17 \\ -0.6605e+17 & 7.1864e+17 & -4.9445e+17 \end{pmatrix}; G_9 = \begin{pmatrix} -1.0385e+18 & 4.4289e+18 & -3.9423e+18 \\ 0.4349e+18 & -2.0566e+18 & 2.1624e+18 \\ -1.2249e+18 & 5.4554e+18 & -4.7973e+18 \end{pmatrix};$$

$$G_{10} = \begin{pmatrix} -2.2221e+19 & 3.8688e+19 & -1.6121e+19 \\ 0.7279e+19 & -1.0259e+19 & 0.0016e+19 \\ -2.6634e+19 & 4.5796e+19 & -1.8309e+19 \end{pmatrix}; G_{11} = \begin{pmatrix} -0.3661e+21 & 0.9626e+21 & -0.5024e+21 \\ 0.1233e+21 & -0.3503e+21 & 0.1881e+21 \\ -0.4213e+21 & 1.1108e+21 & -0.5322e+21 \end{pmatrix}.$$

The comparison of the original and reduced models is given in the following figure.



As it can be seen, this is a good approximation since the two models are almost superposed.

Example 4

As a final example, we have chosen a 6^{th} order 3×3 matrix transfer function with matrix coefficients given as follows:

$$N_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad N_1 = \begin{pmatrix} 425 & 802 & 635 \\ -10 & -321 & 879 \\ 446 & 827 & 434 \end{pmatrix}; \quad N_2 = \begin{pmatrix} -6015 & -6051 & 2332 \\ 1601 & 2622 & 1421 \\ 323 & 89 & 422 \end{pmatrix};$$

$$N_3 = \begin{pmatrix} 2010 & -9440 & 8140 \\ 2550 & 3430 & 5440 \\ 5450 & 6690 & 4020 \end{pmatrix}; \quad N_4 = \begin{pmatrix} 676e+2 & 496e+2 & 887e+2 \\ 235e+2 & 346e+2 & 151e+2 \\ 285e+2 & 365e+2 & 656e+2 \end{pmatrix};$$

$$N_5 = \begin{pmatrix} 489e+2 & 938e+2 & 987e+2 \\ 775e+2 & 303e+2 & 828e+2 \\ 986e+2 & 898e+2 & 699e+2 \end{pmatrix}.$$

and the coefficients of the characteristic matrix polynomial are:

$$D_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad D_1 = \begin{pmatrix} 443.2222 & 2.5972 & -2.2459 \\ -8.9817 & 447.2432 & 6.7297 \\ -4.5700 & 19.0155 & 433.5347 \end{pmatrix};$$

$$D_2 = \begin{pmatrix} 8.1523e+4 & 0.1051e+4 & -0.0839e+4 \\ -0.3363e+4 & 8.3221e+4 & 0.2425e+4 \\ -0.1786e+4 & 0.7140e+4 & 7.8043e+4 \end{pmatrix}; \quad D_3 = \begin{pmatrix} 7.9646e+6 & 0.1690e+6 & -0.1249e+6 \\ -0.5025e+6 & 8.2470e+6 & 0.3483e+6 \\ -0.2781e+6 & 1.0697e+6 & 7.4668e+6 \end{pmatrix}$$

$$D_4 = \begin{pmatrix} 4.3586e+8 & 0.1350e+8 & -0.0926e+8 \\ -0.3745e+8 & 4.5907e+8 & 0.2491e+8 \\ -0.2157e+8 & 0.7994e+8 & 4.0043e+8 \end{pmatrix}; \quad D_5 = \begin{pmatrix} 1.2667e10 & 0.0536e10 & -0.0342e10 \\ -0.1392e10 & 1.3611e10 & 0.0887e10 \\ -0.0834e10 & 0.2980e10 & 1.1412e10 \end{pmatrix};$$

$$D_6 = \begin{pmatrix} 1.5273e+11 & 0.0845e+11 & -0.0504e+11 \\ -0.2065e+11 & 1.6794e+11 & 0.1258e+11 \\ -0.1284e+11 & 0.4433e+11 & 1.3504e+11 \end{pmatrix}.$$

The computed spectral factors with their corresponding eigenvalues are:

$$Q_1 = \begin{pmatrix} -89.5771 & 1.0364 & 0.4908 \\ 1.0992 & -87.6963 & -1.8026 \\ -0.4118 & -0.5339 & -86.7266 \end{pmatrix} \qquad \{-90.0000,\ -88.0000,\ -86.0000\}$$

$$Q_2 = \begin{pmatrix} -83.3883 & 1.1538 & 0.3415 \\ 1.2105 & -82.1737 & -1.2121 \\ -0.8138 & -0.5633 & -80.4381 \end{pmatrix} \qquad \{-84.0000,\ -82.0000,\ -80.0000\}$$

$$Q_3 = \begin{pmatrix} -76.7314 & 1.0210 & 0.3087 \\ 1.7207 & -76.6410 & -1.6227 \\ -0.9049 & -0.6827 & -74.6276 \end{pmatrix} \qquad \{-78.0000,\ -76.0000,\ -74.0000\}$$

$$Q_4 = \begin{pmatrix} -71.3728 & 1.0480 & 0.3827 \\ 1.1941 & -70.2345 & -1.1586 \\ -0.6788 & -0.6445 & -68.3926 \end{pmatrix} \qquad \{-72.0000,\ -70.0000,\ -68.0000\}$$

$$Q_5 = \begin{pmatrix} -66.1529 & 0.4771 & 0.0556 \\ 0.2570 & -64.6643 & -1.1003 \\ -1.1214 & -0.4244 & -65.1828 \end{pmatrix} \qquad \{-66.0000 \pm 0.4890i,\ -64.0000\}$$

$$Q_6 = \begin{pmatrix} -55.9997 & -7.3335 & 0.6666 \\ 3.5002 & -65.8334 & 0.1666 \\ 8.5007 & -16.1667 & -58.1670 \end{pmatrix} \qquad \{-62.0000,\ -60.0000,\ -58.0000\}$$

A reduced model of order 15 is derived for a relative error equal to 0.1 with the following coefficients

$a_0 = 1;\ a_1 = 1.144e-3;\ a_2 = 6.1023e+5;\ a_3 = 2.0134e+8;\ a_4 = 4.5950e+10;\ a_5 = 7.684e+12;$
$a_6 = 9.7264e+14;\ a_7 = 9.4896e+16;\ a_8 = 7.1950e+18;\ a_9 = 4.2394e-20;\ a_{10} = 1.9254e+22;$
$a_{11} = 6.6189e+23;\ a_{12} = 1.6672e+25;\ a_{13} = 2.9049e+26;\ a_{14} = 3.1306e-27;\ a_{15} = 1.5732e+28.$

and

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ G_1 = \begin{pmatrix} 1.1258e+3 & 0.7994e+3 & 0.6372e+3 \\ -0.0010e+3 & 0.3758e+3 & 0.8723e+3 \\ 0.4506e+3 & 0.8080e+3 & 1.1445e+3 \end{pmatrix};$$

$$G_2 = \begin{pmatrix} 5.2002e+5 & 5.3782e+5 & 4.5049e+5 \\ 0.0134e+5 & -0.2223e+5 & 6.2388e+5 \\ 3.2512e+5 & 5.5474e+5 & 5.2852e+5 \end{pmatrix};\ G_3 = \begin{pmatrix} 1.3430e+8 & 1.5959e+8 & 1.4182e+8 \\ 0.0120e+8 & -0.3921e+8 & 1.9837e+8 \\ 1.0414e+8 & 1.6882e+8 & 1.3617e+8 \end{pmatrix};$$

$$G_4 = \begin{pmatrix} 2.1805e+10 & 2.7348e+10 & 2.6114e+10 \\ 0.0397e+10 & -1.0409e+10 & 3.6812e+10 \\ 1.9441e+10 & 2.9881e+10 & 2.2167e+10 \end{pmatrix};\ G_5 = \begin{pmatrix} 2.3424e+12 & 2.9423e+12 & 3.0973e+12 \\ 0.0797e+12 & -1.4802e+12 & 4.3862e+12 \\ 2.3326e+12 & 3.3666e+12 & 2.4089e+12 \end{pmatrix};$$

$$G_6 = \begin{pmatrix} 1.6945e+14 & 1.9123e+14 & 2.4301e+14 \\ 0.1256e+14 & -1.3274e+14 & 3.4362e+14 \\ 1.8691e+14 & 2.3962e+14 & 1.7664e+14 \end{pmatrix};\ G_7 = \begin{pmatrix} 0.8272e+16 & 0.3692e+16 & 1.1941e+16 \\ 0.1839e+16 & -0.8124e+16 & 1.6508e+16 \\ 1.0040e+16 & 0.7922e+16 & 0.8252e+16 \end{pmatrix};$$

$$G_8 = \begin{pmatrix} 2.8097e+17 & -7.9074e+17 & 2.2958e+17 \\ 2.3968e+17 & -3.8874e+17 & 2.4342e+17 \\ 3.4725e+17 & -4.6210e+17 & 1.6175e+17 \end{pmatrix};\ G_9 = \begin{pmatrix} 0.0808e+20 & -1.1707e+20 & -0.1699e+20 \\ 0.2437e+20 & -0.1889e+20 & -0.3416e+20 \\ 0.0606e+20 & -0.9268e+20 & -0.0928e+20 \end{pmatrix}$$

$$G_{10} = \begin{pmatrix} 0.2602e+21 & -9.1826e+21 & -1.9347e+21 \\ 1.7783e+21 & -0.9966e+21 & -3.5920e+21 \\ -0.1190e+21 & -7.6182e+21 & -1.0688e+21 \end{pmatrix};$$

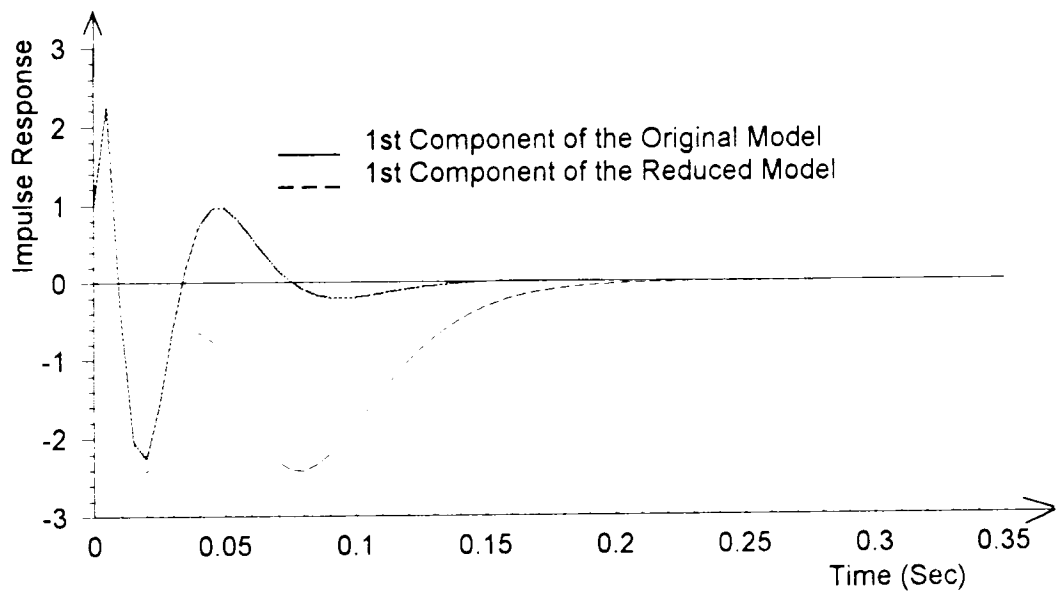
$$G_{11} = \begin{pmatrix} 0.0631e+23 & -4.6624e+23 & -1.0293e+23 \\ 0.8873e+23 & -0.4312e+23 & -1.9459e+23 \\ -0.1554e+23 & -3.9138e+23 & -0.5554e+23 \end{pmatrix};$$

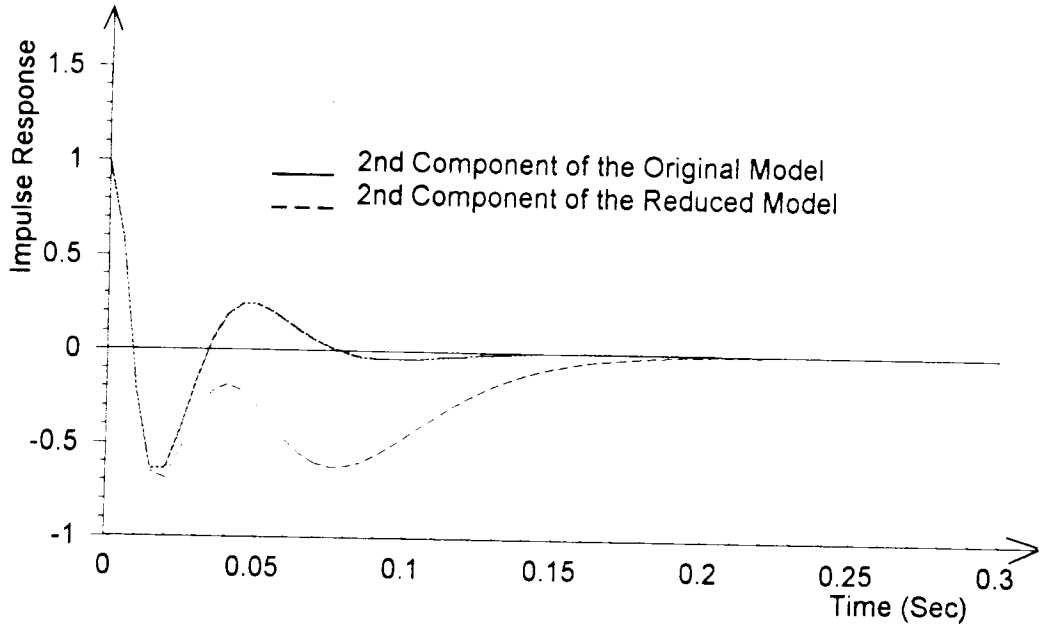
$$G_{12} = \begin{pmatrix} -0.0019e+25 & -1.5178e+25 & -0.3359e+25 \\ 0.2869e+25 & -0.1212e+25 & -0.6559e+25 \\ -0.0722e+25 & -1.2794e+25 & -0.1772e+25 \end{pmatrix};$$

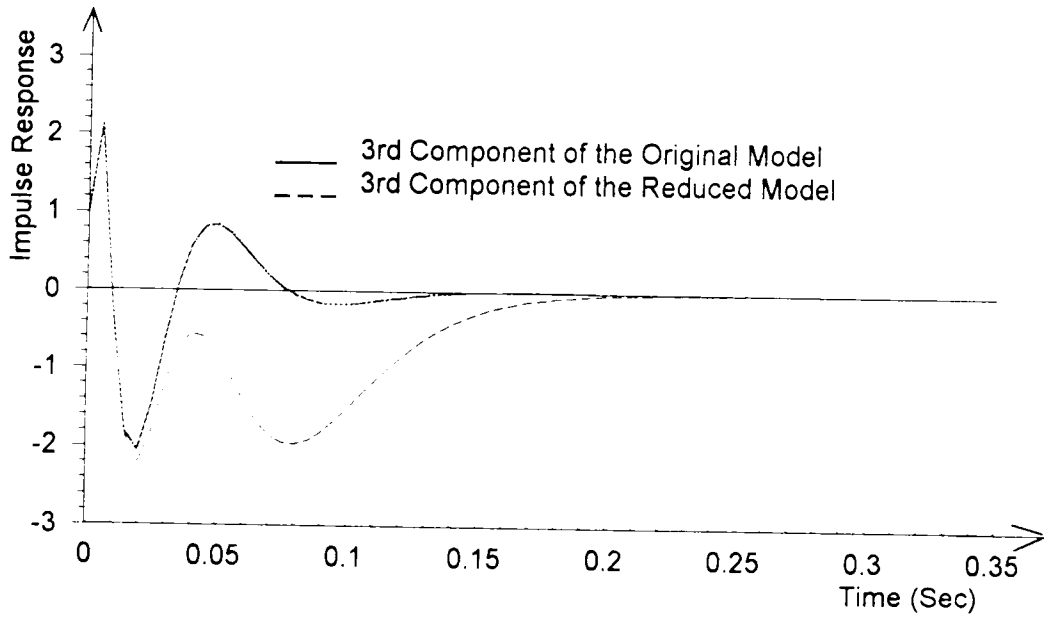
$$G_{13} = \begin{pmatrix} -0.0526e+26 & -2.8797e+26 & -0.6386e+26 \\ 0.5418e+26 & -0.1879e+26 & -1.2894e+26 \\ -0.1795e+26 & -2.4343e+26 & -0.3298e+26 \end{pmatrix};$$

$$G_{14} = \begin{pmatrix} -0.0875e+27 & -2.4220e+27 & -0.5410e+27 \\ 0.4544e+27 & -0.1186e+27 & -1.1273e+27 \\ -0.1886e+27 & -2.0537e+27 & -0.2736e+27 \end{pmatrix}.$$

For clarity purposes we give, in the following, the impulse response curves of the three components of the original and reduced models separately.







As it may be observed, this is a poor approximation due simply to the truncation of the last subsystem corresponding to the spectral factor Q_6. This shows also that the last subsystem can not be neglected in terms of impulse response in front of the other first 5 subsystems. On the other side, the value of the relative error $RE = 0.1$ is meaningful in this sense; Compared to the previous examples this value is large, meaning that the neglected subsys m (one spectral factor) can not be ignored in front of the rest of the model. This validates the choice of the Hankel matrix singular value relative dominance criterion.

Chapter 6

Conclusion

Usually analysis, synthesis and simulation in the classical control theory require the use of a low-order system to approximate a high-order one. It turns out that this trend remains very attractive to modern control systems, in other words, the low dimensionality is often strongly desired.

Model approximation represents a useful means for the construction of models smaller than Kalman's minimal realization which is often too large to be tractable, and develop a partial realization which is "close" to the minimal realization in some sense. In fact Kalman's minimal realization requires a zero relative error ($RE=0$) while model approximation is defined to allow some relative error ($RE>0$), that is only the dominant components of the model are retained in the construction of the reduced model.

The contribution of the present thesis has been first of all the elaboration of two model reduction procedures: the solvents based method and the spectral factors based method. Second, it points attention and promote the use of very promising tools, the principal component analysis and the theory of matrix polynomials, in solving model reduction problems.

The principal component theory is used to provide a relative measure of the system components so that the analyst can go beyond the Kalman's minimal realization. In opposition to results from singular perturbation or classical dominant pole methods, it was found that the modes with slower decay have less effect on the behaviour of the system when the state space

description matrices B and C are taken into consideration in the truncation criterion.

Matrix polynomial theory is used to develop two model reduction approaches from a stable, proper, linear and time-invariant system described by a transfer function in the form of a matrix fraction description. The obtained reduced models are given under rational transfer function forms and are also stable, proper, linear and time-invariant. The expression of the reduced model in a rational form is due to the fact that most of the time the reduced model is not block divisible.

As already mentioned, the first method is based on the dominant solvents of the characteristic matrix polynomial. A block Vandermande similarity transformation and a Hankel matrix relative dominance criterion have been used to develop a stable and efficient algorithm. The application of this algorithm has shown some interesting results as illustrated by the examples given in the previous chapter.

By choosing appropriate coordinate transformations prior to component truncation, a number of different choices for "system components" are defined including those which cause decoupling of the outputs, decoupling of component dynamics and the decoupling of the disturbances. The only shortcoming of this procedure is the numerical cost of the inversion of the Vandermande matrix.

In order to bypass this obstacle and increase the efficiency and stability of the algorithm, a second method based on the dominant spectral factors of the characteristic matrix polynomial, combining both the matrix polynomial division and the block Vandermande similarity transformation is developed. The major advantages offered by this method over the previous one are the avoidance of the inversion of large Vandermande matrices, since the original model is first reduced using polynomial division, then it is block decoupled using the Vandermande similarity transformation, and the fact that it does not require the computation of the complete set of spectral factors, hence, it is also applicable to partially factorizable matrix polynomials. However, the

transformation of the computed spectral factors to solvents has also its cost in terms of space and time but to a lesser degree than inverting the Vandermande matrix.

The determination of the solvents and the spectral factors of the characteristic matrix polynomial is achieved through the use of any matrix polynomial root finding method such as the ones seen in chapter 2. The prerequisite with the proposed methods which can be seen as a restriction, is the need for such matrix polynomial root finding methods that generate the solvents and the spectral factors in a specific order of dominance assuming the existence of a complete set. The comparison with existing methods in the current literature shows, that the proposed methods have, in general, better accuracy, they are easily programmed, save time since they start by determining block reduced models and only the last subsystem is diagonalized to be truncated at the eigenvalue level.

A look to the theoretical development of both methods, reveals that the reduced models are derived from a transfer function given in the form of a right matrix fraction description. However, it is understood that a similar analysis of the left matrix fraction description can be done and similar results will be obtained. It is also clear that the proposed methods are applicable for both SISO systems and MIMO systems.

There are several points in this thesis which may be natural points of departure for further studies. The investigation of other methods other than the matrix polynomial division, for block reduction before the block decoupling, using less memory space provides one such point. Another point of departure that is probably more important than the previous one is the investigation of other methods for block decoupling. An interesting idea would be the use of the algorithm developed in [65] for the inversion of a block Vandermande matrix. This may yield better results and save memory space.

Moreover the search for other means to extract the dominant part of the last added subsystem, and the study of other ways for converting the

obtained reduced model state space triple to matrix transfer function form, may be suggested as two other points for further studies.

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A N N E X

Liste et composition du Jury en vue de la soutenance de memoire de Magister en Ingenierie des Systemes Electroniques par *Mr ZABOT Hakim.*

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