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ON THE CHOICE OF CLOSED-LOOP BLOCK POLES IN MULTIVARIABLE CONTROL DESIGN

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DEDICATION

I dedicate this modest work to my dearest lovely family (parents),

To all my brothers and sisters for their valuable help and moral support, without their motivation and encouragement I would have never achieved this work.

To anybody with whom I exchanged a smile.

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ملخص

في الأنظمة متعددة المتغيرات، تصميم بتعديل الحالات وتصميم المعوض قد يكون تحقيقه باستعمال وضع جماعي للأقطاب. بخلاف الوضع المعتاد لقطب فان الوضع الجماعي للأقطاب يسمح استجابة زمنية جيدة من حيث الأداء و الصلابة. لتكن مجموعة مرغوبة من أقطاب، إنشاء لأقطاب جماعية ليس وحيد. هذا التعدد استعمل للقاء المعايير التالية :

- طويلة ربح المصفوفة لتعديل الحالات صغيرة.
- معوض مناسب مع درجة أدنى في حالة تصميم معوض.
- خصائص الاستجابة الزمنية جيدة.
- وينتج نظام بصلابة جيّدة.

الطرق المستعملة لتصميم بتعديل الحالات ولتصميم المعوض قد أعطيت ووضّحت بعدد كبير من دراسة حالات.

الاستجابة لدالة عتبة لهذه الأنظمة قد رسمت، خصائص الاستجابة الزمنية(POS, Tr, Ts, SSV) ، مصفوفة ربح، معوض مناسب مع درجة أدنى، دالة الحساسية و رقم شرط لكلّ نظام حسبت. وكما زوّدت بثلاثة قياسات لحساب صلابة الاستقرار لكل قيمة مميّزة.

النتائج أنفة الذكر قرنت بعد ذلك لانتقاء الشكل الأفضل المتوافق مع المعايير المذكرة سابقا.

دراسات أنجزت لتبرير الاستنتاجات، المستخلصة من دراسة مقارنة، نظريّا.

Abstract

In multivariable systems, state feedback design and compensator design may be achieved using block-pole placement. Unlike the usual pole placement, block pole placement allows a better tuning of time response performance and robustness. Given a set of desired poles, the construction of block poles is not unique. This nonuniqueness is used to meet the following criteria:

i. Small feedback gain matrix using state feedback design,
ii. A proper compensator with minimal degree using compensator design
iii. The best time response characterisation
iv. And yielding system with good robustness.

The methods for designing state feedback controllers and compensators are given and illustrated by a large number of case studies. The step response of these systems are plotted, the time response characteristics (POS, Tr, Ts, SSV), gain matrix, proper with minimal degree of a compensator, the sensitivity function and the condition number of each system are computed. Three measures are provided to compute the robust stability of all eigenvalues. The above results are then compared to select the best form meeting the required criteria mentioned previously.

Résumé:

Dans les systèmes multivariables, le concept de retour d'état et le concept de compensateur peuvent être réalisé en utilisant le placement de block-pôle. Contrairement au placement de pole, le placement de block-pôle permet une meilleure performance de temps de réponse et une meilleure robustesse du système. Considérant un ensemble de pôles désirés, la construction de block-pôles n'est pas unique. Cette variété est utilisée pour satisfaire les critères suivants:

i. Une norme minimale de la matrice de gain de retour, en utilisant le concept de retour d'état

ii. Un compensateur propre avec un degré minimal, en utilisant le concept de compensateur.

iii. Les meilleures caractéristiques de temps de réponse

iv. et rend le système plus robuste.

Les méthodes pour concevoir les commandes en retour d'état et compensateurs sont donnés et illustrés par un grand nombre d'études de cas. Les réponses en échelon unitaire à ces systèmes sont tracées, les caractéristiques de temps de réponse (POS, Tr, Ts, SSV), la matrice de gain, compensateur propre avec degré minimal, la fonction de sensibilité et le conditionnement de chaque système sont calculés. Trois mesures sont fournies pour calculer la robustesse en stabilité de toutes les valeurs propres. Les résultats ci-dessus sont alors comparés pour sélectionner la meilleure forme qui satisfait les critères mentionnés précédemment.

Des recherches sont développées pour justifier théoriquement les conclusions tirées de l'étude comparative.

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Introduction

State feedback is one of the most popular and well known technique for altering the transient response of systems. This technique is usually used to assign the eigenvalues of the closed-loop system to desired locations under the assumption of complete controllability.

In the case of transfer function the use of the compensator is in order to satisfy specified requirements for steady state error, transient response or closed-loop pole locations.

The design of compensators for block poles placement is based on solving a matrix Diophantine equation. The proposed method in our work allows the computation of proper and minimal degree compensators; the proposed algorithms are based on the search for linearly dependent rows in the Sylvester matrix.

A large- scale MIMO system, described by a state space equation is often decomposed into small subsystems, for which analysis and design can be easily performed, so the dynamic properties of the MIMO system depend on the block-poles of its characteristic matrix polynomial. These block poles are no more than the solvents of the closed-loop denominator matrix polynomial of the considered MIMO system.

The solvents play an important role in the spectral decomposition of λ-matrices. The relationship between the solvents and latent roots of matrix polynomial will be presented in chapter three.

In multivariable systems a transfer function matrix is given either by

$$H_R(s) = N_R(s)D_R^{-1}(s)$$

or

$$H_L(s) = D_L{}^{-1}(s)N_L(s)$$

where

$$D_R(s) = I_m s^n + A_1 s^{n-1} + \ldots + A_n$$
$$N_R(s) = C_1 s^{n-1} + C_2 s^{n-2} + \ldots + C_n$$

and

$$D_L(s) = s^n I_m + s^{n-1} A_1 + \ldots + A_n$$
$$N_L(s) = s^{n-1} C_1 + s^{n-2} C_2 + \ldots + C_n$$

$H_R(s)$ and $H_L(s)$ are the right and left matrix fraction description, respectively. The nonsingular denominator matrix of the right (left) matrix fraction description is called the characteristic matrix polynomial and characterizes the properties of the multivariable control system.

The closed-loop right characteristic λ-matrix is given by $D_R(s) = \sum_{i=1}^{l} R_i s^i$ such that the systems is decomposed into *l* subsystems, the closed-loop poles are the roots of $\det[D_R(s)] = 0$, from the pole assignment point of view, $D_R(s)$ or its matrix coefficients $R_i, i = 1,2,\ldots,l$ are nonunique for a required set of closed-loop poles and associated eigenvectors. This leads to the conclusion that different feedback gains may result from the same set of closed-loop poles but different sets of associated eigenvectors.

This nonuniqueness of the gain matrix offers freedom that permit not only to place the closed-loop system eigenvalues but also to satisfy the closed-loop system robustness to parameter variations which is mainly handled by minimizing the closed-loop system condition number [31].

The robustness of the closed-loop system is one of the most important concerns of control system designers. Variations in system parameters due to component aging might result in system performance deterioration and even in system internal stability concerns. Eigenvalue locations can also be affected by external disturbances and, hence, those disturbances should be considered when designing feedback systems.

In single-input single output, the transfer function size is measured by its magnitude, for multi-input multi-output case we deal with transfer function matrices, i.e., matrices whose elements are transfer functions. There are a variety of methods for measuring the size of such matrices; one measure that has gained acceptance is the singular value of a matrix. In our work the singular values are developed in the study of the robustness of the closed-loop systems.

The sensitivity of the eigenvalues and the robustness of the closed-loop system both in state space and transfer function are presented in chapter five.

Problem Statement:

The choice of the closed-loop block poles in the case of Compensator Design

The design of unity feedback compensators leads to the so-called Diophantine equation [6]. The use of block poles constructed from a desired set of closed-loop poles offers the advantage of assigning a characteristic matrix polynomial rather than a scalar one. The desired characteristic matrix polynomial is first constructed from a set of block poles selected among a class of similar matrices, and then the compensator is synthesized by solving the Diophantine equation. The forms of the block poles used in our work are the diagonal, the controller and the observer forms.

Given a set of desired closed-loop poles $\{\lambda_{1d} \quad \lambda_{2d} \quad . \quad . \quad . \quad \lambda_{nd}\}$, a set of l block poles are constructed each in the form of:

- An $m \times m$ diagonal form matrix
- An $m \times m$ controller canonical form matrix
- An $m \times m$ observer canonical form matrix

Forcing these block poles to be matrix roots of the matrix polynomial $D_f(s)$ will determine the desired closed-loop matrix polynomial described by

$$D_f(s) = Is^l + D_{f1}s^{l-1} + ... + D_{fl}$$

The modified recursive algorithm [31] is used to compute the row index of the given proper rational transfer matrix $H(s)$. The recursive [86] or row searching [6] algorithm is used to solve the compensator equation.

Robustness is assessed, in each case, using the infinity norm, the singular value of the closed-loop transfer matrix and the condition number of the closed-loop transfer matrix.

Time response is assessed by plotting the step response and comparing the time response characteristics.

A comparison study is conducted to determine, in light of the above criteria, the best choice of the form of the block poles.

The choice of the closed-loop block poles in the case of State feedback design

The state equation describing linear time-invariant multivariable systems may be transformed via a similarity transformation to block controller form [69]. If the number of inputs *m* divides exactly the number of states *n*, a state feedback controller may then be designed by assigning block poles to the resulting characteristic matrix polynomial [86]. In the case where *m* does not divide *n*, a two stage procedure may be used: a block pole placement followed by usual pole placement [48].

The characteristic matrix polynomial of the closed-loop system is forced to equal a desired matrix polynomial which may be constructed from a set of desired $m \times m$ block poles .These block poles are to be selected from the class of similar matrices having as eigenvalues a set of desired closed-loop poles. Three forms are selected (diagonal, controller and observer form) and compared as to their effects on robustness, time response and feedback gain magnitude.

Stability robustness is assessed, in each case, using the robustness measures M_1, M_2 and M_3 proposed by Tsui [77]. Performance robustness is measured by subjecting the closed-loop system to small random perturbations, then computing the relative change on each closed-loop eigenvalue.

Time response is assessed by plotting the step response and comparing the time response characteristics.

A comparison study is conducted to determine, in light of the above criteria and the state feedback gain magnitude, the best choice of the form of the block poles.

The organization of the thesis

The thesis is divided into seven chapters;

Chapter one constitutes a brief review of state space representation and different block canonical forms used in multi-input multi-output systems.

Chapter two represents a general review on matrix polynomials theory with some material on solvents since they constitute the basic tools for the present work.

The block pole placement using state feedback is presented in chapter three whereas compensator design using block pole placement is developed in chapter four.

To maintain stability and performance of the closed-loop system, robust stability, robust performance and the sensitivity of the eigenvalues are presented in chapter five.

Investigations are attempted to justify theoretically the conclusions drawn from the comparison developed in chapter six.

Extensive testing on a large set of case studies is conducted in chapter seven for illustrative purposes to choose the best block pole form among different forms proposed.

Finally, we provide the general conclusion of this thesis and suggest topics for further research.

Chapter 1

State Space Variable and Canonical Forms

1.1 Introduction:

The analysis and synthesis of complex physical or engineering systems always start by building up models which realistically describe their behavior. The reason is that once a physical phenomenon has been adequately modeled so as to be a faithful representation of reality, all further analysis can be done on the model and experimentation on the process is no longer required. Because of different analytical methods used, we may often set up different mathematical equations to describe the same system.

The transfer function that describes only the terminal property of a system may be called the *external* or *input-output description* of the system.

The set of differential equations that describe the internal as well as terminal behavior may be called *internal* or *state-variable description* of the system [49].

In this chapter an overview of state space representation and different block canonical forms, which are very useful in the design of state feedback, is given.

1.2 The State-Variable Description:

The state space description of the system provides a complete picture of the system structure showing how all of the internal variables $x_i(t)$ $(i=1,2,...,n)$ interact with one another, how the inputs $u_k(t)$ $(k=1,2,...,m)$ affect the system states $x_i(t)$, and how the outputs $y_j(t)$ $(j=1,2,...,p)$ are obtained from various combinations of the state-variables $x_i(t)$ and the inputs $u_k(t)$.

A linear state model is formed by a set of first order linear differential equations with constant coefficient (1.1.a) and a set of linear equations (1.1.b).

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & (1.1.a) \\ y(t) = Cx(t) + Du(t) & (1.1.b) \end{cases}$$

where

$x(t) = [x_1(t) \; , \; . \; . \; . \; . \; , \; x_n(t)]^T$ is the system state vector.

$x_i(t), i = 1,2,...,n$ are the system state variables.

$u(t) = [u_1(t) \; , \; . \; . \; . \; . \; , \; u_m(t)]^T$ is the system input.

$y(t) = [y_1(t) \; , \; . \; . \; . \; . \; , \; y_p(t)]^T$ is the system output.

(“T “ stands for transpose).

and the system matrices (A, B, C, D) are real, constant and with dimensions $n \times n, n \times m, p \times n$ and $p \times m$, respectively.

In the above model, equation (1.1.a) is called the *dynamic equation* which describes the *dynamic part* of the system and how the initial system state $x(0)$ and system input $u(t)$ will determine the system state $x(t)$. Hence matrix A is called *the dynamic matrix* of the system. Equation (1.1.b) describes how the system state $x(t)$ and system input $u(t)$ will instantly determine system output $y(t)$. This is the output part of the system and is static (memoryless) as compared with the dynamic part of the system.

From the definition of (1.1), parameters m and p represent the number of system inputs and outputs, respectively. If $p > 1$ and if $m > 1$, then we call the corresponding system multi-input multi-output system.[77]

Definition 1.1: [6]

The state of a system at time t_0 is the amount of information at t_0 that, together with $u_{[t_0,\infty)}$ determines uniquely the behaviour of the system for all $t \geq t_0$.

System analysis generally consists of two parts: quantitative and qualitative. In the quantitative study, it is dealt with the search for the exact response of the system to certain input and initial conditions. In qualitative study, the general properties of a system are seeked.

The following section introduces two main qualitative properties of linear dynamical equations: controllability and observability [6].

1.3 Controllability and observability of Linear Systems

Controllability and observability have an important role in both theoretical and practical aspects of modern control, before the control system designer can apply a particular design method to a system, it is necessary to establish to what extent the available inputs influence the system behavior, and to what extent the available outputs indicate the system behavior. The extent to which the input influences the system is defined as the *controllability* of the system and the extent to which the output monitors the system behavior is defined as the *observability* of the system [49].

1.3.1 Controllability of Linear Time Invariant System

1.3.1.1 Controllability Matrix

Definition 1.2: [37]

For the system given by (1.1), if there exists an input $u_{[0,t]}$ which transfers the initial state $x(0) = x_0$ to the zero state $x(t_1) = 0$ in a finite time t_1, the state x_0 is said to be controllable. If all initial states are controllable the system is said to be completely controllable.

The solution of (1.1) is:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau \tag{1.2}$$

If the system is controllable, *i.e.*, there exists an input to make $x(t_1) = x_1 = 0$ at a finite time $t = t_1$, then after premultiplying by e^{-At_1} yields:

$$x_0 = \int_0^{t_1} e^{-A\tau} Bu(\tau) d\tau \tag{1.3}$$

Therefore any controllable state satisfies (1.3), and for a completely controllable system, every state $x_0 \in R^n$ satisfies t_1 (>0) and $u_{[0,t_1]}$.

It is found that complete controllability of a system depends on matrix A and B and is independent of the output matrix C.

Theorem 1.1 : [6]

The n dimensional linear time invariant state equation in (1.1) is controllable if and only if any of the following equivalent conditions is satisfied:

i. All rows of $e^{-At}B$ are linearly independent on $[0,\infty)$ over the field of complex numbers

ii. $w(0,t_1) = \int_0^{t_1} e^{-At} BB^T e^{-A^T t} dt$ is nonsingular for any $t_1 > 0$.

iii. The $n \times nm$ controllability matrix $\Phi = \begin{bmatrix} B & AB & A^2B & , & . & . & . & , & A^{n-1}B \end{bmatrix}$ has rank n.

Proof: see Chen [6].

1.3.2 Observability of Linear Time Invariant System

Dual to controllability, observability studies the possibility of estimating the state from the output. If a dynamical equation is observable all the modes of the equation are observed from the output.

Definition 1.3: [37]

When using the input of the system (1.1) measured from time zero to time t_1, if the initial state $x(0) = x_0$ is uniquely determined, x_0 is said to be observable, when the input is assumed to be completely known. When all states are observable, the system is said to be completely observable.

The output of the system (1.1) is given by:

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{1.4}$$

1.3.2.1 Observability Matrix:

Theorem 1.2: [6]

The n dimensional linear time invariant dynamical equation in (1.1) is observable if and only if any of the following equivalent conditions are satisfied:

i. All columns of Ce^{At} are linearly independent on $[0,\infty)$ over the field of complex numbers.

ii. $w(0,t_1)=\int_0^{t_1} e^{A^T t}C^T Ce^{At}dt$ is nonsingular for any $t_1>0$

iii. The $np\times n$ observability matrix $\Phi o=\begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ . \\ CA^{n-1}\end{bmatrix}$ has rank n.

Proof: see Chen [6].

1.4 Diagonalization in Linear Time-Invariant system

The Diagonalization is more general method for converting the state equation by means of a linear *similarity transformation.* Since the state variables are not unique, the intention is to transform the state vector x to a new vector $\tilde{x}$ by means of a constant, square, nonsingular transformation matrix T so that

$$x=T\tilde{x}$$

Since T is a constant matrix, the differentiation of this equation yields

$$\dot{x}=T\dot{\tilde{x}}$$

Substituting these values into the state equation $\dot{x}=Ax+Bu$ produces

$$T\dot{\tilde{x}}=AT\tilde{x}+Bu$$

Premultiplying by T^{-1} gives

$$\dot{\tilde{x}}=T^{-1}AT\tilde{x}+T^{-1}Bu$$

The corresponding output equation is

$$y=CT\tilde{x}+Du$$

The matrix T is called the *modal matrix* when it is selected so that $T^{-1}AT$ is diagonal, *i.e.*,

$$T^{-1}AT = \Lambda = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ 0 & & & & & \lambda_n \end{bmatrix} \quad (1.5)$$

1.5 Block Companion Form for MIMO System [68]

1.5.1 Block Controllable Form

Consider the n-dimensional linear time-invariant, multivariable dynamical equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (1.6)$$

where A, B, C, D are constant matrices of dimensions $n \times n, n \times m, p \times n$ and $p \times m$ real constant matrices, respectively.

Definition 1.4: [68]

The system is block controllable of index l if the matrix

$w_c = [B \quad AB \quad . \quad . \quad . \quad A^{l-1}B]$ has full rank.

The system (1.6) can be transformed into block controller form if the following conditions are satisfied

i. The number $\frac{n}{m} = l$ must be an integer.

ii. The system is controllable of index l.

Let $w_c = \left[B \quad AB \quad . \quad . \quad . \quad A^{l-1}B\right]$; the system is controllable if rank $(w_c) = n$.

Then we make a change of coordinates

$$x_C = T_C x \quad (1.7)$$

where

$$T_c = \begin{bmatrix} T_{c1} \\ T_{c1}A \\ . \\ . \\ . \\ T_{c1}A^{l-2} \\ T_{c1}A^{l-1} \end{bmatrix} \quad (1.8)$$

and

$$T_{c1} = \begin{bmatrix} 0_m & 0_m & . & . & . & I_m \end{bmatrix} \begin{bmatrix} B & AB & . & . & . & A^{l-1}B \end{bmatrix}^{-1} \tag{1.9}$$

In the new coordinates system, we have

$$\begin{cases} \dot{x}_C t) = A_C x_C(t) + B_C u(t) \\ y(t) = C_C x_C(t) \end{cases} \tag{1.10}$$

where $A_C = T_C A T_C^{-1}$, $B_C = T_C B$ and $C_C = C T_C^{-1}$

or

$$A_c = \begin{bmatrix} 0_m & I_m & . & . & . & 0_m \\ 0_m & 0_m & . & . & . & 0_m \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0_m & 0_m & . & . & . & I_m \\ -A_l & -A_{l-1} & . & . & . & -A_1 \end{bmatrix}, \ B_C = \begin{bmatrix} 0_m \\ 0_m \\ . \\ . \\ . \\ I_m \end{bmatrix}$$

and

$$C_c = \begin{bmatrix} C_l & C_{l-1} & . & . & . & C_1 \end{bmatrix}.$$

0_m and I_m are $m \times m$ null and identity matrices, respectively. A_i and C_i $(i = 1,2,...,l)$ are block elements.

1.5.2 Block Observable Form

Consider the n-dimensional linear time-invariant, multivariable dynamical equation described in (1.6)

Definition 1.5: [68]

The system is block observable of index q if the matrix

$$w_o = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{q-1} \end{bmatrix} \text{ has full rank.}$$

The system (1.6) can be transformed into block observable form if the following conditions are satisfied *iii.* The number $\frac{n}{p} = q$ must be an integer.

iv. The system is observable of index q

Let $w_o = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{q-1} \end{bmatrix}$; the system is observable if rank $(w_o) = n$.

Then we make a change of coordinates

$$x = T_o x_o \Leftrightarrow x_o = T_o^{-1} x \tag{1.11}$$

where

$$T_o = \begin{bmatrix} T_{o1} & AT_{o1} & A^2 T_{o1} & . & . & . & A^{q-1} T_{o1} \end{bmatrix} \tag{1.12}$$

and

$$T_{o1} = \begin{bmatrix} C \\ CA \\ . \\ . \\ . \\ CA^{q-1} \end{bmatrix}^{-1} \begin{bmatrix} 0_p \\ 0_p \\ . \\ . \\ . \\ I_p \end{bmatrix}$$

In the new coordinates system, we have

$$\begin{cases} \dot{x}_o(t) = A_o x_o(t) + B_o u(t) \\ y(t) = C_o x_o(t) \end{cases}$$

where

$$A_o = T_o^{-1} A T_o$$

$$B_o = T_o^{-1} B$$

and

$$C_O = C T_o$$

or

$$A_o = \begin{bmatrix} 0_p & 0_p & . & . & 0_p & -A_q \\ I_p & 0_p & . & . & 0_p & -A_q \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0_p & 0_p & . & . & 0_p & -A_2 \\ 0_p & 0_p & . & . & I_p & -A_1 \end{bmatrix}, \ B_o = \begin{bmatrix} B_1 \\ B_2 \\ . \\ . \\ . \\ B_q \end{bmatrix}$$

and

$$C_o = \lfloor 0_p \quad 0_p \quad . \quad . \quad . \quad I_p \rfloor.$$

0_p and I_p are $m \times m$ null and identity matrices, respectively. A_i and B_i $(i = 1,2,...,q)$ are block elements.

1.5.3 Block Diagonal Canonical Form

Once we have the block controllable canonical forms, we can transform it into block diagonal form using the following similarity transformation

$$x_c = V_R x_R$$

where V_R is a Vandermonde matrix which will be described in the next chapter.

Let $\{R_1, R_2, ..., R_l\}$ a complete set of right solvents, and

$$V_R = \begin{bmatrix} I & I & . & . & . & I \\ R_1 & R_2 & . & . & . & R_l \\ R_1^2 & R_2^2 & . & . & . & R_l^2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ R_1^{l-1} & R_2^{l-1} & . & . & . & R_l^{l-1} \end{bmatrix} \quad (m \times l) \times (m \times l) \tag{1.13}$$

The transformation changes the coordinates systems as follows:

$$x_c = V_R x_R \Leftrightarrow x_R = V_R^{-1} x_c \tag{1.14}$$

Differentiating both sides of the above equation produces

$$\dot{x}_R = V_R^{-1} \dot{x}_c \tag{1.15}$$

and replacing (1.14) in (1.15) yields

$$\dot{x}_R = V_R^{-1}(A_c x_c + B_c u)$$

$$\dot{x}_R = (V_R^{-1} A_c V_R) x_R + (V_R^{-1} B_c) u \tag{1.16}$$

and

$$y = C_c x_c = (C_c V_R) x_R \tag{1.17}$$

Hence, the new coordinates system matrices are:

$$\begin{aligned} A_R &= V_R^{-1} A_c V_R \\ B_R &= V_R^{-1} B_c \\ C_R &= C_c V_R \end{aligned} \tag{1.18}$$

The system may be written in block form as:

$$\dot{x} = \begin{bmatrix} R_1 & & & & & 0_m \\ & R_2 & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ 0_m & & & & & R_l \end{bmatrix} x + \begin{bmatrix} B_1 \\ B_2 \\ . \\ . \\ . \\ B_l \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 & . & . & . & C_l \end{bmatrix} x \tag{1.19}$$

As it can be seen, this is a block decoupled system. Thus it can be decomposed into l independent subsystems.

Chapter 2

Elements of Matrix Polynomial Theory

2.1 Introduction

In linear time-invariant single-input single-output system, the transfer function is a ratio of two scalar polynomials. The system modeling of physical, linear, time-invariant multi-input multi-output control system, results in high degree coupled differential equations, or an *n-th* degree m *-th* order differential equation in the form:

$$X^{(n)}(t) + A_1 X^{(n-1)}(t) + \ldots + A_{n-1} X^{(n)}(t) + A_n X(t) = U(t) \tag{2.1.a}$$

Where $A_i \in \Re^{m\times m}, X^{(i)} \in \Re^{m\times 1}$ represents the i *-th* derivate of the vector $X(t)$, and $U(t) \in \Re^{m\times 1}$ being the input vector.

The output $y(t) \in \Re^{p\times 1}$ is generally given by a differential equation in the form,

$$y(t) = C_1 X^{(n-1)}(t) + C_2 X^{(n-2)}(t) + \ldots + C_{n-1} X^{(1)}(t) + C_n X(t) \tag{2.1.b}$$

Where $C_i \in \Re^{p\times m}$.

The Laplace transformation of (2.1.a) and (2.1.b) with zero initial conditions results in

$$s^n X(s) + A_1 s^{n-1} X(s) + \ldots + A_n X(s) = U(s) \tag{2.2}$$

and

$$Y(s) = C_1 s^{n-1} X(s) + C_2 s^{n-2} X(s) + \ldots + C_n X(s) \tag{2.3}$$

which yields,

$$Y(s) = [C_1 s^{n-1} + C_2 s^{n-2} + \dots + C_n][I_m s^n + A_1 s^{n-1} + \dots + A_n]^{-1} U(s) \tag{2.4}$$

where I_m stands for the $m \times m$ identity matrix.

Equation (2.4) can be written as,

$$Y(s) = N_R(s) D_R^{-1}(s) U(s) \tag{2.5}$$

which yields the $p \times m$ transfer function matrix,

$$H(s) = N_R(s) D_R^{-1}(s) \tag{2.6}$$

Where $D_R(s)$ and $N_R(s)$ are $m \times m$ and $p \times m$ matrix polynomials also called λ-matrices, the complex variable λ is often used in stead of s, defined by:

$$D_R(s) = I_m s^n + A_1 s^{n-1} + \dots + A_n \tag{2.7}$$

$$N_R(s) = C_1 s^{n-1} + C_2 s^{n-2} + \dots + C_n \tag{2.8}$$

The equation (2.6) is the right coprime matrix fraction description (RMFD), or the polynomial matrix description [34] of MIMO system shown in (2.1).

The matrix polynomial $D_R(s)$ in (2.6) is a right denominator matrix [34, 42]

An alternative factorization of $H(s)$ is the left matrix fraction description (LMFD) defined by,

$$H(s) = D_L^{-1}(s) N_L(s) \tag{2.9}$$

where $D_L(s)$ is a $p \times p$ left denominator matrix polynomial and $N_L(s)$ is $p \times m$ left numerator matrix polynomial.

The MFD's can be regarded as extensions of the classical single-input single-output (SISO) transfer functions to the multivariable case with coprime numerator and denominator polynomials. Several methods are available for obtaining MFD's, to mention Wolivich [83], Patel [53].

In this section, we attempt to present some of important results obtained in the theory of matrix polynomials. A more emphasis will be given to the latent structure of these matrix polynomials, which consists mainly of the latent roots and latent vectors as well as solvents.

The algebraic theory of matrix polynomials has been investigated by Dennis *et al*. [14] Gohberg *et al*. [24,25, 26]. Spectral factors of a lambda matrix and right (left) solvents, for a right (left) characteristic matrix polynomial have been defined. The different transformations between right (left) solvents and spectral factors are mainly proposed by Shieh and Tsay [67]

Definition 2.1: The following $m \times m$ matrix:

$$A(\lambda) = \begin{bmatrix} a_{11}(\lambda) & a_{12}(\lambda) & . & . & . & a_{1m}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) & . & . & . & a_{2m}(\lambda) \\ . & . & . & . & . & . \\ a_{m1}(\lambda) & a_{m2}(\lambda) & . & . & . & a_{mm}(\lambda) \end{bmatrix} \tag{2.10}$$

is called a λ-matrix of order m , where $a_{ij}(\lambda)$ are scalar polynomials over the field of complex numbers.

Definition 2.2: The matrix polynomial $A(\lambda)$ is called:

i. Monic if A_0 is the identity matrix.

ii. Comonic if A_n is the identity matrix.

iii. Regular if $\det(A(\lambda)) \neq 0$.

iv. Nonsingular if $\det(A(\lambda))$ is not identically zero.

v. Unimodular if $\det(A(\lambda))$ is nonzero constant.

Other definitions for regularity and nonsingularity may be encountered in matrix polynomials literature. For example [43] defines a regular λ-matrix as one whose determinant is not identically zero and nonsingular λ-matrix as one whose determinant is a nonzero constant, thus making statement (*iv*) and (*v*) of definition 2.2 equivalent. Note that, if A_0 is nonsingular, one can always multiply by A_0^{-1} to get a monic matrix polynomial.

2.2 Latent Structure of Matrix Polynomials

Definition 2.1: [66]

The complex number λ_0 is called a latent root of $A(\lambda)$ if it is a solution of the scalar polynomial equation $\det\left(A(\lambda)\right) = 0$.

The nontrivial vector v, solution of $A(\lambda_0)v = 0$ is called a primary right latent vector associated with λ_0. Similarly the nontrivial vector p, solution of $p^T A(\lambda_0) = 0$ is called a primary left latent vector associated with λ_0.

From the definition we can see that the latent problem of a matrix polynomial is a generalization of the concept of eigenproblem for square matrices. Indeed, we can consider the classical eigenvalues/vector problem as finding the latent root/vector of a linear matrix polynomial $\lambda I - A$.

We can also define the spectrum of a matrix polynomial $A(\lambda)$ as being the set of all its latent roots (notation $\sigma(\lambda)$). It is essentially the same definition as the one of the spectrum of a square matrix.

2.3 Structure and Existence of Solvents of Matrix Polynomials

In this section we are going to see the existence of solvents and how they are important in the study of matrix polynomials.

Let X be $m\times m$ complex matrix, the two matrix polynomials, defined by

$$A_R(X) = A_0 X^l + A_1 X^{l-1} + \ldots + A_{l-1} X + A_l \quad (2.11)$$

and

$$A_L(X) = X^l A_0 + X^{l-1} A_1 + \ldots + X A_{l-1} + A_l \quad (2.12)$$

are referred to as the right and the left matrix polynomials associated with the λ-matrix $A(\lambda)$ respectively.

Definition 2.3: A right solvent R of $A(\lambda)$ is defined by

$$A_R(R) = A_0 R^l + A_1 R^{l-1} + \ldots + A_{l-1} R + A_l = 0_m \qquad (2.13)$$

and the left solvent L of $A(\lambda)$ is defined by

$$A_L(L) = L^l A_0 + L^{l-1} A_1 + \ldots + L A_{l-1} + A_l = 0_m \qquad (2.14)$$

where 0_m is an $m \times m$ null matrix, and R, L are $m \times m$ complex matrices.

The relationship between latent roots, latent vectors, and the solvents can be stated as follows [67]

Theorem 2.1 : If $A(\lambda)$ has n linearly independent right latent vectors $p_{1,} p_2, \ldots, p_n$ (left latent vectors $q_1, q_2, \ldots, q_n$) corresponding to latent roots $\lambda_1, \lambda_2, \ldots, \lambda_n$,then $P \Lambda P^{-1}$ $(Q^{-1} \Lambda Q)$ is a right (left) solvent, where $P = [p_1 \;\; p_2 \;\; . \;\; . \;\; . \;\; p_n]$ $(Q = [q_1 \;\; q_2 \;\; . \;\; . \;\; . \;\; q_n]^T)$ and $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Proof: see [40]

From the above, we can determine the relationship between a right solvent and the corresponding left solvent.

Theorem 2.2 : If $A(\lambda)$ has n latent roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the corresponding right latent vectors $p_1, p_2, \ldots, p_n$ has as well as the left latent vectors $q_1, q_2, \ldots, q_n$ are both linearly independent, then the associated right solvent R and left solvent L are related by

$$R = W L W^{-1}$$

where $W = PQ,$ $\quad P = (p_1, \ldots, p_n) \quad$ *and* $\quad Q = (q_1, \ldots, q_n)^T$

"T " stands for transpose

proof: the proof follows from theorem 2.1

Theorem 2.3 :[28] given $A(\lambda) = A_0\lambda^l + A_1\lambda^{l-1} + ... + A_l$ then ,

- The remainder of the division of $A(\lambda)$ on the right by binomial $\lambda I - R$ is $A_R(R)$ where,

$$A_R(R) = A_0R^l + A_1R^{l-1} + ... + A_{l-1}R + A_l \qquad (2.15)$$

- The remainder of the division of $A(\lambda)$ on the left by the binomial $\lambda I - L$ is $A_L(L)$ where,

$$A_L(L) = L^lA_0 + L^{l-1}A_1 + ... + LA_{l-1} + A_l \qquad (2.16)$$

The theorem above can be used to prove the following corollary.

Corollary 2.1: A matrix R (resp. L) is a right (resp. left) solvent of $A(\lambda)$ if and only if $\lambda I - R$ $(resp.\ \ \lambda I - L)$ divides exactly $A(\lambda)$ on the right (resp.left).

Proof: see Hariche [28]

Theorem 2.4: The generalized right (left) eigenvectors of a right (left) solvent are generalized latent vectors of $A(\lambda)$.

Proof : see Hariche [28]

2.4 Block Companion Form

In analogy with scalar polynomials a useful tool for the analysis of matrix polynomials is the block companion form matrix.

Given a $\lambda - matrix$

$$A(\lambda) = I\lambda^l + A_1\lambda^{l-1} + ... + A_l \qquad (2.17)$$

where $A_i \in C^{mxm}$ and $\lambda \in C$, the associated *lower block companion* form is,

$$A_L = \begin{bmatrix} 0_m & I_m & 0_m & . \; . \; . & 0_m \\ 0_m & 0_m & I_m & . \; . \; . & 0_m \\ . & . & . & . \; . \; . & . \\ 0_m & 0_m & 0_m & . \; . \; . & I_m \\ -A_l & -A_{l-1} & -A_{l-2} & . \; . \; . & -A_1 \end{bmatrix} \tag{2.18}$$

and the associated *right block companion form* is,

$$A_R = \begin{bmatrix} 0_m & 0_m & . \; . \; . & 0_m & -A_l \\ I_m & 0_m & . \; . \; . & 0_m & -A_{l-1} \\ 0_m & I_m & . \; . \; . & 0_m & -A_{l-2} \\ . & . & . \; . \; . & . & . \\ 0_m & 0_m & . \; . \; . & 0_m & -A_2 \\ 0_m & 0_m & . \; . \; . & I_m & -A_1 \end{bmatrix} \tag{2.19}$$

Note that A_L is the block transpose of A_R.

It will be useful to know the form of the eigenvectors of the lower and right block companion matrices. The results are a direct generalization of the scalar case [40].

If λ_i is a latent root of $A(\lambda)$ and p_i and q_i are the corresponding right and left latent vectors respectively, then λ_i is an eigenvalues of A_L and of A_R defined in (2.18) and (2.19),

We have the following result,

- $\begin{bmatrix} p_i \\ \lambda_i p_i \\ . \\ . \\ . \\ \lambda_i^{l-1} p_i \end{bmatrix}$ is the right eigenvector of A_L (2.20.a)

- $\begin{bmatrix} q_i^{l-1} \\ . \\ . \\ . \\ q_i^{(1)} \\ q_i \end{bmatrix}$ is the left eigenvector of A_L (2.20.b)

- $\begin{bmatrix} p_i^{l-1} \\ . \\ . \\ . \\ p_i^{(1)} \\ p_i \end{bmatrix}$ is the right eigenvector of A_R (2.20.c)

- $\begin{bmatrix} q_i \\ \lambda_i q_i \\ . \\ . \\ . \\ \lambda_i^{l-1} q_i \end{bmatrix}$ is the left eigenvector of A_R (2.20.d)

where

$$\frac{A(\lambda)p_i}{\lambda-\lambda_i} \equiv p_i\lambda^{l-1} + p_i^{(1)}\lambda^{l-2} + ... + p_i^{l-1} \tag{2.21}$$

and

$$\frac{A(\lambda)q_i}{\lambda-\lambda_i} \equiv q_i\lambda^{l-1} + q_i^{(1)}\lambda^{l-2} + ... + q_i^{l-1} \tag{2.22}$$

2.5 Block Vandermonde Matrix

The block Vandermonde matrix is of fundamental importance in the theory of matrix polynomials.

Given a set of $m \times m$ matrices $\{R_1, R_2, ..., R_k\}$ which are a complete set of right solvents of a matrix polynomial $A(\lambda)$, the following $km \times km$ matrix

$$V(R_1, R_2, ..., R_k) = \begin{bmatrix} I_m & I_m & . & . & . & I_m \\ R_1 & R_2 & . & . & . & R_k \\ . & . & . & . & . & . \\ R_1^{k-1} & R_2^{k-1} & . & . & . & R_k^{k-1} \end{bmatrix} \tag{2.23}$$

is called *the right block Vandermonde matrix* of order k, and the block transpose of *left block Vandermonde matrix* of order k is a $km \times km$ matrix defined by

$$V^T(L_1, L_2, ..., L_k) = \begin{bmatrix} I_m & L_1 & . & . & . & L_1^{k-1} \\ I_m & L_2 & . & . & . & L_2^{k-1} \\ . & . & . & . & . & . \\ I_m & L_k & . & . & . & L_k^{k-1} \end{bmatrix} \quad (2.24)$$

where $\{L_1, L_2, ..., L_k\}$ represents a set of $m \times m$ left solvents of a matrix polynomial $A(\lambda)$.

The companion matrices A_L and A_R defined in (2.18) and (2.19), can respectively block diagonalized via the right and left block Vandermonde matrices and since the Vandermonde matrices are nonsingular [14], we can write

$$[V(R_1, R_2, ..., R_k)]^{-1} A_R [V(R_1, R_2, ..., R_k)] = diag(R_1, R_2, ..., R_k) \quad (2.25)$$

and

$$[V(L_1, L_2, ..., L_k)] A_L [V(L_1, L_2, ..., L_k)]^{-1} = diag(L_1, L_2, ..., L_k) \quad (2.26)$$

2.6 Complete Set of Solvents

Several methods have been developed for solving complete set of solvents and spectral factors, without prior knowledge of the latent roots and latent vectors of a matrix polynomial, we mention for instance, Shieh *et al.* [66] have derived a generalized Newton's method. Dahimene in [11] proposed a generalization of the Quotient-Difference algorithm for the computation of spectral factors of a matrix polynomial. Tsai *et al.*[91] have obtained several algorithms for solving the complete set of solvents and spectral factors of a matrix polynomial. In this section we shall see that a complete set of solvents can be constructed using the latent roots and the latent vectors of $A(\lambda)$.

Definition 2.4 [66]**:** Given $A(\lambda)$, the set of $m \times m$ matrices $\{R_1, R_2, ..., R_l\}$ is called a *complete set of solvents* if the following conditions are met:

i. $\sigma(R_i) \cap \sigma(R_j) = \emptyset$ for $i \neq j$

ii. $\bigcup_{i=1}^{l} \sigma(R_i) = \sigma(A(\lambda))$

iii. $\det V(R_1, R_2, ..., R_l) \neq 0$

where $\sigma(R_i)$ is the spectrum of R_i and $\sigma(A(\lambda))$ is the spectrum of $A(\lambda)$

Note that in the definition 2.4 the latent roots of $A(\lambda)$ are not required to be distinct, and the concept of complete set has been defined only for the case of distinct latent roots.

The conditions for the existence and uniqueness of the complete set of solvents have been investigated by Lancaster [42], Dennis *et al.* [14] and Gohberg *et al.* [24]

The more general condition can be stated as follows [67]

Theorem 2.5: If the elementary divisors of $A(\lambda)$ are linear, then $A(\lambda)$ has a complete set of right and left solvents.

2.7 Complete Spectral Factorization

Definition 2.5: In the spectral factorization $A(\lambda) = A_1(\lambda)A_2(\lambda)$ in which $A_1(\lambda)$ and $A_2(\lambda)$ are called *spectral divisors* of $A(\lambda)$.

Definition 2.6: If a monic λ-matrix can be decomposed into the product of first-degree linear λ-matrices,

$$A(\lambda) = (\lambda I - Q_l)(\lambda I - Q_{l-1})...(\lambda I - Q_1) \qquad (2.27)$$

then the $m \times m$ matrices $Q_1, Q_2, ..., Q_l$, are called the *spectral factors* of $A(\lambda)$ and the equation (2.27) is called *a complete factorization* of $A(\lambda)$.

Note that Q_1 is a right solvent of $A(\lambda)$, whereas Q_l is a left solvent of $A(\lambda)$; other spectral factors are not, in general, right or left solvents of $A(\lambda)$.

The relationship between solvents and spectral factors are explored by Shieh and Tsay in [67], and various transformations have been developed.

Chapter 3

Block-Pole Placement Using State Feedback

One of the most popular and well known techniques used to assign the eigenvalues of the closed-loop system to desired locations is state feedback. In the case of multivariable systems, the feedback gain matrix permitting the assignment of the desired set of poles is not unique. Pole assignment techniques to modify the dynamic response of linear systems are among the most studied problems in modern control theory.

The fundamental result on pole placement by state feedback in linear time-invariant controllable systems was presented in the 1960s by Wonham [84] who states that the closed-loop eigenvalues of any controllable system may be arbitrarily assigned by state feedback control. Davison in 1970 generalized Wonham's result and showed that if the number of output variables l is less than the order of the system n, then it is always possible, by a constant feedback gain matrix, to assign l poles of the closed-loop system matrix [64]. Song and Ishida developed a method to assign the poles of the system, only one output and only one input in system was used to create the feedback controller [72]. Many different aspects of pole placement via feedback have been studied [1, 50].

One of the most important characteristics of desired performance is stability which can be achieved by locating the system poles (eigenvalues) in the left half of the s-plane [6, 34].

The pole placement discussed above uses the controllable canonical form [6, 34, 13]. However, a large scale multivariable control system described by state equations can be decomposed into small subsystems with lower order state equations, Shieh *et al.* in [69] showed that this decomposition can be achieved via the assignment of the block poles of the closed-loop system state feedback

3.1 Pole Placement for MIMO Systems Using State Feedback

Consider the n-dimensional linear time –invariant, multivariable dynamical equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{3.1}$$

where A, B, C, D are, respectively, $n \times n$, $n \times p, q \times n, q \times p$ constant matrices. In state feedback, the input $u(t)$ in (3.1) is replaced by

$$u(t) = r(t) + Kx(t) \tag{3.2}$$

where $r(t)$ stands for a reference input vector and K is a $p \times n$ real constant matrix, called the feedback gain matrix, and equation (3.1)becomes

$$\begin{cases} \dot{x}(t) = (A - BK)x(t) + Br(t) \\ y(t) = (C - EK)x(t) + Dr(t) \end{cases} \tag{3.3}$$

In the following, we shall show that if the dynamical (3.1) is controllable, then the eigenvalues of $(A - BK)$ can be arbitrarily assigned by a proper choice of K. This will be established by using three different methods.

Method I: [6]

In this method we change the multivariable problem into a single-variable problem and then apply the SISO method.

A matrix A is called cyclic if its characteristic polynomial is equal to its minimal polynomial, *i.e.*, if and only if the Jordan canonical form of A has one Jordan block associated with each distinct eigenvalue. The term of cyclicity arises from the property that if A is cyclic, then there exists a vector b such that (A, b) is controllable.

Theorem 3.1: If (A, B) is controllable, then for almost any $p \times n$ real constant matrix K, all the eigenvalues of $(A - BK)$ are distinct and consequently $(A - BK)$ is cyclic.

Proof: see [6]

Theorem 3.2: If the dynamical equation in (3.1) is controllable, by a linear state feedback of the form (3.2), where K is a $p \times n$ real constant matrix, the eigenvalues of $(A - BK)$ can be arbitrarily assigned provided complex conjugate eigenvalues appear in pairs.

Proof: see [6]

Method II (Controller-Form Method): [6, 34]

In this method, the first step will be to transform the given controllable pair (A, B) into the controllable form, that is, we search the columns of the controllability matrix from left to right until we find n linearly independent vectors, which we then rearrange in the form

$$\left\{b_1 \quad Ab_1 \quad . \quad . \quad . \quad A^{k_1-1}b_1 \quad b_2 \quad . \quad . \quad . \quad A^{k_m-1}b_m\right\} \tag{3.4}$$

then by suitable recombination of these vectors we can find a new basis

$$T_c = \left\{e_{11} \quad . \quad . \quad . \quad e_{1k_1} \quad e_{21} \quad . \quad . \quad . \quad e_{m1} \quad . \quad . \quad . \quad e_{mk_m}\right\} \tag{3.5}$$

with respect to which the pair (A, B) is in controller form, *i.e.*,

$$A_c = T_c A T_c^{-1}, B_c = T_c B \tag{3.6}$$

where A_c and B_c have the forms

$$A_c = \begin{bmatrix} A_{11} & A_{12} & . & . & . & A_{1m} \\ A_{21} & A_{22} & . & . & . & A_{2m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ A_{m1} & A_{m2} & . & . & . & A_{mm} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_1 \\ B_2 \\ . \\ . \\ . \\ B_m \end{bmatrix} \tag{3.7}$$

and C_c is in general form. The block matrices A_{ii}, A_{ij} and B_i are such that:

A_{ii} is of dimension $k_i \times k_i$, A_{ij} is of the dimension $k_i \times k_j$, and B_i is of dimension $k_i \times m$, where $\sum_{i=1}^{m} k_i = n$ and they have the following forms :

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ x & x & x & . & . & x \end{bmatrix}, A_{ij} = \begin{bmatrix} 0 & 0 & 0 & . & . & 0 \\ 0 & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 0 \\ x & x & x & . & . & x \end{bmatrix} \; and \quad B_i = \begin{bmatrix} 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & x & . & . & x \end{bmatrix} \tag{3.8}$$

where the first $i-1$ columns of B_i are zero and x is nontrivial element.

Under the transformation

$$x_c(t) = T_c x(t) \tag{3.9}$$

The state equation (3.1) becomes

$$\dot{x}_c(t) = (A_c - B_c K_c) x_c(t) + B_c r(t) \tag{3.10}$$

where $\{A_c, B_c\}$ are as in (3.7) and

$$K = K_c T_c \tag{3.11}$$

The first step in pole shifting algorithm will be to perform elementary column operations on B_c to zero out the entries marked x in the $\{k_1^{th},\quad (k_1+k_2)^{th},\quad (k_1+k_2+k_3)^{th},\quad ...\}$ rows of B_c. This can be done by elementary transformations because of the appropriately located 1s in these rows. Let us choose the nonsingular matrix D to represent these elementary transformations; *i.e.*, we choose D such that

$$\begin{aligned} B_c D &= block \quad diag\left\{[0 \quad 0 \quad . \quad . \quad . \quad 1]^T, \quad 1\times k_i \quad, i = 1,...,m\right\} \\ &= Eb_c \end{aligned} \tag{3.12}$$

Let us also define

$$\tilde{K}_c = D^{-1} K_c, \quad K_c = D\tilde{K}_c \tag{3.13}$$

so that we shall have

$$B_c K_c = B_c D \tilde{K}_c = Eb_c \tilde{K}_c = \begin{bmatrix} \tilde{k}_{11} & . & . & . & \tilde{k}_{1n} \\ k_{21} & . & . & . & \tilde{k}_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ \tilde{.} & . & . & . & \tilde{.} \\ \tilde{k}_{m1} & . & . & . & \tilde{k}_{mn} \end{bmatrix} \tag{3.14}$$

It then follows that we can make

$$A_c - Eb_c\tilde{K}_c = \left\{ \begin{array}{c} \text{a matrix with arbitrary elements in rows} \left\{ \quad k_1, k_1 + k_2, ..., \sum_{i=1}^{m} k_i \quad \right\} \\ \text{and the other rows just as in } A_c \end{array} \right\} \tag{3.15}$$

then we compute the required K.

That is, by a suitable choice of input transformation D and feedback gain matrix K we can arrange for a controllable pair (A, B) to have an arbitrary n^{th} degree characteristic polynomial. We may choose K so that $(A_c - B_c K_c)$ has blocks of companion form on the diagonal with the orders $k_1, k_2, ..., k_m$ respectively, or only one block companion form with order n.

Algorithm

Consider a multivariable system given by equation (3.1)

1. Transform the given system into controllable form.
2. Compute $\tilde{K}_c$ such that $A_c - Eb_c \tilde{K}_c$ has a set of desired eigenvalues.
3. Compute $K_c = D\tilde{K}_c$ where D is such that $B_c = Eb_c D$.
4. Compute K from K_c, such that $K = K_c T_c$.

Method III: [6]

In this method the feedback gain matrix is computed without transforming A into a controllable form. It will be achieved by solving a Lyapunov equation.

Algorithm

Consider a controllable (A, B), where A and B are, respectively, $n \times n$ and $n \times p$ constant matrices. Find a K so that $(A - BK)$ has a set of desired eigenvalues.

1. Choose an arbitrary $n \times n$ matrix F which has no eigenvalues in common with those of A.
2. Choose an arbitrary $n \times n$ matrix $\bar{K}$ such that $\{F, \bar{K}\}$ is observable.
3. Solve the unique T in Lyapunov equation $AT - TF = -B\bar{K}$.

4. If T is nonsingular, then we have $K = \overline{K}T^{-1}$, and $(A - BK)$ has the same eigenvalues as those of F . If T is singular, choose a different F or a different $\overline{K}$ and repeat the process.

3.2 Block-Pole Placement for MIMO Systems Using State Feedback

In this section, block pole placement in MIMO system is introduced; it is based on Shieh *et al*. results which concern mainly the class of MIMO systems for which the number of inputs m divides exactly the order of the state equation n: it is based on a similarity transformation that converts the state equation into a block controllable companion form [68]. In the case where the number of inputs does not divide exactly the order of the state equation [48], design can be achieved through a new similarity transformation that converts the state equation of the given multivariable system into a block-decoupled form.

To introduce the block poles of a matrix fraction description (MFD) which are the solvents of a characteristic λ-matrix, we define the characteristic λ-matrix of an MIMO system as follows:

3.2.1 Characteristic λ-matrices of MIMO Systems

Consider a linear time-invariant system described by a state equation in general coordinates:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3.16}$$

where $x \in \Re^n$, $y \in \Re^p$, $u \in \Re^m$, $A \in \Re^{nxn}$, $B \in \Re^{nxm}$, and $C \in \Re^{pxn}$.

The system (3.16) is block controllable of index l if the matrix

i. $\Phi = \begin{bmatrix} B & AB & A^2B & . & . & . & A^{l-1}B \end{bmatrix}$ has full rank

ii. $l = n/m$ is an integer

Theorem 3.3: The multivariable control system described in (3.16) can be transformed into a block controller form if two conditions are satisfied:

i. $l = n/m$ is an integer.

ii. The system is block controllable of index l.

If both conditions are satisfied, then the change of coordinates

$$x_c(t) = T_c x(t) \tag{3.17}$$

where:

$$T_c = \begin{bmatrix} T_{c1} \\ T_{c1}A \\ T_{c1}A^2 \\ . \\ . \\ . \\ T_{c1}A^{l-1} \end{bmatrix} \tag{3.18}$$

and

$$T_{c1} = \begin{bmatrix} 0_m & 0_m & . & . & . & I_m \end{bmatrix} \begin{bmatrix} B & AB & . & . & . & A^{l-1}B \end{bmatrix}^{-1} \tag{3.19}$$

transforms the system into the following block controller form

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u(t) \\ y(t) = C_c x_c(t) \end{cases} \tag{3.20.a}$$

where

$$A_c = T_c A T_c^{-1} = \begin{bmatrix} 0_m & I_m & 0_m & . & . & . & 0_m \\ 0_m & 0_m & I_m & . & . & . & 0_m \\ . & . & . & . & . & . & . \\ 0_m & 0_m & 0_m & . & . & . & I_m \\ -A_l & -A_{l-1} & -A_{l-2} & . & . & . & -A_1 \end{bmatrix} \tag{3.20.b}$$

$$B_c = T_c B = \begin{bmatrix} 0_m & 0_m & . & . & . & I_m \end{bmatrix}^T \tag{3.20.c}$$

$$C_c = C T_c^{-1} = \begin{bmatrix} C_l & C_{l-1} & . & . & . & C_1 \end{bmatrix} \tag{3.20.d}$$

where $x_C \in \Re^n, A_i \in \Re^{mxm}, C_i \in \Re^{pxm}, i = 1,2,...,l$, I_m and 0_m are m x m identity and null matrices respectively, and the superscript T (3.20.c) denotes the transpose.

Proof: see Shieh *et al.*[68]

The characteristic polynomial in SISO system is directly obtained from the nonzero elements in the last row of the system matrix, when transformed into the controllable canonical form, and the characteristic polynomial is a scalar polynomial. For multivariable control systems, the characteristic polynomial is a matrix polynomial. The right matrix fraction description (RMFD) of the system can be formulated directly from (3.20) as:

$$H(\lambda) = N_R(\lambda) D_R^{-1}(\lambda) \tag{3.21}$$

where the matrix $D_R(\lambda)$ is the right denominator given by

$$D_R(\lambda) = I_m \lambda^l + A_1 \lambda^{l-1} + ... + A_{l-1}\lambda + A_l \tag{3.22}$$

and the right numerator $N_R(\lambda)$ is given by

$$N_R(\lambda) = C_1 \lambda^{l-1} + C_2 \lambda^{l-2} + ... + C_{l-1}\lambda + C_l \tag{3.23}$$

Note that the matrix coefficients of $D_R(\lambda)$ and $N_R(\lambda)$ can be directly obtained from those nontrivial block entries of the block controllable canonical form in (3.20.b) and (3.20.d).

$D_R(\lambda)$ is referred to as the right characteristic $\lambda -$matrix of the system (3.16). In fact, $D_R(\lambda)$ can be directly determined as

$$D_R^{-1}(\lambda) = (E_1^l)^T (\lambda I_n - A_c)^{-1} = (E_1^l) T_c (\lambda I_n - A)^{-1} B \tag{3.24}$$

where

$$(E_1^l)^T = \begin{bmatrix} I_m & 0_m & . & . & . & 0_m \end{bmatrix} \in \Re^{m \times rm} \tag{3.25}$$

Examining T_C of (3.17) we have the following new result:

$$T_C = P(A_C, B_C) P^{-1}(A, B) \tag{3.26.a}$$

$$P(A,B) = \begin{bmatrix} B & AB & . & . & . & A^{l-1}B \end{bmatrix} \tag{3.26.b}$$

$$P(A_C, B_C) = \begin{bmatrix} B_C & A_C B_C & . & . & . & A_C^{l-1} B_C \end{bmatrix} \tag{3.26.c}$$

Substituting (3.26.a) into (3.24) yields the right characteristic λ-matrix of the system in (3.16),

$$D_R^{-1}(\lambda) = (E_l^l)^T P^{-1}(A,B)(\lambda I_n - A)^{-1} B \qquad (3.27.a)$$

$$(E_l^l)^T = \begin{bmatrix} 0_m & 0_m & . & . & . & 0_m & I_m \end{bmatrix} \in \Re^{m \times rm} \qquad (3.27.b)$$

From the definition of the characteristic λ-matrix, we can introduce the block poles of an MFD from the solvents of a λ-matrix.

3.2.2 Block Decomposition of MIMO Systems

Given an l-th degree m-th order monic λ-matrix

$$D_R(\lambda) = I_m \lambda^l + A_1 \lambda^{l-1} + \ldots + A_{l-1}\lambda + A_l \qquad (3.28.a)$$

The associated left matrix polynomial is given by

$$D_{RL}(\lambda) = X^l + X^{l-1}A_1 + \ldots + XA_{l-1} + A_l \qquad (3.28.b)$$

where $X \in C^{m \times m}$. If there is an $L_i \in C^{m \times m}$ such that $D_{RL}(L_i) = 0_m$ then L_i is referred to as a left solvent of $D_R(\lambda)$.

If there exist a set of left solvents $\{ L_i, i = 1,\ldots,l \}$ such that $\bigcup_{i=1}^{l} \sigma(L_i) = \sigma(D_R(\lambda))$,then $D_R(\lambda)$ has a complete set of left solvents [67].

When $D_R(\lambda)$ has a complete set of left solvents, the RMFD of (3.21) has a block partial fraction expansion as follows.

Lemma 3.1: [68] Let $\{ L_i, i = 1,\ldots,l \}$ be a complete set of left solvents of $D_R(\lambda)$, then

$$H(\lambda) = N_R(\lambda) D_R^{-1}(\lambda) = \sum_{i=1}^{l} H_i(\lambda I_m - L_i)^{-1} \qquad (3.29.a)$$

where

$$H_i = \sum_{j=1}^{l} C_j Z_i L^{l-j}, i = 1,\ldots,l \qquad (3.29.b)$$

and $Z_i \in C^{m \times m}, i = 1,\ldots,l$ can be determined from the following matrix equation:

$$\begin{bmatrix} Z_1 & Z_2 & . & . & . & Z_l \end{bmatrix} = \begin{bmatrix} 0_m & 0_m & . & . & . & 0_m & I_m \end{bmatrix} V^{-B}(L_1, L_2, ..., L_l) \tag{3.29.c}$$

$V^{-B}(L_1, L_2, ..., L_l)$ is the inverse of the block transpose of the left block Vandermonde matrix [66] and is defined in (2.23) .

Lemma 3.1 indicates that the system of (3.16) is decomposed into l parallel subsystems whose RMFD can be expressed as $H_i(\lambda I_m - L_i)^{-1}$. The solvents $L_i, i = 1,...,l$ in (3.29) are called *the right block poles* of the RMFD in (3.21) and H_i are the associated block residues of the block partial fraction of the RMFD.

If an open-loop system does not have a complete set of right block poles, then it cannot be decomposed into (3.29)

In [66] the transformation of a given system into the observable block companion form is obtained and is stated in the following theorem:

Theorem 3.4: The linear time-invariant system described by the state equation (3.16), can be transformed into the observable block companion form,

$$\begin{cases} \dot{x}_0(t) = A_0 x_0(t) + B_0 u(t) \\ y(t) = C_0 x_0(t) \end{cases} \tag{3.30}$$

where

$$A_0 = T_0 A T_0^{-1} = \begin{bmatrix} -A_{01} & I_p & 0_p & . & . & 0_p \\ -A_{02} & 0_p & I_p & . & . & 0_p \\ . & . & . & . & . & . \\ -A_{0(q-1)} & 0_p & 0_p & . & . & I_p \\ -A_{0q} & 0_p & 0_p & . & . & 0_p \end{bmatrix} \tag{3.31.a}$$

$$B_0 = \begin{bmatrix} B_{01}^T & B_{02}^T & . & . & . & B_{0(q-1)}^T & B_{0q}^T \end{bmatrix}^T \tag{3.31.b}$$

$$C_0 = \begin{bmatrix} I_p & 0_p & 0_p & . & . & . & 0_p \end{bmatrix} \tag{3.31.c}$$

by the similarity transformation,

$$x(t) = T_0 x_0(t) \tag{3.32.a}$$

where

$$T_0 = [A^{q-1}P_0^{-1}C_0^T, A^{q-2}P_0^{-1}C_0^T, ..., AP_0^{-1}C_0^T, P_0^{-1}C_0^T] \quad (3.32.b)$$

$$P_0 = \left[(CA^{q-1})^T \quad (CA^{q-2})^T \quad . \quad . \quad . \quad (CA)^T \quad C^T\right] \quad (3.32.c)$$

if and only if :

i. $q = n/p$ is an integer.

ii. The matrix P_0 in (3.32.c) has full rank.

Where $x_0 \in \Re^n, A_{0i} \in \Re^{p\times p}, B_{0i} \in \Re^{p\times m}, i = 1,...,q$ and I_p and 0_p are $p \times p$ identity and null matrices respectively.

Proof: see [66]

The LMFD of the system (3.16) can be directly formulated from the block observable form (3.31) as follows,

$$H(\lambda) = D_L^{-1}(\lambda)N_L(\lambda) \quad (3.33)$$

where the left denominator and numerator matrices are respectively given by

$$D_L(\lambda) = I_p\lambda^q + A_{01}\lambda^{q-1} + ... + A_{0(q-1)}\lambda + A_{0q} \quad (3.34)$$

$$N_L(\lambda) = B_{01}\lambda^{q-1} + B_{02}\lambda^{q-2} ... + B_{0(q-1)}\lambda + B_{0q} \quad (3.35)$$

$D_L(\lambda)$ is called the left characteristic matrix polynomial of the system(3.16).

The left characteristic matrix polynomial $D_L(\lambda)$ of the block observable system is given by

$$D_L^{-1}(\lambda) = C(\lambda I_n - A)^{-1}T_0(E_q^q) \quad (3.36)$$

where $(E_q^q) = \left[0_p \quad 0_p \quad . \quad . \quad . \quad 0_p \quad I_p\right]^T \in \Re^{qp\times p}$ and T_0 is nonsingular matrix defined in (3.32.b)

When $D_L(\lambda)$ has a complete set of right solvents $\{\hat{R}_i, i = 1,...,q\}$, the LMFD in (3.33) has a block partial fraction expansion as follows,

$$H(\lambda) = D_L^{-1}(\lambda) N_L(\lambda) = \sum_{i=1}^{q} (\lambda I_p - \hat{R}_i)^{-1} \hat{H}_i \tag{3.37.a}$$

where

$$\hat{H}_i = \sum_{j=1}^{q} \hat{R}_i^{q-j} \hat{Z}_i B_{0j}, \quad i = 1,...,q \tag{3.37.b}$$

and $\hat{Z}_i \in C^{p\times p}, i = 1,...,q$,can be determined from the following matrix equation:

$$\begin{bmatrix} \hat{Z}_1^T & \hat{Z}_2^T & . & . & . & \hat{Z}_q^T \end{bmatrix}^T = V^{-1}(\hat{R}_1, \hat{R}_2,..., \hat{R}_q) \begin{bmatrix} 0_p & 0_p & . & . & . & I_p \end{bmatrix}^T \tag{3.37.c}$$

$V^{-1}(\hat{R}_1, \hat{R}_2,..., \hat{R}_q)$ is the inverse of the block Vandermonde matrix shown in (2.23).

Similar to the decomposition shown in (3.29), equations (3.37) indicate that the system (3.16) is decomposed into q parallel subsystems whose LMFD can be expressed as $(\lambda I_p - \hat{R}_i)^{-1}\hat{H}_i$. The right solvents $\hat{R}_i, i = 1,...,q$ in (3.37.a) are called the left block poles of the LMFD in (3.33), and $\hat{H}_i$ are the associated block residues of the block partial fraction expansion of the LMFD, the left solvents $\hat{L}_i, i = 1,...,q$ of $D_l(\lambda)$ are simply called block poles of the LMFD.

3.3 Block-Pole Placement by State Feedback

The block pole placement technique, using state feedback, in multivariable control systems is formulated as follows: Given a MIMO system described by the state equation (3.16), with $n = lm$, and a desired matrix polynomial $D_f(\lambda)$ find an $m \times n$ gain matrix K such that under the state feedback operation

$$u(t) = r(t) - Kx(t) \tag{3.38}$$

the matrix $(A - BK)$ in the new state equation

$$\dot{x}(t) = (A - BK)x(t) + Br(t) \tag{3.39}$$

has the desired characteristic matrix polynomial ,

$$D_f(\lambda) = I\lambda^l + D_1\lambda^{l-1} + D_2\lambda^{l-2} + ... + D_{l-1}\lambda + D_l \tag{3.40}$$

Note that the matrix polynomial $D_f(\lambda)$ has to be constructed from a desired complete set of closed-loop block poles.

3.3.1 Block Pole Placement for a Class of MIMO Systems

The pole placement by state feedback is an effective method for the design of closed-loop control systems. In MIMO systems, the block controllable canonical form of (3.20) is especially suitable for the closed-loop block pole placement.

For the class of MIMO systems for which the number of inputs divides exactly the order of the state equation, i.e., $n = lm$, the computation of the state feedback gain matrix, achieving the desired block poles, consists of finding the matrix K_C such that the closed-loop state equation matrix $A_C - B_C K_C$ has the desired right characteristic matrix polynomial $D_f(\lambda)$ in (3.40).

Let the state feedback control law be

$$u(t) = r_c(t) - K_c x_c(t) \tag{3.41}$$

where $r_c(t) \in \Re^m$ is the reference input.

$$K_c = \begin{bmatrix} K_{cl} & K_{cl-1} & . & . & . & K_{c1} \end{bmatrix} \in \Re^{m \times lm} \tag{3.42}$$

and

$$K_c \in \Re^{m \times m}, i = 1, \ldots, l$$

then the closed-loop state equation of (3.26) become

$$\begin{cases} \dot{x}_c(t) = \hat{A}_c x_c(t) + B_c r_c(t) \\ y(t) = C_c x_c(t) \end{cases} \tag{3.43.a}$$

where the closed-loop system matrix $\hat{A}_C$ is given by

$$\hat{A}_c = A_c - B_c K_c = \begin{bmatrix} 0_m & I_m & 0_m & . & . & . & 0_m \\ 0_m & 0_m & I_m & . & . & . & 0_m \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & I_m \\ -\hat{A}_l & -\hat{A}_{l-1} & -\hat{A}_{l-2} & . & . & . & -\hat{A}_1 \end{bmatrix} \tag{3.43.b}$$

and

$$\hat{A}_i = A_i + K_i, i = 1,...,l$$

hence

$$K_i = \hat{A}_i - A_i \tag{3.43.c}$$

From (3.43), we have the closed-loop right characteristic λ-matrix

$$\hat{D}_R(\lambda) = \sum_{i=0}^{l} \hat{A}_i \lambda^{l-1}; \hat{A}_0 = I_m \tag{3.43.d}$$

which is equivalent to the desired characteristic matrix polynomial in (3.40).

3.3.2 Block-Pole Placement for General MIMO Systems

In the previous section, the block pole placement requires that the MIMO system is block controllable of index l i.e., the controllability indices of the system are all equal to l and $n = lm$.When the dimension n of the system matrix described (3.16) is not equal to lm, where l is an integer and m is the number of inputs, the proposed method cannot be directly applied. According to Shieh [69] a set of nondominant stable eigenvalues can be added at the diagonal entries of the system matrix A in (3.16) to enlarge the dimension of A from n to $\hat{n}$ such that $\hat{n} = lm$. As a result, the proposed method can be applied to obtain the block decomposition of the modified MIMO system.

In order to avoid enlarging the dimension of the system matrix A, Loubar [48] proposed a similarity transformation that will decompose the system in (3.16) into two subsystems of dimension $\hat{n} = lm$ and k respectively such that $n = \hat{n} + k$ and $k < m$. In this case, he proposed a two stage design procedure that will achieve the desired block pole placement for the system of dimension $\hat{n}$, and a pole placement for the remaining k eigenvalues through state feedback.

3.3.2.1 The Block-Decoupled Form

Consider a MIMO system described by (3.16) where n/m is not an integer. Since m does not divide exactly n, we can write:

$n = lm + k$ with $k < m$

The desired block-decoupled form is chosen as,

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u(t) \\ y(t) = C_c x_c(t) \end{cases} \tag{3.44}$$

where the matrices A_C and B_C can be written in the following form:

$$A_C = \begin{bmatrix} A_{C1} & 0_{lm,k} \\ 0_{k,lm} & p \end{bmatrix} \tag{3.45.a}$$

$$B_C = \begin{bmatrix} B_{C1} \\ B_{C2} \end{bmatrix} \tag{3.45.b}$$

$$C_C = \begin{bmatrix} C_{C1} & C_{C2} \end{bmatrix} \tag{3.45.c}$$

where $0_{lm,k}, 0_{k,lm}$ are lm x k and k x lm null matrices respectively, and

$$A_{c1} = \begin{bmatrix} 0_m & I_m & 0_m & . & . & . & 0_m \\ 0_m & 0_m & I_m & . & . & . & 0_m \\ . & . & . & . & . & . & . \\ 0_m & 0_m & 0_m & . & . & . & I_m \\ -A_l & -A_{l-1} & -A_{l-2} & . & . & . & -A_1 \end{bmatrix} \tag{3.45.d}$$

$$P = diag(P_1, P_2, ..., P_k) \tag{3.45.e}$$

$$B_C = \begin{bmatrix} 0_m & 0_m & . & . & . & I_m & B_{mk} \end{bmatrix}^T \tag{3.45.f}$$

and B_{mk} is an $m \times k$ matrix.

The desirable similarity transformation which transforms the coordinates x in (3.16) into x_C in (3.44) is defined as

$$x_C = T_C x \tag{3.46.a}$$

where

$$T_c = \begin{bmatrix} T_{c1} \\ T_{c2} \\ . \\ . \\ . \\ T_{cl} \\ T_{cl+1} \end{bmatrix} \tag{3.46.b}$$

with, T_{cl} are $m \times n$ matrices for $i = 1,2,...,l$ and T_{cl+1} is a $k \times n$ matrix.

Hence we obtain,

$$A_C = T_C A T_C^{-1} \tag{3.47.a}$$

$$B_C = T_C B \tag{3.47.b}$$

$$C_C = C T_C^{-1} \tag{3.47.c}$$

Theorem 3.5 : [48]

Given a linear time-invariant multivariable system described by the state equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3.48}$$

where $x \in \Re^n, y \in \Re^p, u \in \Re^m, A \in \Re^{n\times n}, B \in \Re^{n\times m}, C \in \Re^{p\times n}$ *and* $n = lm + k$.

The system described by (3.48) can be transformed by the similarity transformation $x_C = T_C x$, into the following state space equations:

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u(t) \\ y(t) = C_c x_c(t) \end{cases} \tag{3.49}$$

with

$$A_C = T_C A T_C^{-1} = \left[\begin{array}{ccccccc|c} 0_m & I_m & . & . & . & 0_m & & 0_{mk} \\ 0_m & 0_m & . & . & . & 0_m & & 0_{mk} \\ . & . & . & . & . & . & & . \\ 0_m & 0_m & . & . & . & I_m & & 0_{mk} \\ -A_l & -A_{l-1} & . & . & . & -A_1 & & 0_{mk} \\ \hline 0_{km} & 0_{km} & . & . & . & 0_{km} & & P \end{array}\right] \tag{3.50.a}$$

$$P = diag(p_1, p_2, ..., p_k) \tag{3.50.b}$$

$$B_C = T_C B = \begin{bmatrix} 0_m & 0_m & . & . & . & I_m & B_{mk} \end{bmatrix}^T \tag{3.50.c}$$

where B_{mk} is an $m \times k$ matrix satisfying

$$B_{mk}^T = T_{cl+1} B \tag{3.50.d}$$

with T_{cl+1} being a $k \times n$ matrix given in (3.65), if and only if the $n \times n$ matrix

$$\tilde{\Phi} = \begin{bmatrix} B & AB & . & . & . & A^{l-1}B & V_1 & V_2 & . & . & . & V_k \end{bmatrix} \tag{3.51}$$

is nonsingular, with V_i being a right eigenvector of A corresponding to the eigenvalues p_i for $i = 1,2,...,k$.

In this case the similarity transformation T_C exists and it is given by:

$$T_C = \begin{bmatrix} T_{C1} \\ T_{C1}A \\ T_{C1}A^2 \\ . \\ . \\ . \\ T_{C1}A^{l-1} \\ T_{cl+1} \end{bmatrix} \tag{3.52}$$

with

$$T_{C1} = \begin{bmatrix} 0_m & 0_m & . & . & . & I_m & 0_{mk} \end{bmatrix} \begin{bmatrix} B & AB & . & . & . & A^{l-1}B & V_1 & V_2 & . & . & . & V_k \end{bmatrix} \tag{3.53}$$

and

$$T_{cl+1} = \begin{bmatrix} T_1^T & T_2^T & . & . & . & T_k^T \end{bmatrix}^T \tag{3.54}$$

where T_i is a left eigenvector of A corresponding to the eigenvalues p_i for $i = 1,2,...,k$.

Proof: see [48]

Similar to the previous results, a second block- decoupled form can also be obtained; this will be summarized in the following theorem.

Theorem 3.6 :[48]

The linear time-invariant multivariable system described by the state equation (3.48) can be transformed by the similarity transformation $x_C = T_C x$, into the following state space equations:

$$\begin{cases} \dot{x}_C(t) = A_c x_c(t) + B_c u(t) \\ y(t) = C_c x_c(t) \end{cases} \tag{3.55}$$

with

$$A_C = T_C A T_C^{-1} = \left[\begin{array}{c|cccccc} P & 0_{km} & 0_{km} & . & . & . & 0_{km} & 0_{km} \\ \hline 0_{mk} & -A_1 & -A_2 & . & . & . & -A_{l-1} & -A_l \\ 0_{mk} & I_m & 0_m & . & . & . & 0_m & 0_m \\ 0_{mk} & 0_m & I_m & . & . & . & 0_m & 0_m \\ . & . & . & . & . & . & . & . \\ 0_{mk} & 0_m & 0_m & . & . & . & I_m & 0_m \end{array}\right] \tag{3.56.a}$$

$$P = diag(p_1, p_2, ..., p_k) \tag{3.56.b}$$

$$B_C = T_C B = \begin{bmatrix} B_{mk} & I_m & 0_m & . & . & . & 0_m \end{bmatrix}^T \tag{3.56.c}$$

where B_{mk} is an $m \times k$ matrix satisfying

$$B_{mk}^T = T_{cl+1} B \tag{3.56.d}$$

with T_{cl+1} being a $k \times n$ matrix given in (3.60), if and only if the $n \times n$ matrix

$$\tilde{\Phi} = \begin{bmatrix} V_1 & V_2 & . & . & . & V_k & A^{l-1}B & . & . & . & AB & B \end{bmatrix} \tag{3.57}$$

is nonsingular, with V_i being a right eigenvector of A corresponding to the eigenvalues p_i for $i = 1,2,...,k$.

In this case the similarity transformation T_C exists and it is given by:

$$T_C = \begin{bmatrix} T_{cl+1} \\ T_{C1}A^{l-1} \\ T_{C1}A^{l-2} \\ . \\ . \\ . \\ T_{C1}A \\ T_{C1} \end{bmatrix} \tag{3.58}$$

with

$$T_{C1} = \begin{bmatrix} 0_{mk} & I_m & . & . & . & 0_m & 0_m \end{bmatrix} \tilde{\Phi}^{-1} \tag{3.59}$$

and

$$T_{cl+1} = \begin{bmatrix} T_1^T & T_2^T & . & . & . & T_k^T \end{bmatrix}^T \tag{3.60}$$

where T_i is a left eigenvector of A corresponding to the eigenvalues p_i for $i = 1,2,...,k$.

Proof: see [48]

3.3.2.2 Find State Feedback Gain Matrix

Theorem 3.7: Given a linear time-invariant multivariable system described by the state equation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{3.61}$$

where $x \in \Re^n, y \in \Re^p, u \in \Re^m, A \in \Re^{n\times n}, B \in \Re^{n\times m}, C \in \Re^{p\times n}$ *and* $n = lm + k$, with $k < m$.

And given a desired complete set of l block poles: $\{L_1, L_2, ..., L_l\}$ and k poles: $\{p_1, p_2, ..., p_k\}$.

If the system described by (3.61) can be transformed by the similarity transformation $x_C = T_C x$, into the block-decoupled form,

$$A_c = \begin{bmatrix} A_{c1} & 0_{lm,k} \\ 0_{k,lm} & P \end{bmatrix} \quad \text{and} \quad B_C = \begin{bmatrix} B_{C1} \\ B_{C2} \end{bmatrix}$$

where, $0_{lm,k}, 0_{k,lm}$ are $lm \times k$ *and* $k \times lm$ null matrices respectively, and the matrices $A_{C1}, P, B_{C1},$ *and* B_{C2} are given in (3.45).

Then the state feedback gain matrix that achieves the desired set of block poles and poles for the closed-loop system is given by

$$K_C = \begin{bmatrix} K_{C1} + K_{C2}L & K_{C2} \end{bmatrix} \tag{3.62}$$

where K_{C1} is the feedback gain matrix which places the block poles of $(A_{C1} - B_{C1}K_{C1})$ at the desired left solvents $\{L_1, L_2, ..., L_l\}$, and L is a solution of the following *Lyapunov* equation :

$$L(A_{C1} - B_{C1}K_{C1}) - PL = B_{C2}K_{C1} \tag{3.63}$$

and K_{C2} is the feedback gain matrix which places the remaining k poles of $P - (B_{C2} + LB_{C1})K_{C2}$ at the k desired locations.

Algorithm

Let

n : Order of the state equation

m : Number of inputs

l, k are integers satisfying $n = lm + k$ with $k < m$.

Step1 : Input the system matrices A, B, C and the complete set of l left solvents $\{L_1, L_2, ..., L_l\}$ or right solvents $\{R_1, R_2, ..., R_l\}$, and the set of k poles to be assigned.

Step2 : Form the desired matrix polynomial $D_f(\lambda)$,

$$D_f(\lambda) = I\lambda^l + D_1\lambda^{l-1} + D_2\lambda^{l-2} + ... + D_{l-1}\lambda + D_l$$

from the given set of desired solvents using either:

$$\begin{bmatrix} D_l & D_{l-1} & . & . & . & D_1 \end{bmatrix} = -\begin{bmatrix} R_1^l & R_2^l & . & . & . & R_l^l \end{bmatrix} V_R^{-1} \quad (3.64.a)$$

if the matrices $R_1, R_2, ..., R_l$ form a complete set of right solvents,

or,

$$\begin{bmatrix} D_l \\ D_{l-1} \\ . \\ . \\ . \\ D_1 \end{bmatrix} = -V_L^{-B} \begin{bmatrix} L_1^l \\ L_2^l \\ . \\ . \\ . \\ L_l^l \end{bmatrix} \quad (3.64.b)$$

if the matrices $L_1, L_2, ..., L_l$ form a complete set of left solvents.

V_R and V_L^B are the right and the block transpose of the left block Vandermonde matrices respectively.

Step3 : Compute k eigenvalues of A, respectively $p_1, p_2, ..., p_k$, and find their corresponding left T_i and right V_i eigenvectors(for $i = 1,2,....,k$).

Step4 : Check that the matrix

$$\tilde{\Phi} = \begin{bmatrix} B & AB & . & . & . & A^{l-1}B & V_1 & V_2 & . & . & . & V_k \end{bmatrix} \quad (3.65)$$

is nonsingular, if not the system cannot be transformed into the block-decoupled form; hence, select a new set of k eigenvalues and go back to step3.

Step5: Compute the similarity transformation $x_C = T_C x$ shown in (3.69) and transform the system into the following block-decoupled form (block controllable form if $k = 0$)

$$A_c = \begin{bmatrix} A_{c1} & 0_{lm,k} \\ 0_{k,lm} & P \end{bmatrix} \qquad \text{and} \quad B_C = \begin{bmatrix} B_{C1} \\ B_{C2} \end{bmatrix}$$

Step6 : Compute a state feedback gain matrix K_{C1} that places the block poles of $(A_{C1} - B_{C1}K_{C1})$ at the desired l block poles using

$$K_{C1} = [K_l \quad K_{l-1} \quad . \quad . \quad . \quad K_1] \tag{3.66}$$

where $K_i = D_i - A_i$ for $i = 1,2,...,l$ and $A_i\,(i = 1,2,...,l)$ are $m \times m$ matrices obtained from A_{C1} in the block controllable form in (3.45.a).

Step7 : Compute a $k \times lm$ matrix L satisfying the *Lyapunov* equation :

$$L(A_{C1} - B_{C1}K_{C1}) - PL = B_{C2}K_{C1} \tag{3.67}$$

Step8 : Compute a feedback gain matrix K_{C2} that places the k poles of $P - (B_{C2} + LB_{C1})K_{C2}$ at the k remaining desired locations.

Step9 : Compute the state feedback gain matrix using

$$K_C = [K_{C1} + K_{C2}L \quad K_{C2}] \tag{3.68}$$

and compute the state feedback gain matrix in original coordinates using

$$K = K_C T_C \tag{3.69}$$

Chapter 4

Compensator Design Using Block-Pole Placement

4.1 Introduction

The problem of block-poles placement using state feedback is studied in the chapter 3. In this chapter, we consider the problem of assigning the closed-loop block-poles of linear time-invariant multivariable system to achieve a compensator design.

There are many possible feedback configurations: Output feedback, Input-output feedback and Unity feedback, this chapter is based on the last one.

Let us consider the feedback configurations stated above. The design problem is to find a proper compensator that achieves the desired set of poles or block poles for the closed-loop system such that the degree of the compensator is as small as possible.

The main step in the design of compensators, using arbitrary block pole placement for the closed-loop system, is the solution of the compensator equation (Diophantine equation). The solution whose rows have the minimal possible degree is proposed.

The matrix fraction description provides a natural generalization of the scalar rational function, though in multivariable case we have to distinguish between right and left descriptions, some definitions and results concerning matrix fraction description of MIMO systems needed later in this chapter are reviewed in the following section.

4.2 Matrix Fraction Descriptions

Theorem 4.1: Let $H_1(s)$ and $H_2(s)$ be, respectively $q \times p$ and $p \times q$ rational function matrices (not necessary proper), then we have

$$\det\left[(I_P + H_2(s))H_1(s)\right] = \det\left[(I_q + H_1(s))H_2(s)\right]$$

Theorem 4.2: If $\det\left[\left(I_q + H_1(s)\right)H_2(s)\right] \neq 0$, then

$$H_1(s)\left[I_p(s) + H_2(s)H_1(s)\right]^{-1} = \left[I_q + H_1(s)H_2(s)\right]^{-1} H_1(s)$$

Proof: see Chen [6]

Theorem 4.3: Let $H_1(s)$ and $H_2(s)$ be, respectively $q \times p$ and $p \times q$ rational function matrices. Then the closed-loop transfer matrix

$$H(s) = H_1(s)\left[I_p(s) + H_2(s)H_1(s)\right]^{-1}$$

is proper if and only if $I_p + H_2(\infty)H_1(\infty)$ is nonsingular.

Proof: see Chen [6]

Definition 4.1: Consider a proper rational matrix $H(s)$ factored as

$$H(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s).$$

It is assumed that $D_R(s)$ and $N_R(s)$ are right coprime and $D_L(s)$ and $N_L(s)$ are left coprime, then the characteristic polynomial of $H(s)$ is defined as

$$\det D_R(s) \text{ or } \det D_L(s)$$

and the degree of $H(s)$ is defined as

$$\deg H(s) = \deg\det D_R(s) = \deg\det D_L(s)$$

where deg det stands for the degree of the determinant.

Lemma 4.1: $N(s)$ and $D(s)$ will be right coprime if and only if they have no common latent vectors and associated latent roots.

Proof: see Kailath [34]

Let k_i be the degree of the *i-th* column of $D(s)$: if $\deg\det D(s) = \sum_{i=1}^{p} k_i$, we say that $D(s)$ is *column reduced*. If $\deg\det D(s) = \sum_{i=1}^{p} k'_i$ where k'_i is the degree of the *i-th* row of $D(s)$, $D(s)$ is said to be *row reduced*.

In general, we can write

$$D(s) = D_{hc} S(s) + L(s)$$

where

$S(s) = diag\{s^{ki}, i = 1,..., p\}$

D_{hc} = the highest-column-degree coefficient matrix, or the leading (column)coefficient of $D(s)$

$L(s)$ = denotes the remaining terms and is a polynomial matrix with column degrees strictly less than those of $D(s)$.

Then

$$\det D(s) = (\det D_{hc}(s)) S^{\sum ki} + \text{terms of lower degree in s}$$

and therefore it follows that a nonsingular polynomial matrix is column reduced if and only if its leading (column) coefficient matrix is nonsingular.

The following Lemma gives the properness of $N(s)D^{-1}(s)$ when $D(s)$ is column reduced.

Lemma 4.2: If $D(s)$ is column-reduced, then $H(s) = N(s)D^{-1}(s)$ is strictly proper (proper) if and only if each column of $N(s)$ has degree less than (less than or equal to) the degree of the corresponding column of $D(s)$.

Proof: see [34]

4.3 Pole Placement for MIMO Systems Using Design of Compensator

4.3.1 Single-input or Single-output

In this section we discuss the design of compensators to achieve pole placement for single-input multi-output and multi-input single-output systems. The general case (MIMO system) is postponed to the next section.

Consider the unity feedback system shown in figure (4.1) where the plant is described by the $q \times 1$ proper rational matrix $H(s)$:

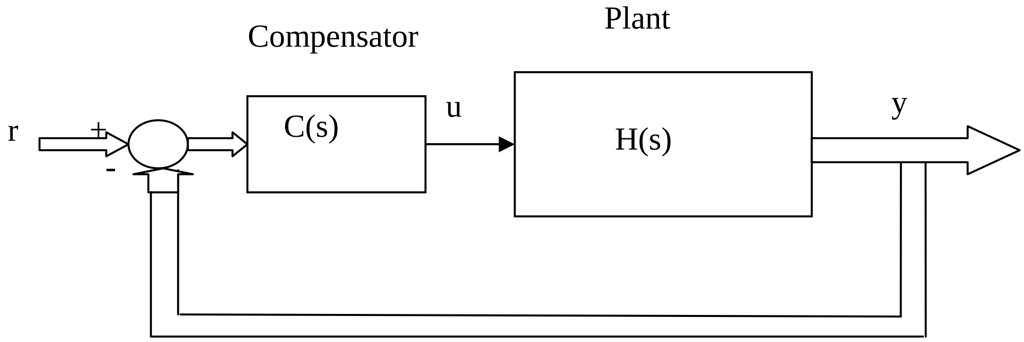


Figure 4.1.a: Single-input Multi-output

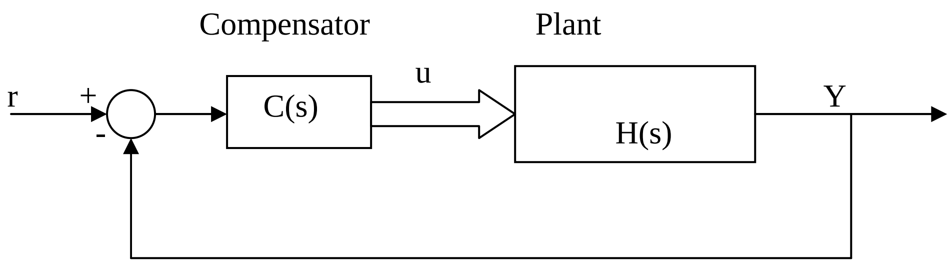


Figure 4.1.b: Multi-input Single-output

$$H(s) = \begin{bmatrix} \frac{N'_1(s)}{D'_1(s)} \\ \frac{N'_2(s)}{D'_2(s)} \\ \cdot \\ \cdot \\ \cdot \\ \frac{N'_q(s)}{D'_q(s)} \end{bmatrix} = \frac{1}{D(s)} \begin{bmatrix} N_1(s) \\ N_2(s) \\ \cdot \\ \cdot \\ \cdot \\ N_q(s) \end{bmatrix} = N(s)D^{-1}(s) \tag{4.1}$$

where $D(s)$ the least common denominator of all elements of $H(s)$.

We assume

$$\begin{aligned} D(s) &= D_0 + D_1 s + D_2 s^2 + \ldots + D_n s^n \qquad D_n \neq 0 \\ N(s) &= N_0 + N_1 s + N_2 s^2 + \ldots + N_n s^n \end{aligned} \tag{4.2}$$

where D_i are constant and N_i are $q \times 1$ constant vectors.

The problem is to find a compensator with a proper transfer matrix of degree m so that $n+m$ number of poles of the feedback system in figure (4.1.a) can be arbitrarily assigned. Furthermore, the degree m of the compensator is required to be as small as possible.

The closed- loop transfer function matrix of the feedback system of figure (4.1.a) is given by

$$H_{cl}(s) = H(s)[1 + C(s)H(s)]^{-1} C(s) \tag{4.3}$$

Let us write the compensator $C(s)$ as

$$C(s) = \frac{1}{D_C(s)} \begin{bmatrix} N_{C1}(s) & N_{C2}(s) & . & . & . & N_{Cq}(s) \end{bmatrix} = D_C^{-1}(s) N_C(s) \tag{4.4}$$

with

$$\begin{aligned} D_C(s) &= D_{C0} + D_{C1} s + \ldots + D_{Cm} s^m \\ N_C(s) &= N_{C1} + N_{C2} s + \ldots + N_{Cm} s^m \end{aligned} \tag{4.5}$$

where D_{ci} are scalars and N_{ci} are $1 \times q$ constant vectors. The substitution of (4.1) and (4.4) into (4.5) yields

$$\begin{aligned} H_{cl}(s) &= N(s) D^{-1}(s) \left[1 + D_C^{-1}(s) N_C(s) N(s) D^{-1}(s)\right]^{-1} D_C^{-1}(s) N_C(s) \\ &= \left[D_C(s) D(s) + N_C(s) N(s)\right]^{-1} N(s) N_C(s) \end{aligned} \tag{4.6}$$

because $N(s)$ and $N_C(s)$ are $q \times 1$ and $1 \times q$ vectors, $N_C(s)N(s)$ is a 1×1 matrix and $N(s)N_C(s)$ is a $q \times q$ matrix. Hence $H_{cl}(s)$ is a $q \times q$ rational matrix. Define

$$D_f(s) = D_C(s) D(s) + N_C(s) N(s) \tag{4.7}$$

Hence the problem of pole placement reduces to solve equation (4.7) which is called the *Diophantine equation* (or the *Compensator equation*).

Theorem 4.4: Consider the feedback system shown in figure (4.1.a) with the plant described by a $q \times 1$ strictly proper (proper) rational matrix $H(s) = N(s)D^{-1}(s)$ with $\deg D(s) = n$. Then for $D_f(s)$ of degree $n+m$, there exists a $1 \times q$ proper (strictly proper) compensator $C(s) = D_C^{-1}(s)N_C(s)$ with $\deg D_C(s) = m$ so that the feedback system has $q \times q$ transfer function matrix $N(s)D_f^{-1}(s)N_C(s)$ if and only if $D(s)$ and $N(s)$ are right coprime and $m \geq v-1$ $(m \geq v)$, where v is the row index of $H(s)$.

Proof: see Chen [6]

Dual to theorem 4.4, we have the following theorem for the feedback system shown in figure (4.1.b).

Theorem 4.5: Consider the feedback system shown in figure (4.1.b) with the plant described by a strictly proper (proper) $1 \times p$ rational matrix $H(s) = D^{-1}(s)N(s)$ with $\deg D(s) = n$. Then for any $D_f(s)$ of degree $n+m$, there exists a $p \times 1$ proper (strictly proper) compensator $C(s) = N_C(s)D_C^{-1}(s)$ with $\deg D_C(s) = m$ so that the feedback system has 1×1 transfer function $N(s)D_f(s)N_C(s)$ if and only if $D(s)$ and $N(s)$ are left coprime and $m \geq \mu - 1$ $(m \geq \mu)$ where μ is column index of $H(s)$

Proof: see Chen [6]

The polynomial equation arising in this theorem is of the form

$$D_f(s) = D(s)D_c(s) + N(s)N_C(s) \tag{4.8}$$

4.3.2 Multi-input Multi-output

In this section, the design technique developed in the previous section will be extended to general proper rational matrices. We extend it first to a special class of rational matrices, called *cyclic* rational matrices, and then to the general case.

4.3.2.1 Pole Placement for Cyclic Rational Matrices

Consider a $q \times p$ proper rational matrix $H(s)$.

Let $\Psi(s)$ and $\Delta(s)$ be the least common denominator of all elements of $H(s)$ and the characteristic polynomial of $H(s)$, respectively. In general, we have $\Delta(s) = \Psi(s)h(s)$ for some polynomial $h(s)$. If $\Delta(s) = \Psi(s)k$ for some constant k, then $H(s)$ is called a *cyclic rational matrix*. For cyclic rational matrices, the characteristic polynomial is equal to the minimal polynomial.

Theorem 4.6: Consider a $q \times p$ cyclic rational matrix $H(s)$. Then for almost all $p \times 1$ and $1 \times q$ real constant vectors t_1 and t_2, we have

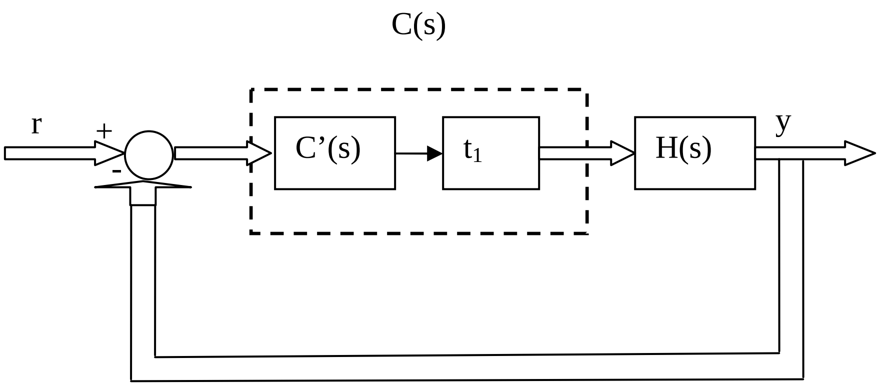
$$\Delta[H(s)] = \Delta[H(s)t_1] = \Delta[t_2 H(s)] \tag{4.9}$$

Where $\Delta(.)$ denotes the characteristic polynomial of a rational matrix.

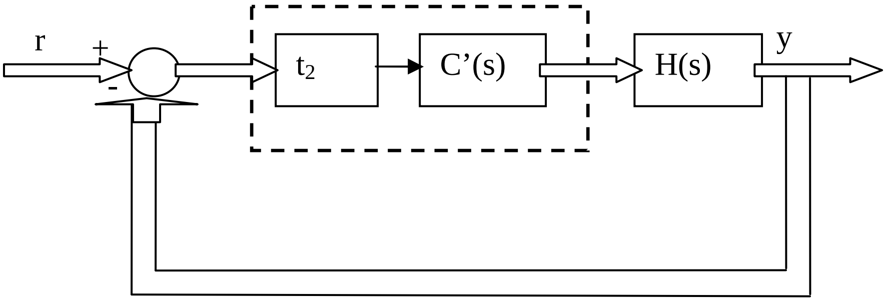
Proof: see Chen [6]

Using theorem 4.6, we can extend the design procedure in theorems 4.4 and 4.5 to cyclic rational matrices.

Theorem 4.7: Consider the feedback system shown in figure (4.2) with the plant described by a $q \times p$ cyclic strictly proper (proper) rational matrix $H(s)$ of degree n. The compensator is assumed to have a $p \times q$ proper (strictly proper) rational matrix $C(s)$ of degree m. If $m \geq \min(\mu - 1, v - 1)$ $[m \geq \min(\mu, v)]$, then all $n + m$ poles of the unity feedback system can be arbitrarily assigned, where μ and v are, respectively, the column index and the row index of $H(s)$.



(a)



(b)

Fig 4.2: design of compensators for plant with cyclic proper rational matrices.

Since $H(s)$ is cyclic, there exists a $p\times 1$ constant vector t_1 such that $\Delta[H(s)]=\Delta[H(s)t_1]$.
Let us write the $q\times 1$ rational matrix $H(s)t_1$ as

$$H(s)t_1 = N(s)D^{-1}(s)$$

then theorem 4.4 implies the existence of a $1\times q$ proper rational matrix $C'(s)=D_C^{-1}(s)N_C(s)$ with $\deg C'(s)=m\geq v-1$ if $H(s)$ is strictly proper, such that $n+m$ poles of

$$D_f(s)=D_C(s)D(s)+N_C(s)N(s) \tag{4.10}$$

can be arbitrarily assigned. It is shown [6] that the $q\times p$ compensator defined by $C(s)=t_1C'(s)=D_C^{-1}(s)t_1N_C(s)$ can achieve arbitrarily pole placement.

The closed-loop transfer function is given by

$$H_{cl}(s)=N(s)D_f^{-1}(s)N_C(s) \tag{4.11}$$

where $D(s)$ and $D_C(s)$ are 1×1 polynomial matrices.

4.3.2.2 Pole Placement for General Rational Matrices

We can now discuss the design of compensators for general proper rational matrices. The procedure consists of two steps: First change a noncyclic rational matrix into a cyclic one and then apply Theorem 4.7.

Theorem 4.8: consider a $q\times p$ proper (strictly proper) rational matrix $H(s)$. Then for almost every $p\times q$ constant matrix K, the $q\times p$ rational matrix

$$H'(s) = [I + H(s)K]^{-1} H(s) = H(s)[I + KH(s)]^{-1}$$

is proper(strictly proper) and cyclic.

Proof: see [6]

With this theorem, the design of a compensator to achieve arbitrarily pole placement for general $H(s)$ consists of two steps: We first introduce a constant gain output feedback K to make $H'(s) = [I + H(s)K]^{-1} H(s)$ cyclic. We then apply Theorem 4.7 to design a compensator $C(s)$. Hence all the poles of the feedback system in figure (4.3) can be arbitrarily assigned.

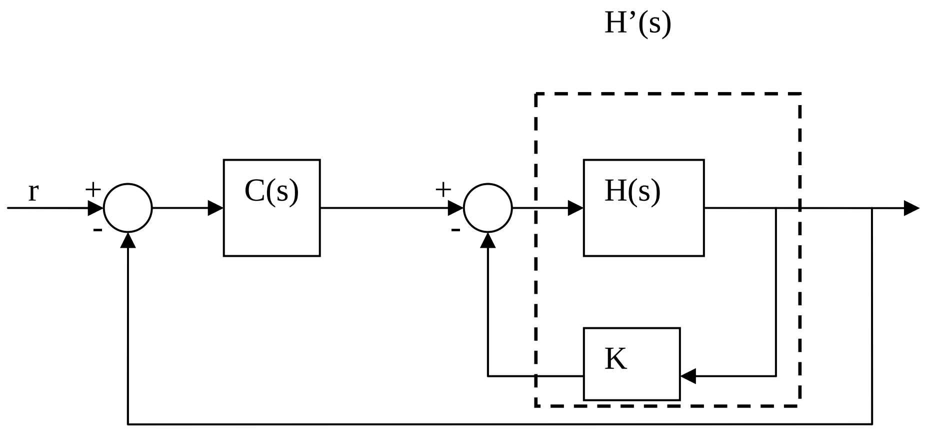


Figure 4.3 :Unity feedback system for noncyclic rational matrices

4.4 Block-Pole Placement for MIMO Systems

In this section, we study the design of compensator to achieve arbitrary block- poles for the closed-loop system; this is equivalent to the assignment of an entire denominator matrix polynomial.

4.4.1 Unity Feedback Systems

Consider the unity feedback system in figure (4.4). The plant is described by a $q \times p$ proper rational matrix.

$$H(s) = N(s)D^{-1}(s) \tag{4.12}$$

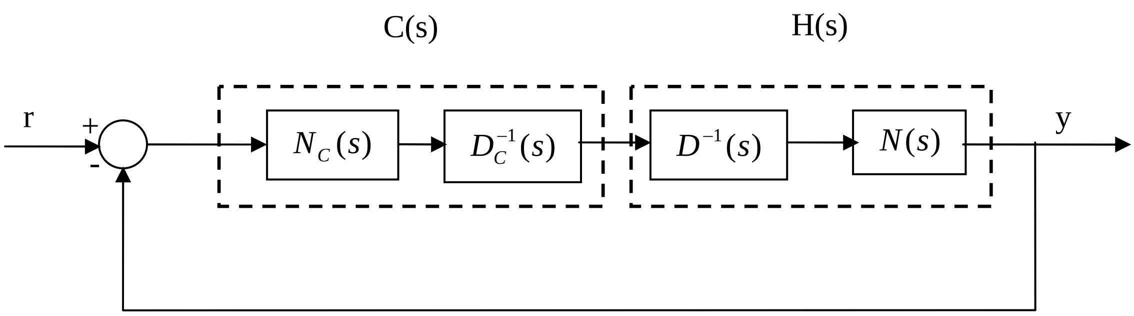


Figure 4.4: Unity feedback for multivariable system

The compensator to be designed is required to have a $p \times q$ proper rational matrix.

$$C(s) = D_C^{-1}(s)N_C(s) \tag{4.13}$$

The closed-loop transfer matrix is given by

$$H_{cl}(s) = \left[I_q + H(s)C(s)\right]^{-1} H(s)C(s) \tag{4.14}$$

Using a theorem 4.1 we obtain,

$$H_{cl}(s) = H(s)\left[I_p + C(s)H(s)\right]^{-1} C(s) \tag{4.15}$$

Replacing (4.12) and (4.13) in (4.15) yields

$$H_{cl}(s) = N(s)D^{-1}(s)\left[I + D_C^{-1}(s)N_C(s)N(s)D^{-1}(s)\right]^{-1} D_C^{-1}(s)N_C(s) \tag{4.16}$$

which can be written as

$$H_{cl}(s) = N(s)\left[D_C(s)D(s) + N_C(s)N\ \ (s)\right]^{-1} N_C(s) \tag{4.17}$$

Define the matrix polynomial,

$$D_f(s) = D_C(s)D(s) + N_C(s)N\ \ (s) \tag{4.18}$$

Then we have

$$H_{cl}(s) = N(s)D_f^{-1}(s)N_C(s) \tag{4.19}$$

Hence the design problem becomes: Given $D(s)$ and $N(s)$ and an arbitrary $D_f(s)$, find $D_C(s)$ and $N_C(s)$ to satisfy the compensator equation (4.18).

From (4.19) we note that the roots of $D_f(s)$ are the poles of the closed-loop transfer matrix $H_{cl}(s)$, and the solvents of $D_f(s)$ are block-poles of $H_{cl}(s)$.

4.4.2 Input-Output Feedback Systems using Design of Compensator

Consider the input-output feedback system shown in figure (4.5). The plant is described by a $q \times p$ proper rational matrix $H(s) = N(s)D^{-1}(s)$. The compensators are denoted by the $p \times p$ proper rational matrix $C_0(s) = D_C^{-1}(s)L(s)$ and $p \times q$ rational matrix $C_1(s) = D_C^{-1}(s)N_C(s)$. The closed-loop transfer matrix can be computed as

$$H_{cl}(s) = N(s)\left[D_C(s)D(s) + L(s)D(s) + N_C(s)N(s)\right]^{-1} D_C(s) \tag{4.20}$$

or
$$H_{cl}(s) = N(s)D_f^{-1}(s)D_C(s) \tag{4.21}$$

where $D_f(s)$ is defined as

$$D_f(s) = D_C(s)D(s) + L(s)D(s) + N_C(s)N(s) \tag{4.22}$$

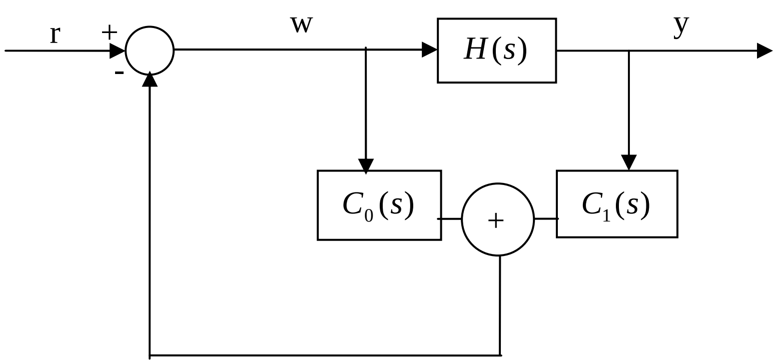


Figure 4.5: Input-Output Feedback

If we let

$$E(s) = D_f(s) - D_C(s)D(s) \tag{4.23}$$

then (4.22) can be written as

$$E(s) = L(s)D(s) - N_C(s)N(s) \tag{4.24}$$

which is the compensator equation.

Note that before solving the compensator equation (4.24), the denominator matrix $D_C(s)$ of the compensators $C_0(s)$ and $C_1(s)$ should be chosen in order to compute $E(s)$ in (4.23).

4.4.3 Output Feedback Systems

Consider the feedback system in figure (4.6). Using the previous results, it can be readily shown that the closed-loop transfer matrix can be written as

$$H_{cl}(s) = N(s)(D_C(s)D(s) + N_C(s)N(s))^{-1} D_C(s) \tag{4.25}$$

or,

$$H_{cl}(s) = N(s)D_f^{-1}(s)D_C(s) \tag{4.26}$$

defining again $D_f(s)$ as

$$D_f(s) = D_C(s)D(s) + N_C(s)N(s) \tag{4.27}$$

it follows that the solvents of $D_f(s)$ are the block poles of the closed-loop transfer matrix.

Note that the main step in the design of compensators is the solution of the compensator equation (Diophantine equation).

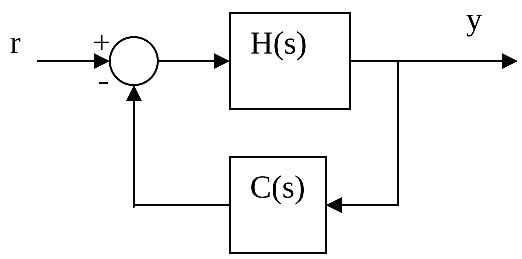


Figure 4.6 : Output Feedback

4.5 Solution of the Diophantine Equation

The compensator design, to achieve arbitrary block pole placement for the feedback configurations described previously, requires the solution of the compensator equation:

$$D_f(s) = D_C(s)D(s) + N_C(s)N(s) \tag{4.28}$$

for a given plant rational transfer matrix $H(s) = N(s)D^{-1}(s)$ and a desired matrix polynomial $D_f(s)$.

and

$$D_f(s) = D(s)D_c(s) + N(s)N_c(s)$$

for a given multivariable system described by a LMFD $H(s) = D^{-1}(s)N(s)$ where

$D(s), D_c(s)$ and $D_f(s)$ are $q \times q$ polynomial matrices, while $N(s)$ and $N_c(s)$ are $q \times p$ and $p \times q$ polynomial matrices, respectively. The desired compensator will be described by the $p \times q$ RMFD,

$$C(s) = N_c(s)D_c^{-1}(s)$$

The following theorem gives the condition for the existence of the solution of (4.28).

Theorem 4.9: Consider a $q \times p$ proper rational matrix with the fraction $H(s) = N(s)D^{-1}(s)$.

Let k_i, $i = 1,2,...,p$, be the column degrees of $D(s)$, and let v be the row index of $H(s)$. If $m \geq v - 1$, then for any $D_f(s)$ with column degrees $m + k_i, i = 1,2,...,p$ or less, there exist $D_C(s)$ and $N_C(s)$ of row degree m or less to meet

$$D_f(s) = D_C(s)D(s) + N_C(s)N(s)$$

if and only if $D(s)$ and $N(s)$ are right coprime and $D(s)$ is column reduced.

Proof: see Chen [6]

Various numerical algorithms, for solving the Diophantine equation, have been developed and different approaches have been attempted [88, 40, 20, 63, 19, 85, 41].

It has been shown in [6, 34] that the coprimeness of $D(s)$ and $N(s)$ ensures the existence of the solution to the Diophantine equation for an arbitrary $D_f(s)$.

The method proposed in this section is developed from the results obtained by Chen [6] and Lai [41]. The idea is basically to transform the given matrices into a set of linear algebraic equations, which leads to the construction of a Sylvester matrix (or a generalized resultant matrix of $\{N(s), D(s)\}$). The solution is obtained by applying searching algorithms for linearly dependent rows of the obtained matrix.

The compensator equation defined in (4.28) can be written [41] as

$$\begin{bmatrix} D_C(s) & N_C(s) & I \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \\ -D_f(s) \end{bmatrix} = 0 \tag{4.29}$$

Let us write

$$D(s) = \sum_{i=0}^{h} D_i s^i \quad ; \quad D_c(s) = \sum_{i=0}^{m} D_{ci} s^i \; ; \; N(s) = \sum_{i=0}^{h} N_i s^i \quad ; \; N_c(s) = \sum_{i=0}^{m} N_{ci} s^i \; ;$$

$$D_f(s) = \sum_{i=0}^{l} D_f s^i \tag{4.30}$$

as a set of linear algebraic equations

The substitution of (4.30) in (4.28) yields

$$\begin{bmatrix} D_{c0} & N_{c0} & I & | D_{c1} & N_{c1} | & D_{c2} & N_{c2} & | . & . & . | & D_{cm} & N_{cm} \end{bmatrix} \hat{S}_m = 0 \tag{4.31}$$

where

$$\hat{S}_m = \left[\begin{array}{cccccccccccc}
D_0 & D_1 & . & . & . & D_h & 0 & 0 & . & . & . & 0 \\
N_0 & N_1 & . & . & . & N_h & 0 & 0 & . & . & . & 0 \\
-D_{f0} & -D_{f1} & . & . & . & -D_{fh} & . & -D_{fl} & 0 & . & . & 0 \\
\hline
0 & D_0 & D_1 & . & . & . & D_h & 0 & . & . & . & 0 \\
0 & N_0 & N_1 & . & . & . & N_h & 0 & . & . & . & 0 \\
\hline
0 & 0 & D_0 & D_1 & . & . & . & D_h & 0 & . & . & 0 \\
0 & 0 & N_0 & N_1 & . & . & . & N_h & 0 & . & . & 0 \\
\hline
. & & & & & . & & & & & & . \\
. & & & & & . & & & & & & . \\
. & & & & & . & & & & & & . \\
\hline
0 & 0 & . & . & . & 0 & D_0 & D_1 & . & . & . & D_h \\
0 & 0 & . & . & . & 0 & N_0 & N_1 & . & . & . & N
\end{array}\right] \qquad \begin{array}{l} \text{1st block (first three rows)} \\ \text{(m+1)th block (last two rows)} \end{array} \tag{4.32}$$

The matrix $\hat{S}_m$ has $m+1$ block rows; the first block contains $(2p+q)$ rows and $q+p$ rows in the *i-th* block, where $2 \le i \le m+1$.

For the solution of the compensator equation we need to search for the linearly dependent rows of $\hat{S}_m$ in order from top to bottom using either row-searching [6] or recursive [86] algorithm.

Let *D*-row denote the rows formed from the rows of D_i 's and let D_μ^α-row denote the $\alpha - th$ D-row in the $\mu - th$ block of (4.32).

Definition 4.2: [41]

A dependent row, say D_μ^α-row, is called a primary dependent row of $\hat{S}_m$ if all the $D_{\tilde{\mu}}^\alpha$-rows are independent rows in $\hat{S}_m$ for $\mu < \tilde{\mu}$.

The general form of the solution of a compensator $C(s) = D_c^{-1}(s) N_c(s)$, instead of

$$\left[D_{c0} \quad N_{c0} \quad I \quad \left| D_{c1} \quad N_{c1} \right| \quad D_{c2} \quad N_{c2} \quad \left| . \quad . \quad . \right| \quad D_{cm} \quad N_{cm}\right] \tag{4.33}$$

will be

$$\left[D'_{c0} \quad N'_{c0} \quad C \quad \left| D'_{c1} \quad N'_{c1} \right| \quad D'_{c2} \quad N'_{c2} \quad \left| . \quad . \quad . \right| \quad D'_{cm} \quad N'_{cm}\right] \tag{4.34}$$

with

$$D'_c(s) = \sum_{i=0}^{m} D'_{ci} s^i \qquad\qquad N'_c(s) = \sum_{i=0}^{m} N'_{ci} s^i$$

Theorem 4.10: [41] Consider a given $D(s)$, $N(s)$ and $D_f(s)$ in (4.28). Then there exists a solution if and only if C in (4.34) is a real constant matrix with $\det(C) \neq 0$.

The solution of (4.31) will be given by the product ,

$$C^{-1}\left[D'_{c0} \quad N'_{c0} \quad C \quad \left|D'_{c1} \quad N'_{c1}\right| \quad D'_{c2} \quad N'_{c2} \quad \left|. \quad . \quad .\right| \quad D'_{cm} \quad N'_{cm}\right] \tag{4.35}$$

$$D_c(s) = C^{-1} D'_c(s)$$
$$N_c(s) = C^{-1} N'_c(s)$$
$$C(s) = D_c^{-1}(s) N_c(s)$$

the obtained compensator will have the minimal degree which is one of the requirement stated previously.

In the determination of the solution of the compensator equation (4.28), the main step is to search for the first linearly dependent rows of $\hat{S}_m$.

Lemma 4.3: [6] If $H(s) = N(s)D^{-1}(s)$ is proper, all D-rows in $\hat{S}_m$, are linearly independent of their previous rows.
Some N-rows in each block, however, may be linearly dependent on their previous rows.

Let r_i be the number of linearly dependent N-rows in the $(i+1)th$ block of $\hat{S}_m$, then because of the structure of $\hat{S}_m$ we have $r_0 \le r_1 \le ... \le r_m \le q$ .let v be the least integer such that $r_v = q$. In this case, we call v the *row index of H(s).*
In the case where the number of inputs is less or equal to the number of outputs, it is sufficient to find the row index of $H(s)$ in order to solve (4.31) with $m=v$.
The following algorithm is a modified version of the recursive algorithm used for finding the row index of the given *H(s).*

4.5.1 Modified Recursive Algorithm for Finding the Row Index

Consider the matrix $\hat{S}_m$ in (4.32) but without the $D_{fi}-$rows, say S_m, and consider a $q\times p$ proper rational matrix $H(s)=N(s)D^{-1}(s)$. In order to improve the recursive algorithm we will make use of the following properties of S_m :

i. The linearly dependent rows appear only in *N*-rows

ii. The addition of the block row to S_m results in the addition of zeros to the right of the previous block row.

According to the definition of the row index, if the number of linearly dependent *N*-rows in the $(j+1)th$ is equal to *q* (number of outputs), then the row index of *H(s)* is equal to *j*.

Let :

S_i : generalized resultant matrix with $(i+1)$ block rows

P_i : projection matrix corresponding to the last row of S_i

r_j : number of linearly dependent *N*-rows in the $(j+1)th$ block row of S_i

v: the row index of *H(s)*

Step1: Initialize $i=0$ and

$$S_0=\begin{bmatrix} D_0 & D_1 & . & . & . & D_h \\ N_0 & N_1 & . & . & . & N_h \end{bmatrix}$$

Step2: Use the recursive algorithm to compute r_i

while $r_i \neq q$ do

Step3: Update

$$S_{i+1}=\begin{bmatrix} S_i & & 0 \\ 0 & D- & rows \\ 0 & N- & rows \end{bmatrix}$$

Step4: Update

$$P_{i+1} = \begin{bmatrix} P_i & 0 \\ 0 & I_m \end{bmatrix}$$

Step5: Update $i = i + 1$ then back to step1

Step6: Finally $v = i$.

4.5.2 Algorithms for Finding the Solution of the Compensator Equation

For the computation of a minimal degree proper compensator that achieves a desired set of block poles for the closed-loop unity feedback systems two algorithms are proposed, and to ensure the existence of q primary dependent rows on $\hat{S}_v$ in the case where the number of inputs is less or equal the number of outputs, both algorithms require the computation of the row index of $H(s)$.

4.5.2.1 Row -Searching Algorithm

Let:

v: the row index of $H(s)$

p: number of inputs

q: number of outputs

Step1: Input D_i and N_i for $i = 1,2,....h$

Input D_{fi} for $i = 1,2,...l$.

Step2: Use the modified recursive algorithm to find the row index v of $H(s)$

step3: Form

$$\hat{S}_v = \begin{bmatrix} D_0 & D_1 & . & . & . & D_h & 0 & 0 & . & . & . & 0 \\ N_0 & N_1 & . & . & . & N_h & 0 & 0 & . & . & . & 0 \\ -D_{f0} & -D_{f1} & . & . & . & -D_{fh} & . & -D_{fl} & 0 & . & . & 0 \\ 0 & D_0 & D_1 & . & . & . & D_h & 0 & . & . & . & 0 \\ 0 & N_0 & N_1 & . & . & . & N_h & 0 & . & . & . & 0 \\ 0 & 0 & D_0 & D_1 & . & . & . & D_h & 0 & . & . & 0 \\ 0 & 0 & N_0 & N_1 & . & . & . & N_h & 0 & . & . & 0 \\ . & & & & & . & & & & & & . \\ . & & & & & . & & & & & & . \\ . & & & & & . & & & & & & . \\ 0 & 0 & . & . & . & 0 & D_0 & D_1 & . & . & . & D_h \\ 0 & 0 & . & . & . & 0 & N_0 & N_1 & . & . & . & N_h \end{bmatrix} \qquad (4.36)$$

and apply the row – searching algorithm to $\hat{S}_V$ to obtain the primary dependent rows.

Step4: Select the first *q* primary dependent rows among the primary dependent rows of $\hat{S}_V$, then form

$$\left[D'_{C0} \quad N'_{C0} \quad C \quad \left|D'_{C1} \quad N'_{C1}\right| \quad D'_{C2} \quad N'_{C2} \quad \left|. \quad . \quad .\right| \quad D'_{Cm} \quad N'_{Cm}\right] \tag{4.37}$$

Using the coefficients of the linear combinations of the chosen primary dependent rows from their previous linearly independent rows in $\hat{S}_V$.

Step5: If the test matrix *C* is singular then back to step 4

Step6: If *C* is nonsingular then compute

$$D_C(s) = C^{-1}D'_C(s)$$
$$N_C(s) = C^{-1}N'_C(s)$$

step7: If $C(s) = D_c^{-1}(s)N_c(s)$ is not a proper compensator the back to step 4.

If the closed loop transfer function matrix is not proper then back to step 4.

Remark: Once the row-searching algorithm is applied to Sylvester matrix $\hat{S}_m$ for searching for the linearly dependent row, the result is a matrix, say $\bar{\hat{S}}_m$, given by

$$K_{n-1}K_{n-2}...K_2K_1\hat{S}_m = K\hat{S}_m = \bar{\hat{S}}_m$$

where $n = (q+p)(m+1)+q$

The rows of $\hat{S}_m$ corresponding to the nonzero rows of $\bar{\hat{S}}_m$ are linearly independent of their previous rows. If a row in $\bar{\hat{S}}_m$ is a zero row, then the corresponding row in $\hat{S}_m$ is linearly dependent of its previous rows, and the corresponding row vector in *K* will give the coefficients of the linear combination.

4.5.2.2 Recursive Algorithm

Using the same notations as the previous algorithm

Step1: Input D_i and N_i for $i = 1,2,....h$

Input D_{fi} for $i = 1,2,...l$.

Step2: Use the modified recursive algorithm to find the row index *v* of $H(s)$

Step3: Form $\hat{S}_V$ as in (4.36), then apply the recursive algorithm to $\hat{S}_V$ to obtain the primary dependent rows.

Step4: Select the first *q* primary dependent rows among the primary dependent rows of $\hat{S}_V$. Then solve the corresponding equation of the form, $XA = B$, to obtain the coefficients of the combination in the form (4.37).

Step5: If the test matrix *C* in (4.37) is singular then back to step4.

Step6: If *C* in (4.37) is nonsingular then compute

$$D_c(s) = C^{-1}D'_c(s)$$
$$N_c(s) = C^{-1}N'_c(s)$$

Step7: If $C(s) = D_c^{-1}(s)N_c(s)$ is not a proper compensator then back to step 4

If the closed –loop transfer function matrix is not proper the back to step 4.

In the case of the multivariable systems described in LMFD, the previous algorithms can be applied for the compensator equation given by

$$D^T{}_f(s) = D_c^T(s)D^T(s) + N_C^T(s)N^T(s)$$

where *T* stands for the matrix transpose.

Chapter 5

Sensitivity and Robust systems

Sensitivity considerations are important in the design of control systems. Since all physical elements have properties that change with environment and age, we cannot always consider the parameters of a control system to be completely stationary over the entire operating life of the system. In general, a good control system should be insensitive to parameter variations but sensitive to the input commands ones [39].

High system performance and low sensitivity are two required properties of control systems. Low sensitivity is defined with respect to the system's mathematical model uncertainty and terminal disturbance called *robustness* [77]. Unfortunately, high performance and robustness are usually contradictory to each other; higher performance systems usually have higher sensitivity and worse robustness properties. Yet both high performance and high robustness are the key properties required by practical control systems.

One of the primary objectives of feedback control or compensator design is to ensure that the system response remains well behaved even under parameter uncertainty and the most important characteristic of desired performance is stability.

There is considerable literature available on robustness analysis of linear systems with parameter perturbation. A method for stability-robustness analysis based on a quadratic Lyapunov function that varies linearly with uncertain parameters is derived in [44].

In control systems the poles dominate the transient response as well as the system stability and so many studies [eg. 6] have addressed pole assignment design. Another important control strategy is the robust stabilisation problem, *i.e.*, the ability to maintain system stability under plant uncertainties. Cruz *et al*. [9] have discussed the robust stabilization of linear feedback systems with time varying nonlinear perturbations in terms of the roles of singular values. However their results are valid only when the plant and the compensator design are stable. Other work [16, 7] uses the spectral norm to formulate an upper bound on the largest singular value of the closed-loop transfer matrix to guarantee robust stability of a multivariable control system under parameter variation. Allowable perturbations are discussed in [87, 90] for maintaining stability of uncertain systems. These results are concerned only with stability robustness; they do not deal with robustness of system performance. Robustness results which do address the performance problem are found in references [29, 30] and a design criterion has been developed to simultaneously consider the performance and the stability robustness of a multivariable feedback system in reference [78].

5.1 Low Eigenvalue Sensitivity

Eigenvalues sensitivity problems have been addressed by many researchers. The selection of the closed-loop eigenvalues is always a tough problem for control engineers, uncertainties are inevitable and always exist in the system models, the eigenvalues would only be assigned within certain specified regions rather that the exact locations. Thus the problem eigenvalues assignment robustness is to decide whether the eigenvalues, both perturbed or not, can be placed in some specified regions [33]. Pole assignment with minimal eigenvalue sensitivities, given in [61] and T.R.Crossley [8], relate changes in the eigenvalues to changes in the elements of matrix *A*. In 1990 Chang derived a criterion for the selection of closed-loop eigenvalues such that the resulting closed-loop system has low sensitivity to the variation of feedback gain [4]. In the case of more than one input $m > 1$, many authors [62, 15, 36] have investigated ways made available by degrees of freedom to achieve low sensitivity of the closed-loop eigenvalues to perturbation in A, B and K (where $A \in \Re^{n\times n}, B \in \Re^{n\times m}$ and $K \in \Re^{m\times n}$ is feedback gain matrix). Different algorithms are proposed in [74] for the robust pole assignment problem, these algorithms are based on the fact that the sensitivity of the eigenvalues of a nondefective[1] matrix to perturbations in its entries is directly related to the condition number of the associated eigenvector matrix.

In situations when an ill-conditioned system is considered, some of the eigenvalues may be very sensitive. The results may then yield large variations for only small uncertainties in the data. The condition of an eigenvalue λ is derived [27] using the right eigenvector V of A corresponding to eigenvalue λ and corresponding left eigenvector T of A^T, *i.e.*,

$$AV = \lambda V, T^T A = \lambda T^T$$

[1] Nondefective: a matrix $M \in \Re^{n\times n}$ be a non-definite matrix if its Jordan matrix is diagonal [71]

with $\|V\| = \|T\| = 1$ and the subscript T denotes the transpose.

Let us examine how the eigenvalue is affected when a matrix A is perturbed.

Consider the eigenvalues $\lambda(\varepsilon)$ and eigenvectors $V(\varepsilon)$ of $(A+\varepsilon F)$ as functions of ε

$$(A+\varepsilon F)V(\varepsilon) = \lambda(\varepsilon)V(\varepsilon) \tag{5.1}$$

By differentiating (5.1) with respect to ε and setting $\varepsilon = 0$, we obtain

$$A\dot{V}(0) + FV = \dot{\lambda}(0)V + \lambda\dot{V}(0) \tag{5.2}$$

Applying T^T to both sides of (5.2) and solving for $\dot{\lambda}(0)$ gives

$$\dot{\lambda}(0) = \frac{T^T FV}{T^T V} \tag{5.3}$$

The absolute value of the factor $\frac{1}{T^T V}$ is known as the *condition of the eigenvalue* λ [3].

If perturbations on the order ε are made to A, then an eigenvalue λ may be perturbed by an amount proportional to the condition value, thus if the condition value is large, the eigenvalue λ is regarded as being ill-conditioned and will have a large sensitivity to changes in A. Additional analytical formulas for eigenvalues perturbation theory are derived in [22].

5.2 Low Eigenvalue Sensitivity Using Eigenstructure Assignment

In order to achieve low eigenvalue sensitivity of closed loop system using eigenstructure assignment, a measure of eigenvalue sensitivity is defined in terms of the closed-loop eigenvectors. By noting the freedom in eigenvector selection, beyond eigenvalues assignment, in multi-input controllable state feedback systems, many algorithms have been proposed to select eigenvectors to improve system robustness. S. Srinathkumar [73] developed design procedures to select both eigenvalues and eigenvectors to improve system robustness.

Three problems of eigenstructure assignment [left, right and simultaneous left and right eigenstructure assignment] in multivariable linear systems via output feedback have been proposed by G.R Duan [17], complete parametric expressions for both the closed-loop eigenvector matrices and the output feedback gain matrix are established in terms of some parameters vectors representing the design degrees of freedom which are used to minimize the condition number of the closed-loop eigenvector matrix for the purpose of obtaining a solution which gives minimum closed-loop eigenvalues sensitivities.

Liu and Patton [47] introduced some performance functions which measure sensitivity of the closed- loop matrix and robustness performance of the closed-loop systems.

5.2.1 Individual Eigenvalue Sensitivity

A measure of individual eigenvalue sensitivities which is particularly well known is found by computing a certain function of the closed-loop right and left eigenvectors [55].
The sensitivity of the *i-th* eigenvalue of a closed-loop matrix *A* to perturbations in some or all of its elements is given by the expression [82]:

$$\eta_i(R,L) = \frac{\|R_i\|_2 \|L_i\|_2}{\left\|L_i^T R_i\right\|_2}$$

where R_i and L_i are the right and left eigenvectors of the closed-loop matrix A, respectively, and $L = R^{-T}$, $\eta_i(R,L) \le 1$ for $i = 1,2,...,n$

Thus, a proper measure μ of individual sensitivities of the closed-loop matrix is given by Patton, Liu and Chen [56] and [18]

$$\mu = \max\{\eta_i\}, i = 1,2,...,n$$

The following quantity is a sensitivity measure of the eigenvalue of the closed-loop matrix *A*:

$$\mu' = \frac{\|\mu\|_2}{\sqrt{n}}$$

where *n* is the dimension of the matrix *A*[18].

If perturbations of order $o(\varepsilon)$ occur in the elements of the matrix closed-loop A, the eigenvalues of the perturbed matrix will satisfy [82]

$$\tilde{\lambda}_i = \lambda_i + o(n\eta_i\varepsilon)$$

where *n* is the dimension of the closed-loop matrix *A*. It is clear that the small eigenvalue sensitivity $\eta_i(R,L)$ will produce relatively small changes in eigenvalue positions if the elements of A are perturbed. An eigenvalue is said to be perfectly conditioned if η_i is equal to unity since it gives the smallest change in the eigenvalue position [55].

5.2.2 Overall Eigenvalue Sensitivity

An overall measure of eigenvalue sensitivity can be derived in terms of the closed-loop right (or left) eigenvectors only [55]

The overall eigenvalue sensitivity of the closed-loop matrix A is defined as [82]

$$\eta(R) = \|R\|_2 \|R^{-1}\|_2$$

where R is the right eigenvector matrix of the closed-loop matrix A.

Patton, Liu and Patel [57] define the whole sensitivity function of the closed-loop matrix A as

$$\eta = \|R\|_2 \|L\|_2$$

Suppose that the right eigenvector matrix R is unitary, *i.e.*, $R^T R = I$. Then $\eta(R) = 1$. This indicates that if R is a unitary matrix then the corresponding eigenvalues are perfectly conditioned and hence minimally sensitive to perturbations or parameter variations.

5.3 System Sensitivity and Robustness using State Feedback

5.3.1 Condition Number

Definition 5.1: [77]

Condition number of a computational problem:

Let A be data and $f(A)$ be the result of a computational problem. Let ΔA be the variation of data A and Δf be the corresponding variation of result $f(A)$ due to ΔA such that

$$f(A + \Delta A) = f(A) + \Delta f$$

Then the condition number $\chi(f)$ of the computational problem $f(A)$ is defined by the following inequality:

$$\|\Delta f\| / \|f\| \le \chi(f) \|\Delta A\| / \|A\| \qquad (5.4)$$

Therefore, $\chi(f)$ is the relative sensitivity of problem f with respect to the relative variation of data A. A small $\chi(f)$ implies low sensitivity of problem f, which is then called *a well-conditioned problem*. On the other hand, a large $\chi(f)$ implies high sensitivity of the problem *f*, which is then called an *ill-conditioned problem* [82].

Matrix eigenvalue sensitivity analysis [82], reveals that the condition number of the matrix *A* defined by

$$1 \le \chi(A) = \|A\|_2 \left\|A^{-1}\right\|_2 \le \infty$$

represents eigensystem robustness, where $\|A\|_2$ is the Euclidean norm of the matrix. Thus, a system tends to be sensitive to parameter perturbation if $\chi(A)$ is large.

It is well known that minimizing the closed-loop eigenvector matrix condition number $\chi(A)$ results in minimizing an upper bound on the closed-loop eigenvalue deviation due to system parameter variations [35, 32] as

$$\delta\lambda \le \chi(V)\|E\|$$

where $\delta\lambda$ is the eigenvalue deviation from its nominal value, *E* is a perturbation matrix and $\chi(V)$ is the condition number defined [31] as:

$$\chi_F(V) = \left\|V^{-1}\right\|_F \|V\|_F$$

With *F* refers to the Frobinious norm.

A minimization of the condition number $\chi_F(V) = 1$ is obtained when the eigenvectors are orthonormal which indicates that one can either minimize the system condition number or adjust the closed-loop eigenvectors to become as orthogonal as possible.

5.3.2 Robust Performance

Robust performance is defined as the low sensitivity of a system performance with respect to system model uncertainty and terminal disturbance.
Any real square matrix *A* can have the eigenstructure decomposition [77] as

$$\begin{aligned} A &= V\Lambda V^{-1} \\ &= T^{-1}\Lambda T \end{aligned} \quad (5.5)$$

where $AV = V\Lambda$

and

$$TA = \Lambda T$$

where V and T are right and left eigenvector matrix of matrix A, respectively, and $\Lambda = diag\{\Lambda_1, \Lambda_2, ..., \Lambda_n\}$ is a Jordan form matrix , whose diagonal matrix blocks $\Lambda_i, i = 1,2,...,n$ are called *Jordan blocks*.

From (5.5)

$$V^{-1}AV = \Lambda$$

Therefore, if A becomes $A + \Delta A$, then

$$V^{-1}(A + \Delta A)V = \Lambda + V^{-1}AV = \Lambda + \Delta\Lambda \quad (5.6)$$

Using the inequality used in the definition 5.1 we will have

$$\|\Delta\Lambda\| \le \|V\| \; \|V^{-1}\| \; \|\Delta A\| = \chi(V) \; \|\Delta A\| \quad (5.7.a)$$

Inequality (5.7.a) indicates that the condition number $\chi(V)$ of eigenvector matrix V can decide the magnitude of $\|\Delta\Lambda\|$.

Based on (5.6), a result using $\chi(V)$ to indicate the variation of eigenvalues was derived in Wilkinson [82]:

$$\min\{|\lambda_i - \lambda'_i|\} = \min\{|\Delta\lambda_i|\} \le \chi(V)\|\Delta A\| \quad (5.7.b)$$

Where $\lambda_i, i = 1,2,...,n$ and λ'_i are an eigenvalue of matrices A and $(A + \Delta A)$, respectively,

Because the left-hand side of (5.7.b) takes the minimum of the difference $\Delta\lambda_i$ between the eigenvalues of A and $(A + \Delta A)$, the upper bound on the right-hand side of (5.7.b) does not apply to other $\Delta\lambda_i$'s.

From (5.7), it is reasonable to use the condition number of eigenvector matrix V of the matrix A, $\chi(V)$, to measure the sensitivity of all eigenvalues (Λ) of matrix $A, s(\Lambda)$.

In other words, we define

$$s(\Lambda) = \chi(V) = \|V\| \; \|V^{-1}\| \quad (5.8)$$

Even though $s(\Lambda)$ is not an accurate measure of the variation (sensitivity) of each individual eigenvalues. The advantage of this measure is that it is valid for large $\|\Delta A\|$ [82].

In order to obtain a more accurate measure of the sensitivity of individual eigenvalues, first order perturbation analysis is applied and the following result is obtained under the assumption of small $\|\Delta A\|$ [82].

Theorem 5.1: [55]

Let λ_i, v_i and t_i be the *i-th* eigenvalue, right and left eigenvectors of matrix *A*, respectively $(i = 1,2,...,n)$. Let $\lambda_i + \Delta\lambda_i$ be the *i-th* eigenvalue of matrix $A + \Delta A, (i = 1,2,...,n)$. Then for small enough $\|\Delta A\|$,

$$|\Delta\lambda_i| \le \|t_i\| \quad \|v_i\| \quad \|\Delta A\| = s(\lambda_i)\|\Delta A\| \qquad i = 1,2,...,n \tag{5.9}$$

Proof: see [55]

This theorem shows clearly that the sensitivity of an eigenvalue is determined by its corresponding left and right eigenvectors.

Relative Change

To study how the eigenvalues are affected by small random perturbations matrix $\|\Delta A\|$, the relative change R_c is computed as:

$$R_c = \frac{|\lambda_i - \lambda'_i|}{|\lambda_i|} = \frac{|\Delta\lambda_i|}{|\lambda_i|} \qquad i = 1,2,...,n$$

5.3.3 Robust Stability

Stability is the foremost system property. Therefore the sensitivity of this property, called *robust stability*, with respect to system model uncertainty is also critically important. Consequently, a generally accurate numerical measure of this sensitivity is also essential to guide robust stability analysis and design.

5.3.4 Existing Methods

Various robustness measures have been investigated in [81], providing upper bounds on perturbations for maintaining the stability of the perturbed system.

Consider the following linear state space model:

Nominal system: $\dot{x} = Ax$ (5.10)

Perturbed system: $\dot{x} = (A + E)x$ (5.11)

Where A is $n \times n$ stable matrix and E is the perturbation matrix.

For perturbed system (5.11) Lyapunov based method of deriving robustness bound measure has been considered as well established by Patel and Toda [54]

The perturbed system (5.11) is stable if

$$\frac{\|\dot{E}x\|}{\|x\|} < \frac{\sigma_m(Q)}{\sigma_M(P)} = \mu_1 \tag{5.12}$$

or $\sigma_m(E) < \mu_1$

where Q is some symmetric positive-definite matrix and P is the symmetric positive-definite matrix that satisfies the Lyapunov equation

$$A'P + PA = -2Q$$

$\sigma_M(.)$ and $\sigma_m(.)$ are the maximum and the minimum singular values of the matrix (.). μ is the robustness measure and $\|.\|$ is the Euclidean norm, A' is transpose of A.

The bound defined in (5.12) is maximum.

It is shown in [54] that the perturbed system is stable if

$$\left|E_{ij}\right| < \frac{1}{n\sigma_M(P)} = \mu_2 \tag{5.13}$$

where P is the solution of the Lyapunov equation

$$A'P + PA = -2I$$

In [80] Wang and Lin studied the robust eigenvalue assignment for systems with parameters perturbation via matrix measures. Their analysis is based on some essential properties of the induced norms and matrix measures to compute some robustness bounds.

Definitions of norm, induced norm and matrix measures and detailed properties can be found in [80, 79].

For a specific norm on C^n, in general, it is not always easy to obtain the explicit expression of the induced norm as well as the matrix measure. However, corresponding to norms $\|\bullet\|_1, \|\bullet\|_2$ and $\|\bullet\|_\infty$ the induced norms and matrix measures have explicit expressions as shown in the following table:

P	Norm on C^n	Induced norm on $C^{n\times n}$	Matrix measure on $C^{n\times n}$
1	$\max_j \lvert x_j \rvert$	$\max_j \left(\sum_i \lvert a_{ij} \rvert \right)$	$\max_j \left(\operatorname{Re} a_{jj} + \sum_{i\neq j} \lvert a_{ij} \rvert \right)$
2	$\sqrt{\sum_i x_i^2}$	$\max\left(\sqrt{\lambda(A^T A)} \right)$	$\dfrac{\max \lambda \ (A^T + A)}{2}$
∞	$\max_i \lvert x_i \rvert$	$\max_i \left(\sum_j \lvert a_{ij} \rvert \right)$	$\max_i \left(\operatorname{Re} a_{ii} + \sum_{j\neq i} \lvert a_{ij} \rvert \right)$

Where $x \in C^n$ and $A \in C^{n\times n}$.

Piou and Sobel [59] extend the matrix measure results of Wang and Lin [80] to compute the robustness bounds.

Consider the linear time-invariant multivariable system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{5.14}$$

where A, B and C are real constant matrices.

Suppose that the system is subject to uncertainties in the entries of A, B described by dA and dB, respectively, where

$$\begin{aligned} \dot{x}(t) &= (A + dA)x(t) + (B + dB)u(t) \\ y(t) &= Cx(t) \end{aligned} \tag{5.15}$$

Further, suppose that bounds are available on the absolute values of the elements of dA and dB, that is

$$\begin{aligned} \lvert da_{ij} \rvert &\le (a_{ij})_{\max}, i = 1,2,\ldots,n,\ j = 1,2,\ldots,n \\ \lvert db_{ij} \rvert &\le (b_{ij})_{\max}, i = 1,2,\ldots,n,\ j = 1,2,\ldots,m \end{aligned} \tag{5.16}$$

define dA^+ and dB^+ as the matrices obtained by replacing the entries of dA and dB by their absolute values. Also, define $A_{\max}$ and $B_{\max}$ as the matrices with entries $(a_{ij})_{\max}$ and $(b_{ij})_{\max}$ then

$$\begin{aligned} dA &: dA^+ \le A_{\max} \\ dB &: dB^+ \le B_{\max} \end{aligned} \tag{5.17}$$

and where $"\le"$ is applied element by element to matrices and $A_{\max} \in \Re_+^{n\times n}$ and $B_{\max} \in \Re_+^{n\times m}$ where $\Re_+$ is the set of non-negative numbers.

Consider the control law described by

$$u(t) = -Kx(t)$$

then
$$\dot{x}(t) = (A - BK)x(t) \tag{5.18}$$

and the uncertain closed-loop system is given by

$$\dot{x}(t) = (A - BK)x(t) + dA + dBK \tag{5.19}$$

Let $\mu_{ip}(M)$ be the matrix measure defined by [80]

$$\mu_{ip}(M) = \lim_{\varepsilon\to 0} \frac{\left|I + \varepsilon M\right|_{ip} - 1}{\varepsilon} ;1 \le p < \infty \tag{5.20}$$

Theorem 5.2: [59]

Suppose that closed-loop system described by (5.19) has its eigenvalues in the R region of figure 5.1. Further, suppose that the matrix $A - BK$ in (5.19) is non-defective. The eigenvalues of the closed-loop system with uncertainty described by equation (5.20) will be in R region for all uncertainty described by (5.17) if

$$\max\left[\mu_{\rho 1}, \mu_{\rho 2}\right] < 1 \tag{5.21}$$

where

$$\mu_{pl} = \frac{\mu_{ip}(A_{\max} - B_{\max}K^+)}{a_l \cos\theta_l - \mu_{ip}\left[(A - a_l I - BK)\cos\theta_l\right] - \mu_{ip}\left[(-A + a_l I + BK)\, j\sin\theta_l\right]} \tag{5.22}$$

where
$$l = 1 : \theta_1 = 0, a_1 = a_1$$
$$l = 2 : \theta_2 = \theta_2, a_2 = 0$$

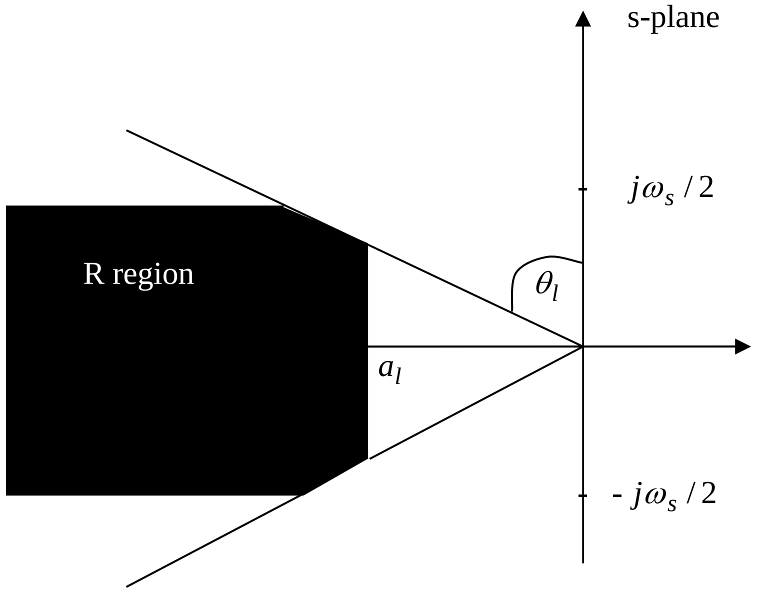


Figure 5.1 : S-plane performance region

5.3.5 Proposed Method

The most basic criterion of system stability is that every matrix eigenvalue has a negative real part. Hence the sensitivity of these eigenvalues with respect to system model uncertainty should be the most direct and critical factor in measuring the sensitivity of system stability (robust stability).

Let us compare the Routh-Hurwitz criterion of system stability, where the system characteristic polynomial must be first computed. The sensitivity of this step of computation can be as high as the direct computation of the eigenvalues (see Wilkinson [82]). The Routh-Hurwitz criterion requires additional determination based on the characteristic polynomial coefficients and on the basic stability criterion. This indirectness will reduce the accuracy of both the stability determination and the measure of robust stability.

Compared to the above stability measure of classical control theory, the sensitivity of eigenvalues (poles) is used to measure robust stability which has the ability to accommodate pole assignment and thus to guarantee performance.

There are three robust stability measures using the sensitivity of system poles. In [77] they are called M_1, M_2 and M_3. We will analyse and compare the general accuracy of these three measures.
Let us introduce these three measures.

5.3.5.1 The Robust Stability Measure M$_1$ [15]

Consider the multivariable linear closed-loop system which is given by

$$\dot{x}(t) = Qx(t) \tag{5.23}$$

where $Q = A - BK, Q$ is an $n \times n$ real matrix .
Assume that under variation in the parameters of Q, the system model is now given by

$$\dot{x}(t) = (Q + E)x(t) \tag{5.24}$$

where E is an $n \times n$ real matrix which represents the model uncertainty.
The robustness problem will be the following. Let the system given by (5.23) be stable, namely, the eigenvalues of Q are located in the open LHP, then the system is robust if under variations in the parameters of Q, the eigenvalues of the system given by (5.24)are still in the open LHP.

If one of the eigenvalues of $Q + E$, say λ_p, $p = 1,2,....,n$, is located on the imaginary axis, namely, $\lambda_p = \pm j\omega_p$, then the matrix $\lfloor j\omega_p I - (Q + E)\rfloor$ is singular, namely,

$$\sigma_m \lfloor j\omega_p I - (Q + E)\rfloor = 0 \tag{5.25}$$

where for a matrix A, σ_m [A] denotes the smallest singular value of a matrix A.
Since Q is nonsingular and since $\sigma_m[A] = \sigma_m[-A]$, then the condition

$$\sigma_m[Q - j\omega I] > \sigma_M[E] \quad \forall \omega \geq 0 \tag{5.26}$$

is sufficient for the system given by (5.24) to be robust, and increasing $\sigma_m[Q - j\omega I]$ will enable one to cope with larger uncertainties in the sense of (5.26), let

$$M_1 = \min_{0 \leq \omega < \infty} \sigma_m[Q - j\omega I] \tag{5.27}$$

denote the robustness measure, namely, the largest perturbation's spectral norm for which stability is guaranteed in the sense of (5.26).

Theorem 5.3: [35]

The stability robustness measure M_1 is given by

$$M_1 = \min_{0 \le \omega < \infty} \sigma_m [Q - j\omega I]$$

where $Q = A - BK$ and I denotes the $n \times n$ unit matrix.

5.3.5.2 The Robust Stability Measure M₂ [45]

Consider the linear time-invariant multivariable system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{5.28}$$

with $x \in \Re^{n \times 1}$, the state vector and $u \in \Re^{m \times 1}$ the input vector $, 1 \le m \le n$. We assume that (A, B) is completely controllable, B has full rank, and we denote by K the state feedback gain matrix

$$u(t) = -Kx(t) \tag{5.29}$$

so that the closed-loop system is

$$\dot{x}(t) = (A - BK)x(t) \tag{5.30}$$

We introduce the set $L = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ of desired closed-loop characteristic values, where the system is assumed stable i.e. $\text{Re}\{\lambda_i\} < 0, i = 1,2,...,n$

We shall order the characteristic values according to their real parts as follows:

$$\text{Re}\{\lambda_1\} \le .... \le \text{Re}\{\lambda_{n-l}\} < \text{Re}\{\lambda_{n-l+1}\} = ... = \text{Re}\{\lambda_n\} = -\lambda_0 < 0 \tag{5.31}$$

indicating that the last $l, 1 \le l \le n$ characteristic values have identical real parts. Note that λ_0 is that minimal distance, in the complex plane, between the set L and the imaginary axis, *i.e.*,

$$\min_{1 \le k \le n} |\text{Re}\{\lambda_k\}| = \lambda_0 \tag{5.32}$$

Since (A, B) is completely controllable, there exist one or more matrices K, which achieve a closed-loop pole location at L.

Given a stable closed-loop system (5.23) and perturbed system (5.24)

Definition 5.2: [45]

Stability robustness measure: we denote by $\rho(A, B, L, K)$ the stability robustness measure of the quadruple (A, B, L, K)

$$\rho(A, B, L, K) = \sup_{\alpha > 0} \{\alpha \mid \|E\|_2 < \alpha\}$$

where $\|.\|$ is the 2-norm.

Now we define the maximal stability robustness

Corollary 5.1: [45]

Taking the supremum of both sides of (5.27) yields

$$\rho_m(A, B, L) = \sup_K \left\{ \min_\omega [\sigma_m(Q - j\omega I)] \right\}$$

The following theorem states an upper bound for ρ_m.

Theorem 5.4: [45]

The maximal stability robustness measure ρ_m of the triple (A, B, L), satisfies the following upper bound:

$$\rho_m(A, B, L) \le \lambda_0 \tag{5.33}$$

Proof: see Lewkowicz [45].

It is shown that the robustness margins are given by the eigenvalues closest to the imaginary axis [5].

The approach in [35] considers the robust stability as a part of the robustness of all eigenvalues, J.Kaustky [35] states that Q is a normal matrix if and only if, it has a nonsingular eigenvector's matrix V, so that the following relations hold:

$$V^{-1}QV = diag\{\lambda_1, \lambda_2, ..., \lambda_n\} \qquad \chi(V) = 1$$

where $\chi(V) = \sigma_M(V)/\sigma_m(V)$ is the condition number of the matrix *V*. A few algorithms for minimizing $\chi(V)$ for a given (A, B, L) are presented in [35]. The smallest $\chi(V)$ is, the more reluctant are the characteristic values of $M \in C^{n\times n}$ to move as a result of a perturbation [45].

From [2], for $M, E \in C^{n\times n}$ and $1 \le k \le n$, we have

$$\left|\lambda_k(M) + \lambda_k(M+E)\right| \le \chi(V)\|E\|_2$$

where *V* is the eigenvector's matrix of *M*.

It is known [35] that the stability robustness measure

$$\rho(A, B, L, K) \ge \frac{\lambda_0}{\chi(V)} \tag{5.34}$$

hence, minimization of $\chi(V)$ is a desired property .

An upper bound for stability robustness measure is based on the characteristic values of the system [45], and the maximal stability robustness as it shown in the theorem 5.4 is equal to the smallest distance between a set *L* and the imaginary axis.

Using (5.34) and the theorem 5.4 then, the stability robustness measure M_2 is given by $M_2 = \dfrac{\lambda_0}{\chi(V)}$ for which $\chi(V)$ is minimized.

5.3.5.3 The Robust Stability Measure M$_3$

M$_3$ is developed in the early 90's [76, 75] and is given by

$$M_3 = \min_{1\le i\le n}\left\{s(\lambda_i)^{-1}\left|\mathrm{Re}(\lambda_i)\right|\right\} \tag{5.35}$$

Let us analyze these three measures in the following.

Consider the multivariable linear time-invariant closed-loop system which is given by

$$\begin{aligned}\dot{x}(t) &= (A - BK)x(t)\\ &= Qx(t)\end{aligned} \tag{5.36}$$

Assuming all its eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ are stable $(\text{Re}\{\lambda_i\} < 0, i = 1,2,..,n)$ and are already assigned for guaranteed performance. The three stability robustness measures are

$$\begin{aligned} M_1 &= \min_{1 \le \omega \le \infty} \{\sigma_m (Q - j\omega I)\} \\ M_2 &= s(\Lambda)^{-1} |\text{Re}\{\lambda_n\}|, \qquad (|\text{Re}\{\lambda_n\}| \le ... \le |\text{Re}\{\lambda_1\}|) \\ M_3 &= \min_{1 \le i \le n} \{s(\lambda_i)^{-1} |\text{Re}(\lambda_i)|\} \end{aligned} \tag{5.37}$$

where $s(\Lambda)$ is defined in (5.8)

Because σ_m indicates the smallest possible norm of matrix variation norm for a matrix to become singular, see the following the theorem

Theorem 5.5: [77]

If the singular values computed from a given matrix $A + \Delta A$ are $\sigma_1 \ge \sigma_2 \ge ... \ge \sigma_n > 0$ $(r = n)$ (r is the rank of matrix A, n is the rank of the matrix ($A + \Delta A$) then the necessary condition for the rank of the original matrix A to be less than n (or σ_n of $A = 0$) is $\|\Delta A\| \ge s_n$,and the necessary condition for the rank of A to be less than r (or σ_r of $A = 0$) is $\|\Delta A\| \ge s_r\,(r = 1,2,...,n)$.

Proof: see [77]

M_1 equals the smallest possible matrix variation norm for the matrix Q to have an unstable and pure imaginary eigenvalue $j\omega$.

In the measure M_2, the term $|\text{Re}\{\lambda_n\}|$ is the shortest distance between the unstable region and eigenvalues λ_i. Thus, M_2 equals this distance divided by the sensitivity of all the eigenvalue matrix Λ. The lower the sensitivity $s(\Lambda)$, the greatest M_2. In other words, M_2 may be considered as the likelihood margin for λ_n to become unstable.

There exist several general and numerical algorithms which can compute state feedback gain matrix K such that the value of $s(\Lambda)^{-1}$ or M_2 is maximized, with arbitrarily assigned eigenvalues in matrix Q [35]. However, M_2 seems to be less accurate in measuring the likelihood margin for λ_n to become unstable, because $s(\Lambda)$ is not an accurate measure of the sensitivity of λ_n .

In the definition of the measure M_3, the likelihood margins for every eigenvalue to become unstable are considered. The likelihood margin for each λ_i equals $|\mathrm{Re}\{\lambda_i\}|$ divided by its corresponding sensitivity $s(\lambda_i), i = 1,2,...,n$.

M_1 and M_2 consider only the likelihood margin for λ_n to become unstable, while the instability of any eigenvalue can cause system instability , the $s(\Lambda)$ of M_2 is generally not an accurate measure of individual eigenvalues sensitivity and is not as accurate as the sensitivity $s(\lambda_i)$ of λ_i itself in measuring the sensitivity of λ_i for $\forall i$ (including $i = n$). Hence, M_3 is more accurate than M_1 and M_2 , and reflects the instability likelihood of all eigenvalues.

$$s(\Lambda) = \|V\| \ \|V^{-1}\| > \|v_i\| \ \|t_i\| = s(\lambda_i) \geq 1, \qquad i = 1,2,...,n \tag{5.38}$$

$$M_2 = s(\Lambda)^{-1} |\mathrm{Re}\{\lambda_n\}| \leq M_3 \leq |\mathrm{Re}\{\lambda_n\}| \tag{5.39}$$

From (5.38) and (5.39), if the overall eigenvalue sensitivity $s(\Lambda)$ is at the lower possible value (=1), then all three measures $M_i, i = 1,2,3$ will reach their common highest possible value $|\mathrm{Re}\{\lambda_n\}|$. A lower $s(\Lambda)$ does not necessary imply a higher M_1 or M_3 [35] which implies that M_1 and M_3 have higher accuracy than M_2.

5.4 System Sensitivity and Robustness using Compensator Design

In most practical situations, the given mathematical model (either state space or transfer function) of the plant system is inaccurate because the parameters of practical physical system are difficult to measure accurately. So there is a difference between the actual plant and its mathematical model $H(s)$.This difference is called *model uncertainty* and is defined as $\Delta H(s)$. Therefore, it is essential that the control systems have low sensitivity to $\Delta H(s)$.

Let $\Delta H_{cl}(s)$ be the uncertainty of the overall control system $H_{cl}(s)$, which is the closed-loop transfer function, caused by the plant uncertainty $\Delta H(s)$. In single variable system, we use relative plant system model uncertainty $\Delta H(s)/H(s)$ and relative closed-loop transfer function uncertainty $\Delta H_{cl}(s)/H_{cl}(s)$ to measure the overall control system sensitivity versus plant system model uncertainty.

Definition 5.2: [77]

The sensitivity of a control system $H_{cl}(s)$ to $\Delta H(s)$ is defined as

$$s(H_{cl}(s))\Big|_H = \left|\frac{\Delta H_{cl}(s)/H_{cl}(s)}{\Delta H(s)/H(s)}\right|$$

for small enough $\Delta H(s)$ and $\Delta H_{cl}(s)$

$$s(H_{cl}(s))\Big|_H \approx \left|\frac{\partial H_{cl}(s)H(s)}{\partial H(s)H_{cl}(s)}\right|$$

MIMO systems have transfer function matrices instead of scalar transfer functions. There are different ways to measure the size or magnitude of a matrix, the singular value of the matrix can be used to measure the size of a matrix. In [10] J.B.Cruz showed that, in multivariable systems, there exists a matrix S which is defined as *sensitivity matrix*. And given by

$$S = [I + H(s)C(s)]^{-1}$$

The sensitivity function S is a very good indicator of closed-loop performance, both for SISO and MIMO systems [70].

Considering the unity feedback for multivariable system shown in figure 4.4, the sensitivity transfer function and the complementary transfer function can be represented as

$$S(s) = [I + H(s)C(s)]^{-1}$$

and

$$T(s) = H(s)C(s)[I + H(s)C(s)]^{-1}$$

These transfer functions are function of s, where $(s = j\omega)$, and the singular values of these matrices are functions of frequency. Therefore, the singular value plays an important role in the frequency domain analysis of multivariable systems [51].

The performance of a feedback system indicates that the system performance can be expressed in terms of the performance specifications of the sensitivity function and complementary functions.

5.4.1 Condition Number [70]

We define the condition number of a matrix as the ratio between the maximum and minimum singular values,

$$\chi(H) = \sigma_M(H)/\sigma_m(H)$$

A matrix with a large condition number is said to be ill-conditioned. If the condition number is large then this may indicate control problem [70]:

1. A large condition number may be caused by a small value of $\sigma_m(H)$, which is generally undesirable (on the other hand, a large value of $\sigma_M(H)$ need not necessary be a problem).
2. A large condition number does imply that the system is sensitive to unstructured input uncertainty, but this kind of uncertainty often does not occur in practice. We therefore cannot generally conclude that a plant with a large condition number is sensitive to uncertainty.

5.4.2 Robust Stability

Theorem 5.5: [70, 58, 12]

Assume that the system $M(s)$ is stable and that the perturbations $\Delta(s)$ are stable. Then $M\Delta$ -system in figure 5.1is stable for all perturbations Δ satisfying $\|\Delta\|_\infty \leq 1$ if and only if

$$\sigma_M\left(M(j\omega)\right) < 1 \quad \forall\omega \quad \Leftrightarrow \quad \|M\|_\infty < 1 \tag{5.40}$$

Condition (5.41) may be rewritten as

$$\text{Robust stability} \Leftrightarrow \sigma_M\left(M(j\omega)\right)\ \sigma_M\left(\Delta(j\omega)\right) < 1, \quad \forall\omega, \forall\Delta$$

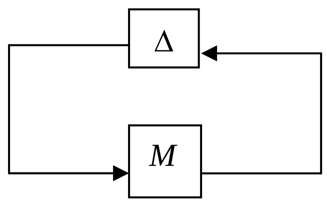


Figure 5.2 : $M\Delta$ -structure for robust stability

5.4.3 Robust Performance

Robust performance means that the performance objective is satisfied for all possible plant in the uncertainty set.

It says [20] that a robust performance problem is equivalent to a robust stability with augmented uncertainty Δ_f as shown in figure 5.3

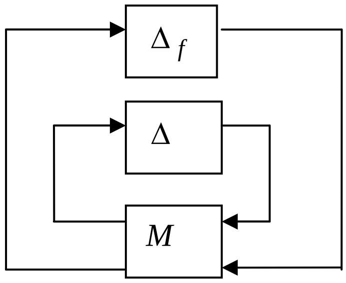


Figure 5.3 : Robust performance versus robust stability

Chapter 6

Proposed Approach

The contribution of this thesis is concerned with the choice of the closed-loop block poles in multivariable systems. Given a multivariable system described by a state space equations or a transfer function, we want to find the appropriate forms for the closed-loop block poles to be assigned. Among the criteria used to select these forms, we have:

i. Time response characteristics.
ii. Robustness.
iii. Magnitude of feedback gains.

6.1 Time Domain Specifications [39]

The transient portion of the time response is the part which goes to zero (for stable systems) as time becomes large. Nevertheless, the transient response of a control system is necessarily important, since both the amplitude and time duration of the transient response must be kept within prescribed limits.
Performance criteria commonly used for the characterization of linear control systems in the time domain are defined as follows:

i. *Maximum overshoot*: Let $y(t)$ be the unit-step response. Let y_{max} denotes the maximum value of $y(t)$, and y_{ss} be the steady-state of $y(t)$, and $y_{\max} \geq y_{ss}$.

The maximum overshoot of $y(t)$ is defined as

$$\textit{Maximum overshoot} = y_{\max} - y_{ss}$$

The maximum overshoot is often represented as a percentage of the final value of the step response, that is,

$$\text{Percent maximum overshoot} = \frac{\text{maximum overshoot}}{y_{ss}} \times 100\%$$

a system with large overshoot is usually undesirable

ii. *Delay time* t_d: is defined as the time required for the step response to reach 50 percent of its final value.

iii. *Rise time* T_r : is defined as the time required for the step response to reach 10 to 90 percent of its final value.

iv. *Settling time Ts*: is defined as the time required for the step response to decrease and stay within a specified percentage (2% or 5%) of its final value or it is the smallest value T_s such that:

$$|y(t) - y_{ss}| \le 0.02 y_s \text{ or } 0.05 y_{ss} \text{ for all } t \ge T_s$$

The four quantities just defined give a direct measure of the transient characteristics of a control system in terms of the unit-step response. The rise time and settling time are measures of the speed of the response, whereas the overshoot, steady state are related to the quality of the response.

The unit step response is a measure for SISO systems, for this we have adapted its characteristics to MIMO systems.

Maximum overshoot is the highest deviation from steady state value (which is not single in the case of MIMO systems).

6.2 Proposed Procedure

Given a multivariable system described by the following state equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

where A, B, C are, respectively, $n \times n$, $n \times m, q \times n$ constant matrices. The feedback control law is $u(t) = -Kx(t)$. The given system can be converted into block controller form if it is block controllable of index l where $l = n/m$ is an integer. The block controller form is as follows

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c u(t) \\ \quad y(t) = C_c x_c(t) \end{cases}$$

where

$$A_c = \begin{bmatrix} 0_m & I_m & 0_m & . & . & . & 0_m \\ 0_m & 0_m & I_m & . & . & . & 0_m \\ . & . & . & . & . & . & . \\ 0_m & 0_m & 0_m & . & . & . & I_m \\ -A_l & -A_{l-1} & -A_{l-2} & . & . & . & -A_1 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0_m & 0_m & . & . & . & I_m \end{bmatrix}^T$$

$$C_c = \begin{bmatrix} C_l & C_{l-1} & . & . & . & C_1 \end{bmatrix}$$

Using State feedback we will have

$$A_c - B_c K_c = \begin{bmatrix} 0_m & I_m & 0_m & . & . & 0_m \\ 0_m & 0_m & I_m & . & . & 0_m \\ . & . & . & . & . & . \\ 0_m & 0_m & 0_m & . & . & I_m \\ -A_l & -A_{l-1} & -A_{l-2} & . & . & -A_1 \end{bmatrix} - \begin{bmatrix} 0_m \\ . \\ . \\ 0_m \\ I_m \end{bmatrix} \begin{bmatrix} K_{c1} & K_{c2} & . & . & . & K_{cl} \end{bmatrix}$$

$$= \begin{bmatrix} 0_m & I_m & 0_m & . & . & 0_m \\ 0_m & 0_m & I_m & . & . & 0_m \\ . & . & . & . & . & . \\ 0_m & 0_m & 0_m & . & . & I_m \\ -A_l - K_{c1} & -A_{l-1} - K_{c2} & -A_{l-3} - K_{c3} & . & . & -A_1 - K_l \end{bmatrix}$$

Where $K_c = \begin{bmatrix} K_{c1} & K_{c2} & . & . & . & K_{cl} \end{bmatrix}$ and $K_{ci}, i = 1,2,...,l$ are $m \times m$ matrices

Since $(A_c - B_c K_c)$ is in block companion form, its characteristic matrix polynomial equation is given by:

$$\Delta(s) = I_m s^l + (A_l + K_{c1}) s^{l-1} + ... + (A_1 + K_{cl})$$

The desired matrix polynomial constructed from desired solvents is

$$\Delta_d(s) = I_m s^l + D_{d1} s^{l-1} + ... + D_{dl}$$

where $D_i, i = 1,2,...,l$ are an $m \times m$ matrices.

By forcing $\Delta(s) = \Delta_d(s)$, then the matrices $K_{c1}, K_{c2}, ..., K_{cl}$ are given by $K_{ci} = D_{di} - A_i$ for $i = 1,2,...,l$.

Given a set of right solvents $\{R_i\}$ of $\Delta_d(s)$ which satisfy

$$\Delta_d(R_i) = I_m R_i^l + D_{d1} R_i^{l-1} + ... + D_{dfl} = 0_m \text{ for } i = 1,2,...,l$$

The coefficients of the desired matrix polynomial are given by

$$\begin{bmatrix} D_{dl} & D_{d(l-1)} & . & . & . & D_{d1} \end{bmatrix} = -\begin{bmatrix} R_1^l & R_2^l & . & . & . & R_l^l \end{bmatrix} \begin{bmatrix} I_m & I_m & . & . & I_m \\ R_1 & R_2 & . & . & R_l \\ R_1^2 & R_2^2 & . & . & R_l^2 \\ . & . & . & . & . \\ R_1^{l-1} & R_2^{l-1} & . & . & R_l^{l-1} \end{bmatrix}^{-1} \quad (6.1)$$

For a set of left solvents $\{L_i\}$of $\Delta_d(s)$ satisfies:

$$\Delta_d(L_i) = L_i I_m + L_i^l D_{d1} + ... + D_{dl} = 0_m$$

The coefficients of the desired matrix polynomial are as follows

$$\begin{bmatrix} D_{fl} \\ D_{f(l-1)} \\ . \\ . \\ D_{f1} \end{bmatrix} = -V_L^{-B} \begin{bmatrix} L_1^l \\ L_2^l \\ . \\ . \\ L_l^l \end{bmatrix} \quad (6.2)$$

where V_L^B is the block transpose of the left block Vandermonde matrix.

The block Vandermonde matrix is not necessary nonsingular for any choice of $R_1, R_2, .., R_l$, a necessary but not sufficient condition is that a set $\{R_i\}$ for $i = 1,2,...,l$ form a complete set of solvents.

From the same given set of desired eigenvalues, different structures of solvents can be constructed; the well known forms are the following:

6.2.1 Diagonal Form

Given a set of n distinct eigenvalues $\{\lambda_1 \quad \lambda_2 \quad . \quad . \quad \lambda_n\}$, the construction of the solvents in diagonal form is as follows:

$$R_i = \begin{bmatrix} \lambda_1 & 0 & . & . & 0 \\ 0 & \lambda_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \lambda_n \end{bmatrix} \quad (6.3.a)$$

6.2.2 Jordan Form

If some eigenvalues are repeated, say λ_1 with multiplicity μ then the constructed solvents has the form is

$$R_i = \begin{bmatrix} \lambda_1 & 1 & & & & & & & 0 \\ 0 & \lambda_1 & & & & & & & \\ & & . & 1 & & & & & \\ & & & . & 1 & & & & \\ & & & & \lambda_1 & 0 & & & \\ & & & & 0 & \lambda_2 & & & \\ & & & & & & . & & \\ & & & & & & & . & 0 \\ 0 & & & & & & & 0 & \lambda_n \end{bmatrix} \tag{6.3.b}$$

6.2.3 Solvents Constructed through Modal Matrices

In the case where some eigenvalues are complex conjugate pairs, i.e., $\lambda_i = \sigma + j\omega$ and $\lambda_{i+1} = \sigma - j\omega$, the block poles are given as:

$$R_i = \begin{bmatrix} \sigma & \omega & . & . & 0 \\ -\omega & \sigma & . & . & 0 \\ . & . & \lambda_3 & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \lambda_n \end{bmatrix}$$

6.2.4 Companion Form

The characteristic equation constructed from a given set of n eigenvalues is

$$\Delta(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) = \lambda^n + a_1\lambda^{n-1} + ... + a_n$$

Two different structures of solvents can be constructed

6.2.4.1 Controllable Companion Form

$$R_i = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & . & . & -a_1 \end{bmatrix}$$

or (6.3.c)

$$R_i = \begin{bmatrix} -a_1 & -a_2 & . & . & -a_{n-1} & -a_n \\ 1 & 0 & . & . & 0 & 0 \\ 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 1 & 0 \end{bmatrix}$$

6.2.4.2 Observable Canonical Form

$$R_i = \begin{bmatrix} 0 & 0 & . & . & 0 & -a_n \\ 1 & 0 & . & . & 0 & -a_{n-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & .. & 1 & -a_1 \end{bmatrix}$$

or (6.3.d)

$$R_i = \begin{bmatrix} -a_1 & 1 & . & . & 0 & 0 \\ -a_2 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & 1 \\ -a_n & 0 & . & . & 0 & 0 \end{bmatrix}$$

In the case of compensator design using block pole placement the proposed approach is as follows:

Consider the unity feedback system in figure (6.1). The plant is described by a $q \times p$ proper rational matrix.

$$H(s) = N(s)D^{-1}(s)$$

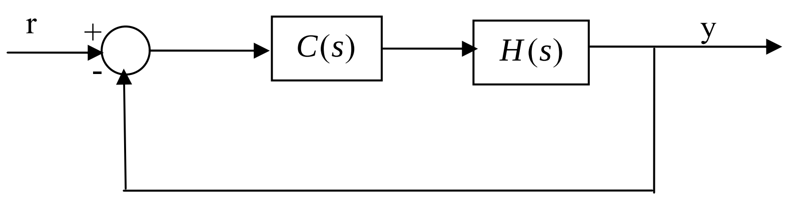


Figure 6.1: Unity feedback for multivariable system

We want to find the compensator $C(s) = D_C^{-1}(s)N_C(s)$ which is a $p \times q$ proper rational matrix that achieves the desired block poles in the desired positions so that the closed- loop system meets the different criteria stated before.

Given the coefficients matrices of the plant of $N(s)$ and $D(s)$,

$$D(s) = D_n s^n + D_{n-1}s + ... + D_1$$

and

$$N(s) = N_n s^n + N_{n-1} s^{n-1} + ... + N_1$$

Find $C(s)$ such that the closed-loop system is given by

$$H_{cl}(s) = N(s) D_f^{-1}(s) N_C(s)$$

or

$$H_{cl}(s) = N(s)(D_C(s)D(s) + N_C(s)N\ \ (s))^{-1} N_C(s)$$

yields

$$D_f(s) = D_C(s)D(s) + N_C(s)N\ \ (s)$$

so that

$$D_f(s) = D_{fn} s^n + D_{f(n-1)} s^{n-1} + ... + D_{f1}$$

Forcing $D_f(s) = \Delta_d(s)$ which is the desired matrix polynomial constructed from desired solvents that is $D_{fi} = D_{di}$ for $i = 1,2,...n$

The coefficients D_{di} are constructed as in (6.1) (6.2) and the solvents by the matrices described in (6.3).

The coefficients of $D_c(s)$ are found by solving the Diophantine equation using either recursive or row searching algorithm, i.e., find the primary linearly dependent rows in Sylvester matrix.

To assess the stability robustness of the closed-loop system using state feedback, the three following measures are proposed by Tsui [77] using the sensitivity of the eigenvalues say M_1, M_2 and M_3,

where $$M_1 = \min_{0 \le \omega < \infty} \left\{ \sigma_m \quad (A - j\omega I) \quad \right\}$$

M_1 is the smallest possible matrix variation norm for the dynamic matrix to have an unstable and pure imaginary eigenvalues

$$M_2 = s(\Lambda)^{-1} \left| \mathrm{Re}\{\lambda_n\} \right|, \qquad (\left|\mathrm{Re}\{\lambda_n\}\right| \le ... \le \left|\mathrm{Re}\{\lambda_1\}\right|)$$

The term $\left|\mathrm{Re}\{\lambda_n\}\right|$ is the shortest distance between the unstable region and the eigenvalues λ_i, M_2 equals this distance divided by the sensitivity of all eigenvalues matrix Λ or may be considered as the likelihood margin for λ_n to become unstable

$$M_3 = \min_{1 \le i \le n} \left\{ s(\lambda_i)^{-1} \quad \left|\mathrm{Re}\{\lambda_i\}\right| \right\}$$

M_3 is defined as the likelihood margins for every eigenvalues to become unstable

For the robust performance, the closed-loop system is subjected to small random perturbation then the relative change of the eigenvalues is computed.

For each form used in different block poles, the step response of the closed-loop system is plotted and the time response characteristics (Maximum overshoot, settling time, rise time and steady state value) are computed. The robustness of the closed-loop system as well as the norm of the state feedback gain matrix, the results are then compared to select the form of the solvents so that the closed-loop system meet the required criteria (good robustness, small transient response, and small feedback gain matrix).

6.3 Effect of Eigenstructure on Time Response

In this section it is shown that the feedback gain matrix K determines the eigenvectors as well as the eigenvalues of the closed-loop plant matrix $A - BK$ and both these quantities determine the time response.

For the system represented by the closed-loop state equation

$$\dot{x} = (A - BK)x = Qx \tag{6.4}$$

the eigenvalue spectrum $\sigma(Q)$ is the set of roots of the characteristic equation which is formed from

$$\Delta(\lambda) = |\lambda I - Q| = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_0 = 0 \tag{6.5}$$

When all the eigenvalues of Q are distinct, the modal matrix T can be determined such that

$$T^{-1}QT = \Lambda \tag{6.6}$$

The matrix Λ is a diagonal matrix in which the eigenvalues appear in the diagonal. The eigenvectors v_i are the columns of T and satisfy the equation

$$[\lambda_i I - Q]v_i = 0 \tag{6.7}$$

The rows of T^{-1} are the row vectors w_i^T , which are called the reciprocal or left eigenvectors and satisfy the equation

$$w_i^T [\lambda_i I - Q] = 0 \tag{6.8}$$

Thus

$$T = [v_1 \quad v_2 \quad . \quad . \quad . \quad v_n] \qquad T^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ . \\ . \\ . \\ w_n^T \end{bmatrix} \tag{6.9}$$

Since $TT^{-1} = I$, the sets of eigenvectors v_i and reciprocal eigenvectors w_i^T are orthogonal, i.e.,

$$w_i^T v_i = \begin{cases} 1 & for\, i = j \\ 0 & for\, i \neq j \end{cases} \tag{6.10}$$

Solving for Q in equation (6.6) yields $Q = T\Lambda T^{-1}$, which can be substituted into the solution of the state equation

$$x(t) = e^{Qt}x(0) + \int_0^t e^{Q\tau}Bu(t-\tau)d\tau \tag{6.11}$$

Thus it is apparent that the state transition matrix e^{Qt} can be expressed in terms of the eigenvectors and reciprocal eigenvectors. Using the series representation of e^{Qt} yields

$$\begin{aligned} e^{(Q^{-1}\Lambda Q)t} &= I + (Q^{-1}\Lambda Q)t + \frac{(Q^{-1}\Lambda Q)^2 t^2}{2!} + ... \\ &= Q^{-1}(I + \Lambda t + \frac{\Lambda^2 t^2}{2!} + ...)Q \\ &= Q^{-1}e^{\Lambda t}Q \end{aligned} \tag{6.12}$$

In the case of distinct eigenvalues the matrix $e^{\Lambda t}$ has the diagonal form

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & & & 0 \\ & e^{\lambda_2 t} & & & & \\ & & . & & & \\ & & & . & & \\ & & & & . & \\ 0 & & & & & e^{\lambda_k t} \end{bmatrix} \tag{6.13}$$

In case where the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ are repeated with multiplicity $\mu_1, \mu_2, ..., \mu_k$ respectively,

$$e^{\Lambda t} = \begin{bmatrix} e^{J_1 t} & & & & 0 \\ & e^{J_2 t} & & & \\ & & . & & \\ & & & . & \\ & & & & . \\ 0 & & & & e^{J_k t} \end{bmatrix}$$

Where $J_i = \begin{bmatrix} \lambda_i & 1 & . & . & 0 \\ 0 & \lambda_i & . & . & . \\ . & . & . & . & . \\ . & . & . & . & 1 \\ 0 & 0 & . & . & \lambda_i \end{bmatrix}$ and $e^{J_i t} = \begin{bmatrix} 1 & t & t^2/2 & . & . & t^{\mu_i - 1}/(\mu_i - 1)! \\ 0 & 1 & t & . & . & t^{\mu_i - 2}/(\mu_i - 2)! \\ 0 & 0 & 1 & . & . & t^{\mu_i - 3}/(\mu_i - 3)! \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & 1 \end{bmatrix} e^{\lambda_i t}$

If $J_i = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$ then $e^{J_i t} = \begin{bmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{bmatrix}$

Therefore from (6.13) the state transition matrix can be written as

$$e^{Qt} = \sum_{i=1}^{n} v_i e^{\lambda_i t} w_i^T \tag{6.14}$$

The output equation, when the dimension of the input *u* is *m*, is given by

$$y(t) = \sum_{i=1}^{k} C v_i e^{\lambda_i t} w_i^T x(0) + \sum_{j=1}^{m} \sum_{i=1}^{n} C v_i w_i^T b_j \int_0^t e^{\lambda_i \tau} u_j (t - \tau) d\tau \tag{6.15}$$

The transient response of the system is therefore a linear combination of *n* functions of the form

$$v_i e^{\lambda_i t}, \quad i = 1,2,...,n \tag{6.16}$$

which describe the dynamical modes of the system. From equation (6.15), the entire eigenstructure determines the time response of the system: *i.e.*, the eigenvalues λ_i, the associated eigenvectors v_i and the left eigenvectors w_i all contribute to time response. The terms $c_k^T v_i$, $w_i^T x(0)$ and $w_i^T b_i$ are scalars and determine the magnitude of the modal responses $e^{\lambda_i t}$. The ability to select v_i and w_i^T provides the potential for adjusting the magnitude of each mode which appear in each of the outputs.

For a matrix *A*, in companion form, the eigenvector associated with λ_i has the following form:

$$\begin{bmatrix} 1 & \lambda_i & \lambda_i^2 & . & . & \lambda_i^{n-1} \end{bmatrix}^T, \quad i = 1,2,...,n$$

if a matrix A is in diagonal form its eigenvector is of the form

$$[0 \;\; . \;\; . \;\; 0 \;\; 1 \;\; 0 \;\; . \;\; . \;\; 0]$$

From the structure of the eigenvectors, the norm of the eigenvectors associated with the matrix of companion form is larger than that of diagonal form, Hence the magnitude of the dynamical mode $e^{\lambda_i t}$ decreases in diagonal form than is in companion form. However, as shown later, it yields less overshoot and less settling time which gives rise to better time response.

6.4 The Effect of the Eigenvalues and the Associated Eigenvectors on the Feedback Gain Matrix

Given a closed-loop matrix $(A - BK)$, the purpose in applying state feedback is to assign both closed-loop eigenvalue spectrum

$$\sigma(A - BK) = \{\lambda_1, \lambda_2, ..., \lambda_n\}$$

and an associated set of eigenvectors

$$v(A - BK) = \{v_1, v_2, .., v_n\}$$

which are selected to achieve the desired time response characteristics.

The closed-loop eigenvalues and eigenvectors are related by the equation

$$(A - BK)v_i = \lambda_i v_i \tag{6.17}$$

This equation can be put in the form

$$[A - \lambda_i I \quad B]\begin{bmatrix} v_i \\ q_i \end{bmatrix} = 0 \; for \quad i = 1,...,n \tag{6.18}$$

Where v_i is the eigenvector and

$$q_i = Kv_i \tag{6.19}$$

In order to satisfy equation (6.18), the vector $[v_i^T \quad q_i^T]$ must lie in the kernel or null space of the matrix

$$S(\lambda_i) = [A - \lambda_i I \quad B] \; for \quad i = 1,2,...,n$$

The notation ker $S(\lambda_i)$ is used to define the null space which contains all the vectors $[v_i^T \quad q_i^T]$ for which equation (6.18) is satisfied.

Equation (6.19) can be used to form the matrix equality

$$\begin{aligned}[q_1 \quad q_2 \quad . \quad . \quad . \quad q_n] &= [Kv_1 \quad Kv_2 \quad . \quad . \quad . \quad Kv_n] \\ &= K[v_1 \quad v_2 \quad . \quad . \quad . \quad v_n]\end{aligned} \tag{6.20}$$

hence

$$K = [q_1 \quad q_2 \quad . \quad . \quad . \quad q_n]\,[v_1 \quad v_2 \quad . \quad . \quad . \quad v_n]^{-1} = QV^{-1} \tag{6.21}$$

If the eigenvalues of $(A - BK)$ are specified and the associated eigenvectors are selected to satisfy equation (6.18), then equation (6.21) specifies the required state feedback matrix K. The selected eigenvectors must be linearly independent so that the inverse matrix V^{-1} in equation (6.21) exists.

6.5 Sensitivity of Eigenstructure [46]

If λ is an eigenvalue of a matrix A and its associated right and left eigenvectors are V and T respectively, it is shown that

$$|\lambda - \lambda'| \le \varepsilon \|T\|_2$$

where λ' is an eigenvalue of a slightly perturbed matrix $(A+E)$ with $\varepsilon = \|E\|_2$, the Euclidean norm of E. We notice that the sensitivity of λ is determined by the norm of the corresponding left eigenvector. Hence, $\|T\|_2$ is a condition number for the eigenvalue λ.

And we have

$$\|V - V'\|_2 \le \frac{\varepsilon}{\min|\lambda_k - \lambda|}$$

where V' is an eigenvector of $(A+E)$ and λ_k an eigenvalue of A other than λ.

It is clear from above that the left eigenvector T play an important role in the sensitivity of the eigenvalue λ.

In multivariable system, both closed-loop eigenvalues and eigenvectors are assigned. Given a perturbed closed-loop matrix as $(A - BK) + \Delta A$, the idea is the select the norms of the left eigenvectors of the corresponding closed-loop eigenvalues to minimize the effect of the perturbation ΔA of A.

Given a closed-loop matrix

$$A - BK = \begin{bmatrix} 0_m & I_m & 0_m & . & . & 0_m \\ 0_m & 0_m & I_m & . & . & \\ . & . & . & . & . & . \\ . & . & . & . & . & I_m \\ -D_l & -D_{l-1} & . & . & . & -D_1 \end{bmatrix}$$

Its matrix characteristic polynomial is $\Delta(\lambda) = I_m \lambda^l + D_1 \lambda^{l-1} + ... + D_l$

The left eigenvector of $A - BK$ is defined as

$$T=\begin{bmatrix} q_1^{l-1} & . & . & . & q_n^{l-1} \\ q_1^{l-2} & . & . & . & q_n^{l-2} \\ . & . & . & . & . \\ q_1^1 & . & . & . & q_n^1 \\ q_1 & . & . & . & q_n \end{bmatrix}=\begin{bmatrix} Q^{T(l-1)} \\ Q^{T(l-2)} \\ . \\ . \\ Q^T \end{bmatrix} \quad where \;\; Q^T=\begin{bmatrix} q_1 & . & . & . & q_n \end{bmatrix}$$

$$T_i=\begin{bmatrix} q_i^{l-1} \\ . \\ . \\ . \\ q_i^{(1)} \\ q_i \end{bmatrix}$$

The norm of the left eigenvector is given by

$$\|T\|=\left\|\begin{bmatrix} q_1^{l-1} & . & . & . & q_n^{l-1} \\ q_1^{l-2} & . & . & . & q_n^{l-2} \\ . & . & . & . & . \\ q_1^1 & . & . & . & q_n^1 \\ q_1 & . & . & . & q_n \end{bmatrix}\right\|=\left\|\begin{bmatrix} Q^{T(l-1)} \\ Q^{T(l-2)} \\ . \\ . \\ Q^T \end{bmatrix}\right\|$$

Hence

$$\|T_i\|=\left\|\begin{bmatrix} q_i^{l-1} \\ . \\ . \\ . \\ q_i^{(1)} \\ q_i \end{bmatrix}\right\|$$

We have a latent vector is a subvector of the left eigenvector and the norm of the left eigenvector depends on the norm of the latent vector.
We have $L_j=Q_j^{-1}\Lambda_j Q_j$ where $Q_j=\begin{bmatrix} q_{1j} & . & . & . & q_{mj} \end{bmatrix}^T$ for $j=1,2,...,l$; hence the latent vector are related to the left solvent so the norm of the left eigenvector depends on the norm of the solvent.
The minimal norm of the left eigenvectors is given by the minimal norm of the solvent which is no more than the solvent in diagonal form.

6.6 The Effect of the Block Pole on the Magnitude of the State Feedback Gain Matrix

Given a multivariable system $\dot{x}=Ax+Bu$ with the characteristic equation

$$\Delta(s)=Is^l+A_{l-1}s^{l-1}+...+A_0$$

It is desired to find the state feedback gain matrix so that the block controllable matrix $(A_c - B_c K_c)$ has the following desired characteristic equation

$$\Delta_d(s) = Is^l + D_{l-1}s^{l-1} + \ldots + D_0$$

or

$$\Delta_d(s) = Is^l + (A_{l-1} + K_{(l-1)c})s^{l-1} + \ldots + (A_0 + K_{0c})$$

where

$$D_i = A_i + K_{ic}$$

or

$$K_{ic} = D_i - A_i$$

and we have

$$\|K_{ic}\| = \|D_i - A_i\| \le \|D_i\| + \|A_i\|$$

Our purpose is to find the norm of the state feedback gain matrix as small as possible, since $\|A_i\|$ cannot be selected, we seek to get $\|D_i\|$ minimum.

Let $\{R_i\}$ be a set of right solvent of the desired matrix polynomial $\Delta_d(s)$, we can write:

$$\Delta_d(s) = Q(s)(\lambda I - R_i)$$

where

$$Q(s) = s^{l-1}Q_0 + s^{l-2}Q_1 + \ldots + Q_{l-1}$$
$$Q_0 = I$$

hence

$$\Delta_d(s) = Q(s)(\lambda_i I - R_i) = Is^l + D_{l-1}s^{l-1} + \ldots + D_0$$

To get $\|D_i\|$ minimum, R_i must be selected so that $\|R_i\|$ is minimized.

Using the fact that the norm of the solvents in companion form is larger than the norm of the solvents in diagonal form, the solvents R_i must be selected in diagonal form to have the norm of the desired closed-loop block poles D_i minimum, hence the norm of the state feedback gain matrix is minimum since we have $\|K_{ic}\| \le \|D_i\| + \|A_i\|$.

6.7 Conclusion

i. From the above discussions, we notice that the choice of the form of the closed-loop block pole minimizes the norm of the state feedback gain matrix and is given, as it is shown later, by a block pole in diagonal form.

ii. We notice that both eigenvalues and corresponding left and right eigenvectors can be selected to provide better time response.

iii. The magnitude of the dynamical mode $e^{\lambda_i t}$ decreases in diagonal form which leads to less settling time and smaller percent overshoot.

iv. Left eigenvector T play an important role in the sensitivity of the eigenvalue λ.

Chapter 7

Simulation Results

A large number of case studies are presented to test the proposed approach described in chapter 6 using the software package MATLAB.

For multivariable state feedback, both cases n/m is an integer and n/m is not an integer are considered.

The placement of block poles in multivariable system using either state feedback or compensator design requires the construction of a matrix polynomial from a given a set right or left solvents. The different right and left solvents are constructed using different canonical forms: controllable, observable and diagonal canonical forms.

Let $D_f(s)$ represent the desired monic matrix polynomial,

$$D_f(s) = Is^l + D_{f1}s^{l-1} + ... + D_{fl}$$

then the complete set of right solvents R_i and left solvents L_i satisfy, respectively, the following matrix polynomial

$$R_i^l + D_{f1}R_i^{l-1} + ... + D_{f(l-1)}R_i + D_{fl} = 0_m, \quad i = 1,2,...,l$$

and

$$L_i^l + L_i^{l-1}D_{f1} + ... + L_i D_{f(l-1)} + D_{fl} = 0_m, \quad i = 1,2,...,l$$

hence, the coefficient matrices of the desired matrix polynomial can be obtained by using either:

$$\begin{bmatrix} D_{fl} & D_{f(l-1)} & . & . & . & D_{f1} \end{bmatrix} = -\begin{bmatrix} R_1^l & R_2^l & . & . & . & R_l^l \end{bmatrix} V_R^{-1}$$

or
$$\begin{bmatrix} D_{fl} \\ D_{f(l-1)} \\ . \\ . \\ . \\ D_{f1} \end{bmatrix} = -V_L^{-B} \begin{bmatrix} L_1^l \\ L_2^l \\ . \\ . \\ . \\ L_l^l \end{bmatrix}$$

where V_R and V_L^B are the right Vandermonde and the block transpose of the left Vandermonde matrices, respectively, given in (2.23) and (2.24) mentioned in chapter 2.

To ensure the stability and the performance robustness of the block poles to be assigned the proposed methods given in chapter 5 are used.

7.1 The Case of the Block Pole Placement using State Feedback

Case Study 1:

Consider the following open-loop system with 2-inputs and 2-outputs and the system is of order 4 given by the following matrices:

$$A = \begin{bmatrix} -0.501 & -0.985 & 0.174 & 0 \\ 16.83 & -0.575 & 0.0123 & 0 \\ -3227 & 0.321 & -2.1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0.109 & 0.007 \\ -132.8 & 27.19 \\ -1620 & -1240 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, $n = 4 = 2 \times 2 = lm$, i.e., l is an integer, it follows that we can assign two block poles of dimension 2×2.

We want to design a state feedback controller such that the closed-loop system *A-BK* has the following set of desired eigenvalues: $-53, -54, -13.3333 \pm 14.8897i$.

Since rank $\Phi_c = rank[B \quad AB] = 4$, i.e., the controllability matrix has full rank, the pair (A, B) is block controllable. Therefore the pair (A, B) can be converted into multivariable block controllable companion form (A_c, B_c).

The pair (A_c, B_c) and C_c are as follows:

$$A_c = T_c A T_c^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -48.1221 & -77.2688 & -0.2358 & 0.6516 \\ -330.3420 & -530.4239 & -2.1174 & -2.9402 \end{bmatrix},$$

$$B_c = T_c B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$and \quad C_c = C T_c^{-1} = \begin{bmatrix} -0.1511 & -0.2426 & 0.0001 & 0.0000 \\ 0.0845 & 0.1357 & -0.1328 & 0.0272 \\ 0 & -0.0000 & -1.6200 & -1.2400 \\ -1.6200 & -1.2400 & 0.0000 & 0.0000 \end{bmatrix} * 10^3$$

where T_c is the required similarity transformation.

The characteristic matrix polynomial of this block companion form is determined by the last 2×4 block row

$$\Delta c = \begin{bmatrix} -48.1221 & -77.2688 & -0.2358 & 0.6516 \\ -330.3420 & -530.4239 & -2.1174 & -2.9402 \end{bmatrix}$$

The state feedback gain K_c is to be selected so that:

$$A_c - B_c K_c = A_D \ ,$$

where A_D is a desired closed-loop matrix whose eigenvalues are the set of desired eigenvalues.

7.1.1 State feedback Using Block Poles in Diagonal Form.

The desired block poles are constructed in diagonal form as

$$R_1 = \begin{bmatrix} -13.3333 & 14.8897 \\ -14.8897 & -13.3333 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} -53 & 0 \\ 0 & -54 \end{bmatrix}$$

The corresponding 2×2 desired right denominator matrix polynomial of degree 2 is:

$$D_f(s) = Is^2 + D_{f1}s + D_{f2}$$

where

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = -\begin{bmatrix} R_1^2 & R_2^2 \end{bmatrix} V_R^{-1}$$

i.e.,

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = \begin{bmatrix} 713.0690 & -786.6613 & 66.4541 & -14.5678 \\ 806.6448 & 713.4733 & 15.2197 & 67.2125 \end{bmatrix}$$

Then we can have K_c which given by

$$K_c = \begin{bmatrix} 664.9468 & -863.9301 & 66.2183 & -13.9162 \\ 476.3029 & 183.0494 & 13.1023 & 64.2723 \end{bmatrix}$$

Computing the state feedback gain matrix, that places the block poles of the closed-loop system to the desired locations, in original coordinates, yields

$$K = \begin{bmatrix} 10.5375 & -0.4952 & 0.0004 & -1.4191 \\ 1.6660 & 0.4220 & -0.0426 & -0.4274 \end{bmatrix}$$

The norm of feedback gain matrix is: $\|K\|_2 = 10.7773$

The closed loop matrix using solvents in diagonal form will be:

$$(A - BK)_{diagonal} = \begin{bmatrix} -0.0002 & -0.0001 & 0.0000 & 0.0000 \\ 0.1371 & -0.0078 & 0.0001 & -0.0177 \\ 1.5910 & -0.0279 & -0.0054 & -0.2829 \\ 0 & 0 & 0.0001 & 0 \end{bmatrix} * 10^4$$

The following table summarizes the time response for this choice:

Inputs	Transient steady state specifications	Maximum overshoot (M_p)	Percent overshoot (POS)	Settling time (T_s)	Rise time (T_r)	Steady State Value (SSV)
U_1	y_1	0.0835	8.5826%	0.361s	0.149s	0.0769
	y_2	0.128	197.6744%	0.308s	0.000185s	-0.043
	y_3	5.45	/	0.376s	0s	0
	y_4	-0.722	430.8824%	0.385s	0.0104s	-0.136
U_2	y_1	-0.282	10.5882%	0.277s	0.0504s	-0.255
	y_2	0.184	28.6713%	0.39s	0.00315s	0.143
	y_3	1.29	/	0.34s	0s	0
	y_4	-2.03	7.26%	0.289s	0.0599s	-1.89

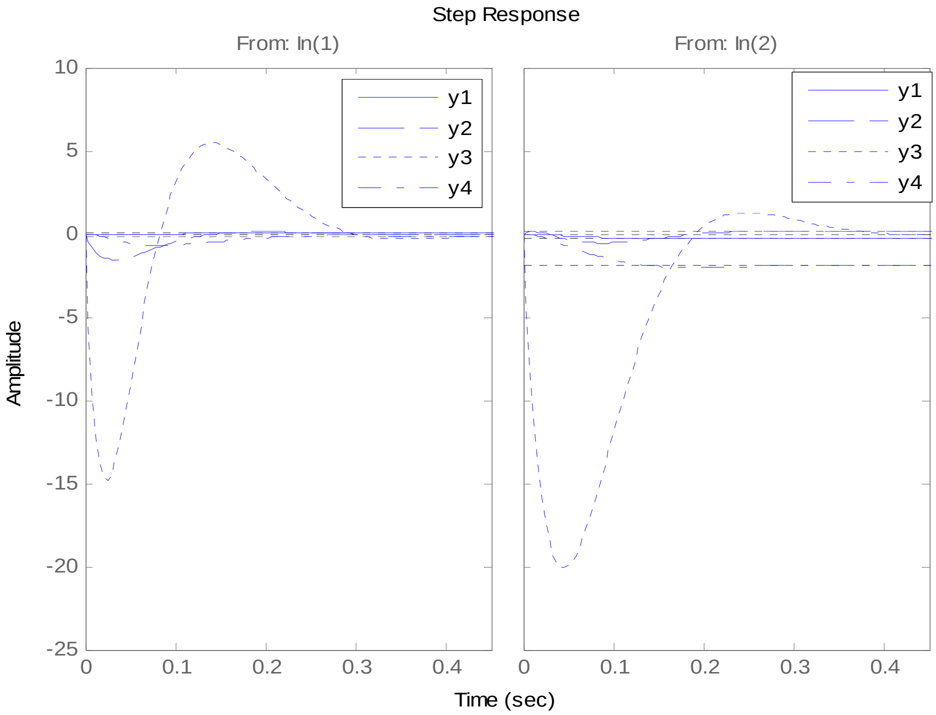


Figure1: Time response for diagonal form

Initial response for $x_0 = [1;1;1;1]$ in diagonal Form

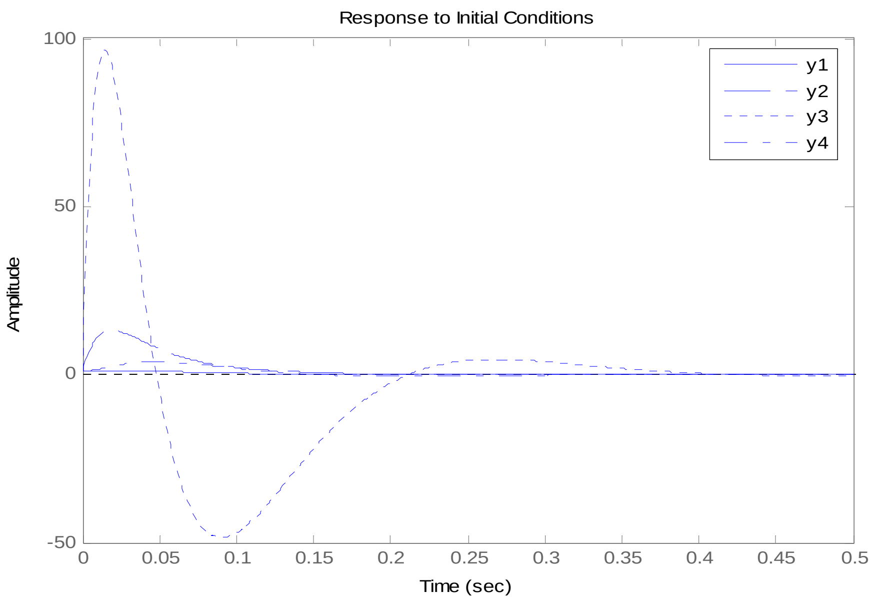


Figure2: Response to initial condition x_0= [1;1;1;1]

The response to initial condition of the closed-loop matrix for diagonal form is summarized in the following table

Transient steady state specifications	Maximum overshoot (M_p)	Percent overshoot (POS)	Settling time (T_s)	Steady State Value (SSV)
y_1	1.2262	5.7069%	0.316s	0.00362
y_2	15.1100	12.7612%	0.324s	0.00268
y_3	165.5000	49.0991%	0.299s	0.000222
y_4	6.2250	5.8673%	0.382s	0.00207

7.1.1.a Robust Stability

For the study of the robustness of the system, three measures stated in the chapter 5 are computed.

Let us compute the right and the left eigenvector of the closed-loop matrix

$$V = \begin{bmatrix} -0.0026-0.0065i & -0.0026+0.0065i & 0.0018 & -0.0036 \\ 0.0449-0.0463i & 0.0449+0.0463i & -0.0827 & -0.0199 \\ 0.9966 & 0.9966 & -0.9964 & 0.9996 \\ -0.0333-0.0371i & -0.0333+0.0371i & 0.0188 & -0.0185 \end{bmatrix}$$

its norm is $\|V\|_2 = 1.9968$

The norms of $v_i, i = 1,2,3,4$ are equal to 1

The norm of the left eigenvector is $\|T\|_2 = 532.4127$

The norm of $t_i, i = 1,2,3,4$

$\|t_1\|_2 = 254.3486$, $\|t_2\|_2 = 254.3486$, $\|t_3\|_2 = 382.0304$, $\|t_4\|_2 = 93.8477$

The sensitivity of all the eigenvalues is

$s(\Lambda) = \|V\|_2 \|T\|_2 = 1.0631*10^3$ its inverse is given by $s(\Lambda)^{-1} = 9.4063*10^{-4}$

The sensitivity of every eigenvalue is computed as follows:

$s(\lambda_i) = \|v_i\|_2 \|t_i\|_2, i = 1,2,3,4$

yields

$$s(\lambda_1 = -13.3333 + 14.8897i) = \|v_1\|_2 \|t_1\|_2 = 254.3486$$
$$s(\lambda_2 = -13.3333 - 14.8897i) = \|v_2\|_2 \|t_2\|_2 = 254.3486$$
$$s(\lambda_3 = -53) = \|v_3\|_2 \|t_3\|_2 = 382.0304$$
$$s(\lambda_4 = -54) = \|v_4\|_2 \|t_4\|_2 = 93.8477$$

Finally we compute the stability robustness measures

Computing $M_1 = \min_{0 \le \omega \le \infty} \{\underline{\sigma}(A - j\omega I)\}$ we have $M_1 = 0.0986$

Computing $M_2 = s(\Lambda)^{-1} |\mathrm{Re}\{\lambda_n\}|$, $(|\mathrm{Re}\{\lambda_n\}| \le \ldots \le |\mathrm{Re}\{\lambda_1\}|)$ we have $M_2 = 0.0125$

Finally for $M_3 = \min_{1 \le i \le n} \{ s(\lambda_i)^{-1} \quad |\mathrm{Re}\{\lambda_i\}| \}$ we have

$$s(\lambda_1 = -13.3333 + 14.8897i)^{-1} \times |-13.3333 + 14.8897i| = 0.0524$$
$$s(\lambda_2 = -13.3333 - 14.8897i)^{-1} \times |-13.3333 - 14.8897i| = 0.0524$$
$$s(\lambda_3 = -53)^{-1} \times |-5| = 0.1387$$
$$s(\lambda_4 = -54)^{-1} \times |-2| = 0.5754$$

hence $M_3 = 0.5754$

7.1.1.b Robust Performance

The following perturbation is generated randomly using MATLAB

$$\Delta A = \begin{bmatrix} 0.0935 & 0.0058 & 0.0139 & 0.0272 \\ 0.0917 & 0.0353 & 0.0203 & 0.0199 \\ 0.0410 & 0.0813 & 0.0199 & 0.0015 \\ 0.0894 & 0.0010 & 0.0604 & 0.0747 \end{bmatrix}$$

With $\|\Delta A\| = 0.1933$

The new closed-loop matrix, after perturbation, is:

$$(A - BK + \Delta A)diagonal = \begin{bmatrix} -1.5678 & -0.9282 & 0.1882 & 0.1849 \\ 137.1 & -77.7715 & 1.2465 & -176.8103 \\ 15910 & -278.4754 & -54.1787 & -2828.8 \\ 0.0894 & 0.001 & 1.0604 & 0.0747 \end{bmatrix}$$

with eigenvalues: -12.3260 +15.6215i, -12.3260 -15.6215i, -54.7737, -54.0176 .

The relative change of the eigenvalues of the closed-loop matrix due to the perturbation is

$r_i = \left|\frac{\lambda_i - \lambda'_i}{\lambda_i}\right|$ where λ_i is the eigenvalue of the closed-loop matrix and λ'_i the eigenvalue of the perturbed closed-loop matrix. This leads

$r_1 = 0.0623$, $r_2 = 0.0623$, $r_3 = 0.0335$, $r_4 = 3.2615*10^{-4}$.

7.1.2 State Feedback Using Block Poles in Controllable Form

The desired block poles are constructed in controller form as

$$R_1 = \begin{bmatrix} 0 & 1.0000 \\ -399.4801 & -26.6666 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} -107 & -2862 \\ 1 & 0 \end{bmatrix}$$

hence

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = \begin{bmatrix} 2.8590 & 0.0838 & 0.1070 & 0.0062 \\ -0.0117 & 0.3996 & -0.0009 & 0.0266 \end{bmatrix} *10^3$$

K_c is given by

$$K_c = \begin{bmatrix} 2.8109 & 0.0065 & 0.1068 & 0.0068 \\ -0.3420 & -0.1309 & -0.0030 & 0.0237 \end{bmatrix} *10^3$$

The required feedback gain matrix in the original coordinate systems is

$$K = \begin{bmatrix} 16.5763 & -0.5718 & -0.0179 & -3.3109 \\ -0.9190 & 0.2011 & -0.0147 & 0.3073 \end{bmatrix}$$

The norm of feedback gain matrix is: $\|K\|_2 = 16.9411$

The closed loop matrix using solvents in controller form will be as follows:

$$(A - BK)_{controllable} = \begin{bmatrix} -0.0002 & -0.0001 & 0.0000 & 0.0000 \\ 0.2243 & -0.0082 & -0.0002 & -0.0448 \\ 2.2487 & -0.0677 & -0.0049 & -0.4983 \\ 0 & 0 & 0.0001 & 0 \end{bmatrix} *10^4$$

The following table summarizes the time response obtained for this choice:

Inputs	Transient steady state specifications	M_p	POS	T_s	T_r	SSV
U_1	y_1	-0.0557	0.798%	0.101s	0.0326s	0.0553
	y_2	0	0%	0.128s	0.133s	0.0309
	y_3	0.0201	/	0.129s	0.171s	0
	y_4	-0.581	0.335%	0.105s	0.0317s	-0.579
U_2	y_1	-0.63	5.86%	0.301s	0.0704s	-0.596
	y_2	0.917	175%	0.359s	0.00594s	0.333
	y_3	-27.5	/	0.359s	0.0274s	0
	y_4	-3.15	5.71%	0.301s	0.071s	-2.98

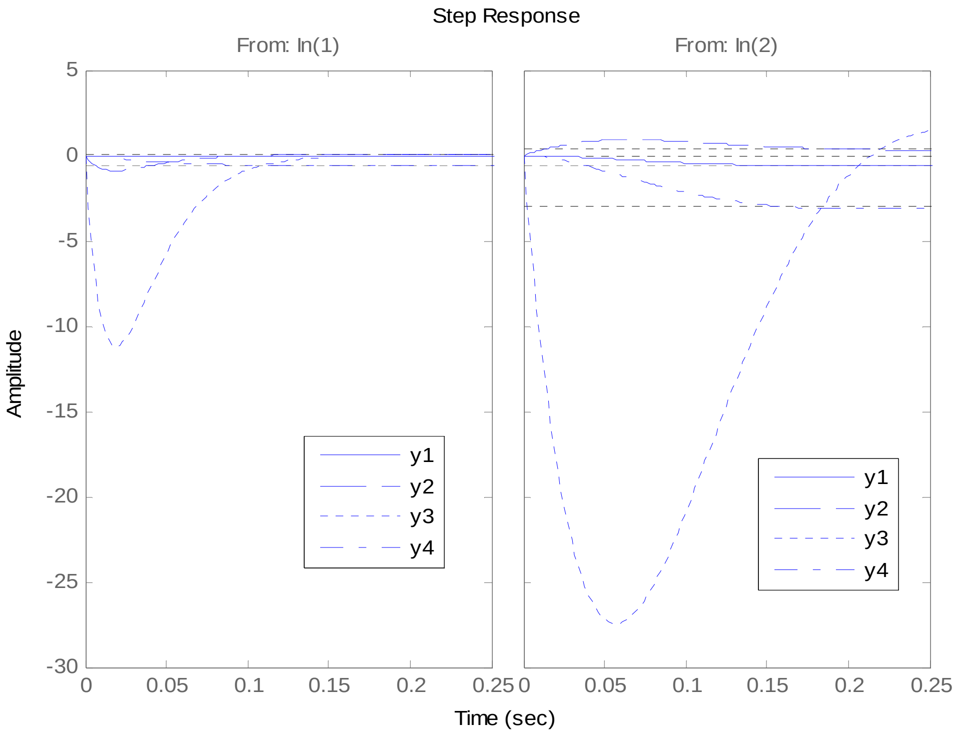


Figure3: Time response for controller form

Response to Initial Condition x_0= [1;1;1;1] using Controller Form

Transient steady state specifications	Maximum overshoot (M_p)	Percent overshoot (POS)	Settling time (T_s)	Steady State Value (SSV)
y_1	1.1577	9.2170%	0.349s	0.00263
y_2	13.1	0%	0.187s	0.006
y_3	144.7000	50.2596%	0.43s	-0.255
y_4	4.2400	12.4668%	0.351s	0.013

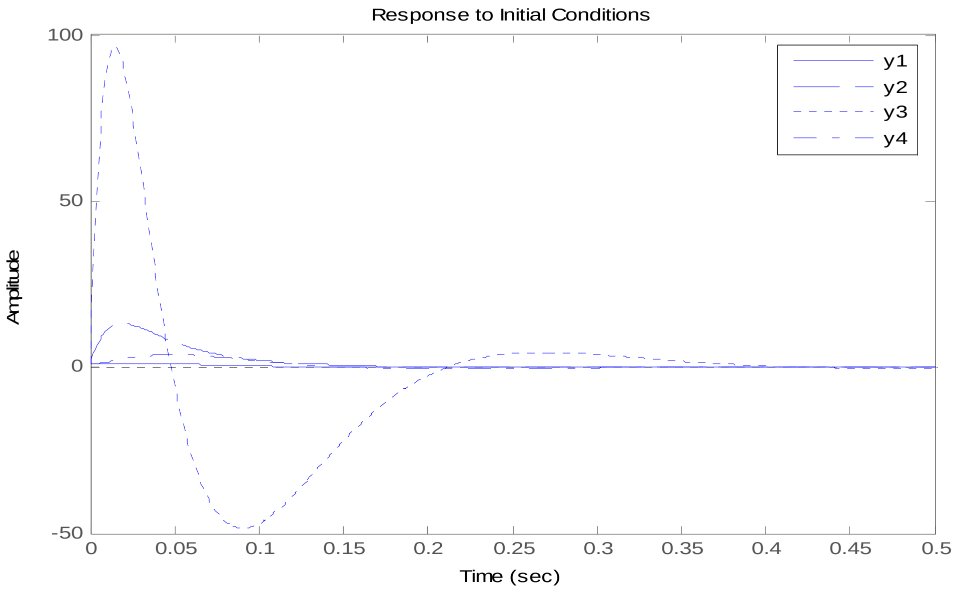


Figure4: Response to initial condition x_0= [1;1;1;1]

7.1.2.a Robust Stability

The right eigenvector of the closed-loop is given by

$$V = \begin{bmatrix} 0.0018 & -0.0018 & -0.0065-0.0076i & -0.0065+0.0076i \\ -0.0841 & 0.0842 & -0.0227-0.0012i & -0.0227+0.0012i \\ -0.9963 & 0.9963 & 0.9984 & 0.9984 \\ 0.0184 & -0.0188 & -0.0333-0.0372i & -0.0333+0.0372i \end{bmatrix}$$

its norm is $\|V\|_2 = 1.9964$

the norms of $v_i, i = 1,2,3,4$ are equal to 1

The norm of the left eigenvector is $\|T\|_2 = 3.7068*10^4$

The norm of $t_i, i = 1,2,3,4$ with t_i are columns of the left eigenvector T

, $\|t_1\|_2 = 2.6218*10^4$, $\|t_2\|_2 = 2.6204*10^4$, $\|t_3\|_2 = 123.7316$, $\|t_4\|_2 = 123.7316$

The sensitivity of all the eigenvalues is $s(\Lambda) = \|V\|_2\|T\|_2 = 7.4001*10^4$

its inverse is given by: $s(\Lambda)^{-1} = 1.3513*10^{-5}$

The sensitivity of every eigenvalue is as follows:

$$s(\lambda_i) = \|v_i\|_2\|t_i\|_2, i = 1,2,3,4$$

yields

$$s(\lambda_1 = -13.3333 + 14.8897i) = \|v_1\|_2\|t_1\|_2 = 123.7316$$
$$s(\lambda_2 = -13.3333 - 14.8897i) = \|v_2\|_2\|t_2\|_2 = 123.7316$$
$$s(\lambda_3 = -53) = \|v_3\|_2\|t_3\|_2 = 2.6204*10^4$$
$$s(\lambda_4 = -54) = \|v_4\|_2\|t_4\|_2 = 2.6218*10^4$$

Now we can compute the stability robustness measures

$$M_1 = 0.1848$$

$$M_2 = 1.8018*10^{-4}$$

we have:

$$s(\lambda_1 = -13.3333 + 14.8897i)^{-1} \times |-13.3333 + 14.8897i| = 0.1078$$
$$s(\lambda_2 = -13.3333 - 14.8897i)^{-1} \times |-13.3333 - 14.8897i| = 0.1078$$
$$s(\lambda_3 = -5)^{-1} \times |-53| = 0.0020$$
$$s(\lambda_4 = -2)^{-1} \times |-54| = 0.0021$$

hence $M_3 = 0.0020$

7.1.2.b Robust Performance

The closed-loop matrix after perturbation is given by:

$$(A - BK + \Delta A)_{controllable} = \begin{bmatrix} -2.2079 & -0.9183 & 0.1900 & 0.3859 \\ 2243.2 & -81.9393 & -1.9493 & -448.0220 \\ 22487 & -676.5500 & -49.3707 & -4982.5 \\ 0.0894 & 0.001 & 1.0604 & 0.0747 \end{bmatrix}$$

with eigenvalues: -59.1879, -47.8118, -13.2218 +15.6415i, -13.2218 -15.6415i

The relative change of the eigenvalues of the closed-loop matrix due to the perturbation is $r_i = \left|\frac{\lambda_i - \lambda'_i}{\lambda_i}\right|$ where λ_i is the eigenvalue of the closed-loop matrix and λ'_i the eigenvalue of the perturbed closed-loop matrix. This leads

$$r_1 = 0.0961, r_2 = 0.0979, r_3 = 0.0380\ ,\ r_4 = 0.0380.$$

7.1.3 State Feedback Using Block Poles in Observable Form

The desired block poles constructed in observer form as:

$$R_1 = \begin{bmatrix} 0 & -399.4801 \\ 1 & -26.6666 \end{bmatrix}, R_2 = \begin{bmatrix} -107 & 1 \\ -2862 & 0 \end{bmatrix}$$

this gives

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = \begin{bmatrix} 0.3973 & 0.0803 & 0.0267 & 0.0021 \\ -0.0803 & 2.8612 & 0.0008 & 0.1069 \end{bmatrix} * 10^3$$

K_c is given by

$$K_c = \begin{bmatrix} 0.3492 & 0.0030 & 0.0265 & 0.0028 \\ -0.4106 & 2.3308 & -0.0013 & 0.1040 \end{bmatrix} * 10^3$$

The required feedback gain matrix in original coordinate systems is

$$K = \begin{bmatrix} 2.0066 & -0.1345 & -0.0052 & -0.4097 \\ -20.3864 & 0.8029 & -0.0664 & 2.1967 \end{bmatrix}$$

The norm of feedback gain matrix is: $\|K\|_2 = 20.6216$

The closed loop matrix using solvents in observer form is given by:

$$(A - BK)_{observable} = \begin{bmatrix} -0.5770 & -0.9760 & 0.1750 & 0.0293 \\ 837.6180 & -40.2698 & 1.1280 & -114.1392 \\ -2.5255 * 10^4 & 778.0471 & -92.8198 & 2060.1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The time response is shown in the following figure:

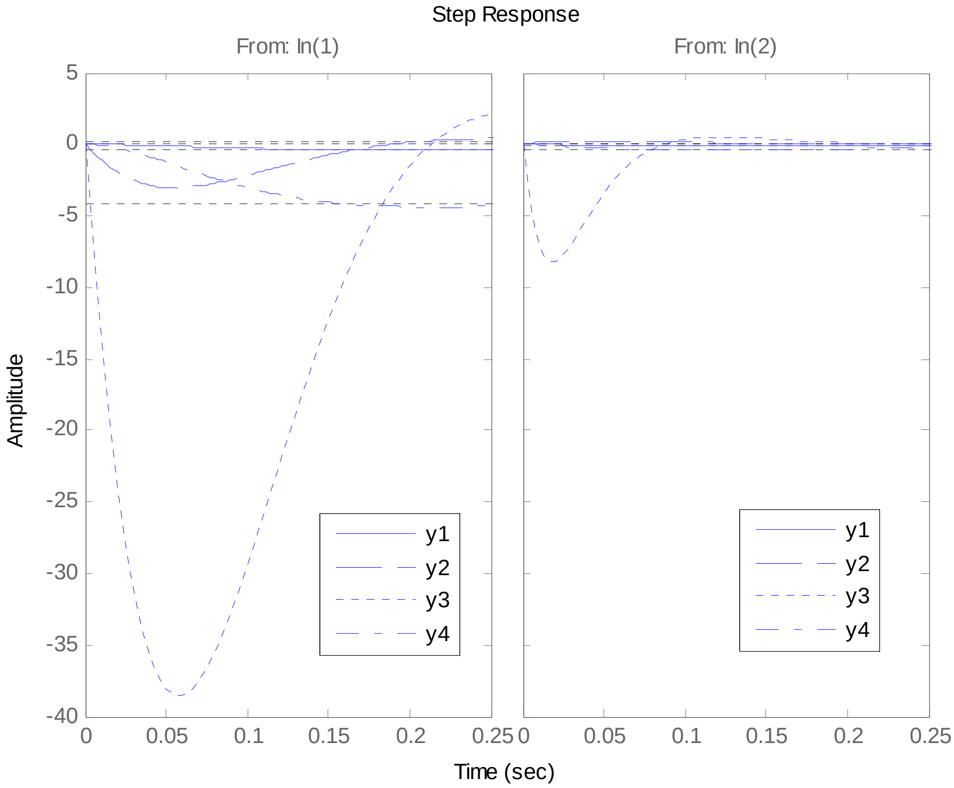


Figure5: Time response for observer form

The following table summarizes the time response obtained for the observer form:

Inputs	Transient steady state specifications	Maximum overshoot (M_p)	Percent overshoot (POS)	Settling time (T_s)	Rise time (T_r)	Steady State Value (SSV)
U_1	y_1	-0.418	5.76%	0.304s	0.0727s	-0.395
	y_2	0.417	88.6878%	0.355s	0.197s	0.221
	y_3	2.29	/	0.359s	0s	0
	y_4	-4.38	5.88%	0.302s	0.0711s	-4.14
U_2	y_1	-0.0753	2.14%	0.117s	0.028s	-0.0737
	y_2	0.238	477%	0.325s	0.000848s	0.0412
	y_3	0.45	/	0.184s	0s	0
	y_4	-0.347	9.5%	0.248s	0.0245s	-0.317

Response to Initial Condition x_0= [1;1;1;1] in Observer Form

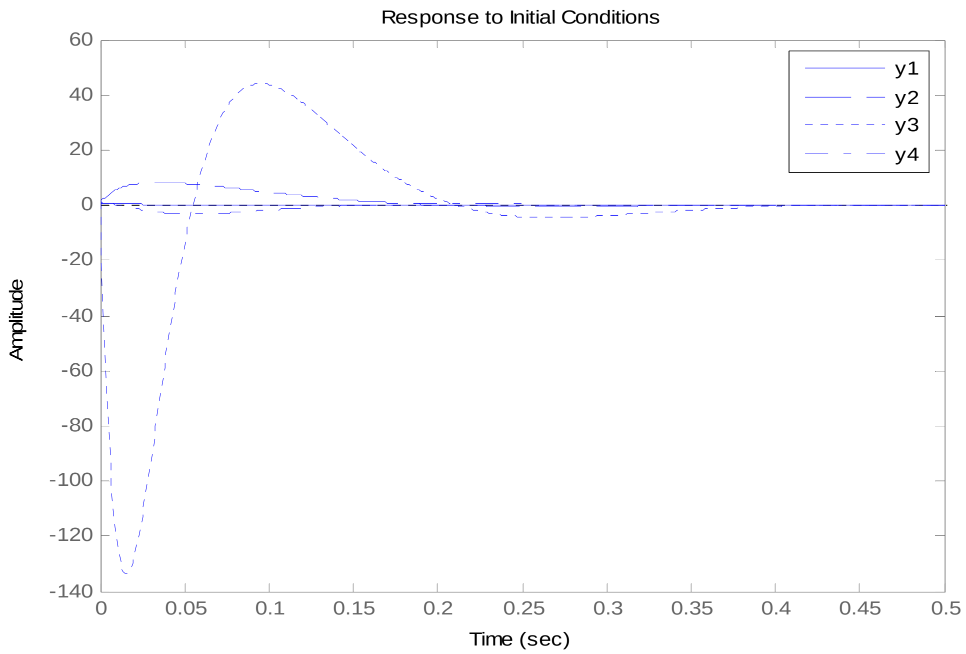


Figure 6: Response to initial condition x_0= [1;1;1;1]

Time response for the initial condition in case of observer form:

Transient steady state specifications	Maximum overshoot (M_p)	Percent overshoot (POS)	Settling time (T_s)	Steady State Value (SSV)
y_1	1.2	20%	0.312s	-0.00129
y_2	8.6590	4.4511%	0.364s	0.0209
y_3	-89	33.0827%	0.238s	0.251
y_4	-2.7760	14.5846%	0.367s	-0.0134

7.1.3.a Robust stability

The right eigenvector of the closed-loop is given by

$$V = \begin{bmatrix} 0.0036 & 0.0037 & -0.0034-0.0035i & -0.0034+0.0035i \\ 0.0174 & 0.0174 & 0.0808+0.0047i & 0.0808-0.0047i \\ -0.9997 & -0.9997 & 0.9955 & 0.9955 \\ 0.0185 & 0.0189 & -0.0332-0.0371i & -0.0332+0.0371i \end{bmatrix}$$

its norm is $\|V\|_2 = 1.9968$

the norms of $v_i, i = 1,2,3,4$ are equal to 1

The norm of the left eigenvector is $\|T\|_2 = 4.0851*10^4$

The norm of $t_i, i = 1,2,3,4$ with t_i are columns of the left eigenvector T

$\|t_1\|_2 = 2.8893*10^4$, $\|t_2\|_2 = 2.8893*10^4$, $\|t_3\|_2 = 141.3501$, $\|t_4\|_2 = 141.3501$.

The sensitivity of all the eigenvalues is

$s(\Lambda) = \|V\|_2\|T\|_2 = 8.1572*10^4$ its inverse is given by: $s(\Lambda)^{-1} = 1.2259*10^{-5}$

The sensitivity of every eigenvalue is as follows:

$s(\lambda_i) = \|v_i\|_2\|t_i\|_2, i = 1,2,3,4$

yields

$$s(\lambda_1 = -13.3333 + 14.8897i) = \|v_1\|_2\|t_1\|_2 = 141.3501$$
$$s(\lambda_2 = -13.3333 - 14.8897i) = \|v_2\|_2\|t_2\|_2 = 141.3501$$
$$s(\lambda_3 = -53) = \|v_3\|_2\|t_3\|_2 = 2.8878*10^4$$
$$s(\lambda_4 = -54) = \|v_4\|_2\|t_4\|_2 = 2.8893*10^4$$

Now we can compute the stability robustness measures

$$M_1 = 0.1658$$

$$M_2 = 1.6346*10^{-4}$$

we have

$$s(\lambda_1 = -13.3333 + 14.8897i)^{-1} \times |-13.3333 + 14.8897i| = 0.0943$$
$$s(\lambda_2 = -13.3333 - 14.8897i)^{-1} \times |-13.3333 - 14.8897i| = 0.0943$$
$$s(\lambda_3 = -53)^{-1} \times |-53| = 0.0018$$
$$s(\lambda_4 = -54)^{-1} \times |-54| = 0.0019$$

hence $M_3 = 0.0018$

7.1.3.b Robust performance

The closed-loop matrix after perturbation is as follows:

$$(A - BK + \Delta A)_{observable} = \begin{bmatrix} -0.4835 & -0.9702 & 0.1889 & 0.0565 \\ 837.7097 & -40.2345 & 1.1483 & -114.1193 \\ -2.5255 & 778.1284 & -92.7999 & 2060.1 \\ 0.0894 & 0.001 & 1.0604 & 0.0747 \end{bmatrix}$$

its eigenvalues are: -53.3944 +14.2210i, -53.3944 -14.2210i, -13.3272 +14.7359i,
-13.3272 -14.7359i

The relative change of the eigenvalues of the closed-loop matrix due to the perturbation is $r_i = \left|\frac{\lambda_i - \lambda'_i}{\lambda_i}\right|$ where λ_i is the eigenvalue of the closed-loop matrix and λ'_i the eigenvalue of the perturbed closed-loop matrix. This leads

$$r_1 = 0.2636, r_2 = 0.2684, r_3 = 0.0077, r_4 = 0.0077.$$

7.1.4 Comparison of the results

Now we gather the results in the following tables to facilitate the comparison

7.1.4.1 Time response:

		Diagonal Form	Controller Form	Observer Form
U1	y1_ MP	0.0835	-0.0557	-0.418
	y1_ POS	8.5826%	0.798%	5.76%
	y1_Ts	0.361s	0.101s	0.304s
	y1_Tr	0.149s	0.0326s	0.0727s
	y1-SSV	0.0769	0.0553	-0.395
	y2_ MP	0.128	0	0.419
	y2_ POS	197.6744%	0%	88.6878%
	y2_Ts	0.308s	0.128s	0.355s
	y2_Tr	0.000185s	0.133s	0.197s
	y2-SSV	0.043	0.0309	0.221
	y3_MP	5.45	0.0201	2.29
	y3_POS	/	/	/
	y3_Ts	0.376s	0.129s	0.359s
	y3_Tr	0s	0.171s	0s
	y3_SSV	0	0	0
	y4_MP	-0.722	-0.581	-4.38
	y4_POS	430.8824%	0.335%	5.88%
	y4_Ts	0.385s	0.105s	0.302s
	y4_Tr	0.0104s	0.0317s	0.0711s
	y4_SSV	-136	-0.579	-4.14
U2	y1_ MP	-0.282	-0.63	-0.0753
	y1_ POS	10.5882%	5.86%	2.14%
	y1_Ts	0.277s	0.301s	0.117s
	y1_Tr	0.0504s	0.0704s	0.028s
	y1-SSV	-0.255	-0.596	-0.0737
	y2_ MP	0.184	0.917	0.238
	y2_ POS	28.6713%	175%	477%
	y2_Ts	0.39s	0.359s	0.325s
	y2_Tr	0.00315s	0.00594s	0.000848s
	y2-SSV	0.143	0.333	0.0412

	y3_MP	1.29	-27.5	0.45
	y3_POS	/	/	/
	y3_Ts	0.34s	0.359s	0.184s
	y3_Tr	0s	0.0274s	0s
	y3_SSV	0	0	0
	y4_MP	-2.03	-3.15	-0.347
	y4_POS	7.26%	5.71%	9.5%
	y4_Ts	0.289s	0.301s	0.248s
	y4_Tr	0.0599s	0.071s	0.0245s
	y4_SSV	-1.89	-2.98	-0.317

7.1.4.2 Robust Stability:

Stability Measures		Diagonal Form	Controllable Form	Observable Form
M1	M1	0.0986	0.1848	0.1658
M2	M2	0.0125	$1.8018*10^{-4}$	$1.6346*10^{-4}$
M3	M31	0.0524	0.0021	0.0019
	M32	0.0524	0.0020	0.0018
	M33	0.1387	0.1078	0.0943
	M34	0.5754	0.1078	0.0943
	M3	0.0524	0.0020	0.0018

7.1.4.3 Robust Performance

	$A-BK$	$(A-BK)+\Delta A$	Relative Change
Diagonal Form	-13.3333 +14.8897i -13.3333 -14.8897i -53.0000 -54.0000	-12.3242 +15.6225i -12.3242 -15.6225i -54.7785 -54.0163	0.0624 0.0624 0.0336 $3.0275*10^{-4}$
Controllable Form	-54.0000 -53.0000 -13.3333 +14.8897i -13.3333 -14.8897i	-59.1879 -47.8118 -13.2218 +15.6415i -13.2218 -15.6415i	0.0961 0.0979 0.0380 0.0380
Observable Form	-54.0000 -53.0000 -13.3333 +14.8897i -13.3333 -14.8897i	-53.3944 +14.2210i -53.3944 -14.2210i -13.3272 +14.7359i -13.3272 -14.7359i	0.2636 0.2684 0.0077 0.0077

Finally we can make the comparison between different forms as follows:

In this case study and following the tables given before we can say that the block pole in controller form yields smaller percent overshoot and smaller settling time. The smallest relative change and smallest norm of the feedback gain matrix are given by the block pole in diagonal form. The block form giving the likelihood margin for the dominant eigenvalue and for every eigenvalues of the closed-loop matrix to become unstable is the diagonal form.

Case Study 2

Consider the following 2-input, 5-output system of order 5 given by its matrices

$$A=\begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.3060 & -2.1320 & 0.9807 & 0 & 0 \\ 0 & 1.5950 & -3.1490 & 1.5470 & 0 \\ 0 & 0.0355 & 2.6320 & -4.2570 & 1.8550 \\ 0 & 0.0023 & 0 & 0.1636 & -0.1625 \end{bmatrix}, B=\begin{bmatrix} 0 & 0 \\ 0.0638 & 0 \\ 0.0838 & -0.1496 \\ 0.1004 & -0.2060 \\ 0.0063 & -0.0128 \end{bmatrix}$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence $n = 5 = 2\times 2+1 = lm+k$ *i.e.*, $l = 2$ and $k = 1$

It follows that we can assign two block poles of dimension 2×2 and one remaining pole. So we can transform a given system into the block- decoupled form; we need to compute arbitrary eigenvalues of matrix A with their corresponding left and right eigenvectors. The eigenvalues of A are: -5.9822,-2.8408,-0.8953,-0.0143, -0.0773. This leads to $\lambda' = -5.9822$ with the corresponding right eigenvector V' and left eigenvector T' given by

$$V'=\begin{bmatrix} -0.0015 \\ 0.1362 \\ -0.5326 \\ 0.8350 \\ -0.0235 \end{bmatrix} \text{ and } T'=\begin{bmatrix} -0.0623 & 0.2801 & -0.6874 & 0.6357 & -0.2026 \end{bmatrix}$$

We form the matrix $\tilde{\Phi}$ as follows

$$\tilde{\Phi} = \begin{bmatrix} B & AB & V' \end{bmatrix}$$

Since $\tilde{\Phi}$ is nonsingular, the given system can be transformed into the following block-decoupled form

$$A_c = T_c A T_c^{-1} = \left[\begin{array}{cccc:c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0.0523 & -0.3917 & -0.6849 & -2.6039 & 0 \\ 0.0472 & -0.2999 & 0.1869 & -3.1428 & 0 \\ \hdashline 0 & 0 & 0 & 0 & -5.9822 \end{array}\right] \text{ and } P = -5.9822$$

$$B_c = T_c B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \hdashline 0.0228 & -0.0255 \end{array}\right]$$

and

$$C_c = \begin{bmatrix} 0.0038 & 0.0002 & 0.0000 & 0.0000 & -0.0015 \\ 0.0067 & 0.0003 & 0.0605 & 0.0037 & 0.1449 \\ 0.0128 & -0.0249 & 0.0967 & -0.1641 & -0.5666 \\ 0.0175 & -0.0436 & 0.0801 & -0.1833 & 0.8884 \\ 0.0193 & -0.0521 & 0.0069 & -0.0134 & -0.0250 \end{bmatrix}$$

Hence the wanted structure is given which is as follows:

$$A_c = \begin{bmatrix} A_{c1} & 0_{lm,k} \\ 0_{k,lm} & P \end{bmatrix} \text{ and } B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix}$$

Let construct the desired block poles with a following desired eigenvalues:

$-0.2, -0.5\ , -1 \pm i\ , -1$

7.2.1 State Feedback using Block Poles in the Diagonal Form

The desired block poles constructed in diagonal form

$$R_1 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.5 \end{bmatrix} \qquad R_2 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

The corresponding 2×2 desired right denominator matrix polynomial of degree 2 is

$$D_f(s) = Is^2 + D_{f1}s + D_{f2}$$

where

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = -\begin{bmatrix} R_1^2 & R_2^2 \end{bmatrix} V_R^{-1}$$

this gives

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = \begin{bmatrix} 0.2429 & -0.5857 & 1.4143 & -1.1714 \\ 0.1786 & 0.3929 & 0.8929 & 1.2857 \end{bmatrix}$$

The remaining closed-loop pole is to be assigned at -1.

Now we compute 2×4 state feedback gain matrix K_{c1} that places the block poles of $(A_{c1} - B_{c1}K_{c1})$ at D_{f1} and D_{f2}.

$$K_{c1} = \begin{bmatrix} 0.2952 & -0.9774 & 0.7294 & -3.7753 \\ 0.2258 & 0.0930 & 1.0797 & -1.8571 \end{bmatrix}$$

Then we compute the 1×4 matrix L by solving the Lyapunov equation

$$L(A_{c1} - B_{c1}K_{c1}) - PL = B_{c2}K_{c1}$$

This yields

$$L = \begin{bmatrix} -0.0002 & -0.0042 & -0.0036 & -0.0065 \end{bmatrix}$$

Next we compute a 2×1 state feedback gain matrix K_{c2} that places the eigenvalue of $P - (B_{c2} + LB_{c1})K_{c2}$ at the desired closed-loop pole -1.

$$K_{c2} = \begin{bmatrix} 0 \\ 155.8956 \end{bmatrix}$$

Using $K_c = \begin{bmatrix} K_{c1} + K_{c2}L & K_{c2} \end{bmatrix}$

Its yields

$$K_c = \begin{bmatrix} 0.2952 & -0.9774 & 0.7294 & -3.7753 & 0 \\ 0.1982 & -0.5606 & 0.5168 & -2.8627 & 155.8956 \end{bmatrix}$$

Using $K = K_c T_c$, where T_c is the similarity transformation, the required state feedback gain matrix in the original coordinate system is given by

$$K = \begin{bmatrix} 3.0044 & -19.2565 & 10.8008 & 10.1741 & 4.9741 \\ 4.0307 & 28.4686 & -98.8280 & 106.9219 & -31.4029 \end{bmatrix}$$

The norm of the state feedback gain matrix is given by $\|K\|_2 = 151.7639$

The closed loop matrix using solvents in diagonal form will be as follows:

$$(A-BK)_{diagonal} = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ 1.1143 & -0.9034 & 0.2916 & -0.6491 & -0.3173 \\ 0.3512 & 7.4676 & -18.8388 & 16.6899 & -5.1147 \\ 0.5287 & 7.8334 & -18.8110 & 16.7474 & -5.1134 \\ 0.0327 & 0.4880 & -1.3330 & 1.4681 & -0.5958 \end{bmatrix}$$

The following table summarizes the time response for this choice:

Inputs	Transient steady state specifications	M_p	POS	T_s	T_r	SSV
U_1	y_1	-0.0269	- 0.7435%	21.9s	5.96s	- 0.0269
	y_2	- 0.0736	56.9%	19.5s	1.77s	- 0.0469
	y_3	0.202	677%	14.7s	0.0937s	0.0261
	y_4	0.257	230%	10.4s	0.244s	0.078
	y_5	0.151	44.3%	23s	1.3s	0.104
U_2	y_1	0.0685	0.1460%	21.4s	5.3s	0.0685
	y_2	0.181	51.3%	20.4s	1.13s	0.119
	y_3	- 0.223	- 621%	16.9s	4.36s	0.0428
	y_4	- 0.313	3.1×10^3%	10.6s	0.0188s	- 0.00981
	y_5	- 0.172	332%	25.8s	0.541s	- 0.0398

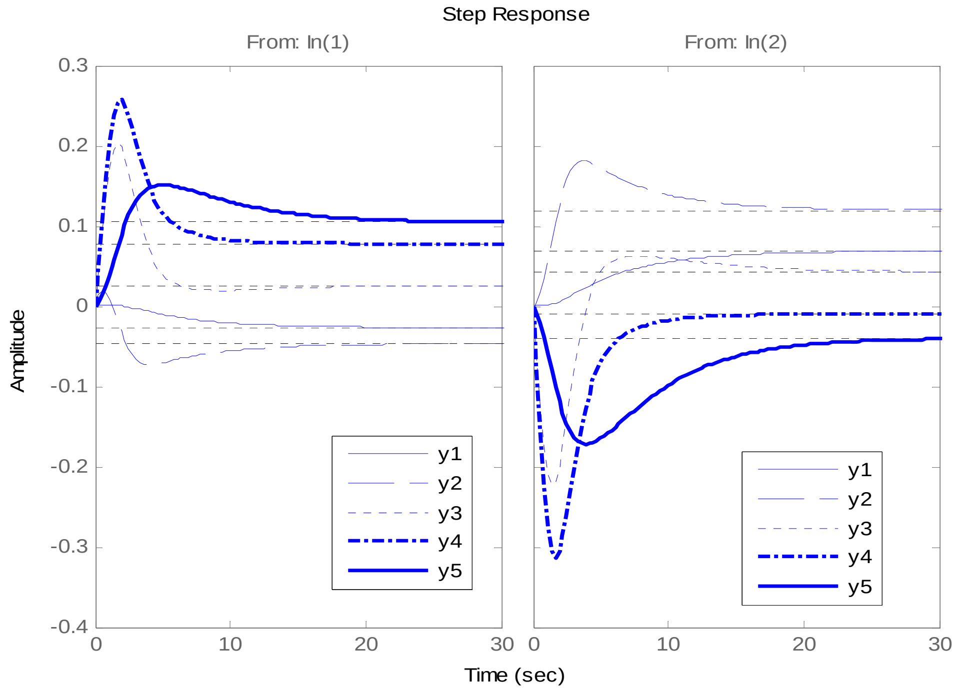


Figure 7: Time response for diagonal form

Response to Initial Condition x_0 =[1;1;1;1;1] in Diagonal Form

Transient steady state specifications	M_p	*POS*	T_s	*SSV*
y_1	1	0%	22.2s	0.00345
y_2	1.831	83.10%	20.7s	-0.00476
y_3	0.0235	13.2340%	16.7s	-0.00554
y_4	31.2	0%	15.6s	0.000848
y_5	2.06	0%	23.9s	0.0158

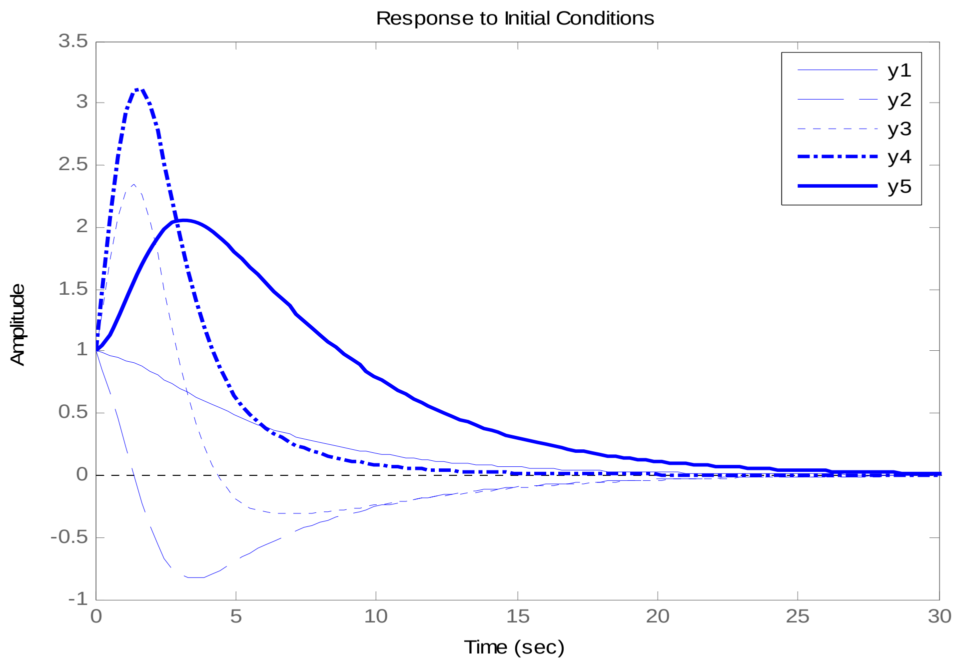


Figure 8: Response to initial condition x_0= [1;1;1;1;1]

7.2.1.a Robust Stability

For the study of the stability robustness of the system, let us compute the right and the left eigenvector of the closed-loop matrix

$$V=\begin{bmatrix} 0.4127-0.0126i & 0.4127+0.0126i & 0.4333 & 0.4303 & -0.4251 & 0.4284 \\ -0.6434 & -0.6434 & -0.6553 & -0.6534 & 0.6459 & -0.6493 \\ 0.4313+0.0350i & 0.4313-0.0350i & 0.3920 & 0.3984 & -0.4133 & 0.4050 \\ 0.1846+0.0056i & 0.1846-0.0056i & 0.1797 & 0.1806 & -0.1876 & 0.1848 \\ -0.2729+0.0008i & -0.2729-0.0008i & -0.2769 & -0.2764 & 0.2723 & -0.2744 \\ 0.3459+0.0025i & 0.3459-0.0025i & 0.3466 & 0.3467 & -0.3491 & 0.3483 \end{bmatrix}$$

its norm is $\|V\|_2 = 1.8692$

the norms of $v_i, i=1,2,3,4,5$ are equal to 1

The norm of the left eigenvector is $\|T\|_2 = 44.7178$

The norm of $t_i, i=1,2,3,4,5$

$\|t_1\|_2 = 26.5911$, $\|t_2\|_2 = 26.5911$, $\|t_3\|_2 = 7.3349$, $\|t_4\|_2 = 20.4448$, $\|t_5\|_2 = 17.2025$.

The sensitivity of all the eigenvalues is

$s(\Lambda) = \|V\|_2 \|T\|_2 = 83.5864$ its inverse is given by $s(\Lambda)^{-1} = 0.0120$

The sensitivity of every eigenvalue is computed as follows:

$s(\lambda_i) = \|v_i\|_2 \|t_i\|_2, i=1,2,3,4,5$

yields

$$s(\lambda_1 = -1+i) = \|v_1\|_2 \|t_1\|_2 = 26.5911$$
$$s(\lambda_2 = -1-i) = \|v_2\|_2 \|t_2\|_2 = 26.5911$$
$$s(\lambda_3 = -1) = \|v_3\|_2 \|t_3\|_2 = 7.3349$$
$$s(\lambda_4 = -0.5) = \|v_4\|_2 \|t_4\|_2 = 20.4448$$
$$s(\lambda_5 = -0.2) = \|v_5\|_2 \|t_5\|_2 = 17.2025$$

Now we can compute the stability robustness measures

Computing $M_1 = \min_{0\le\omega\le\infty} \{\underline{\sigma}(A - j\omega I)\}$ we have $M_1 = 0.0611$

Computing $M_2 = s(\Lambda)^{-1} |\mathrm{Re}\{\lambda_n\}|$, $(|\mathrm{Re}\{\lambda_n\}| \le \dots \le |\mathrm{Re}\{\lambda_1\}|)$ we have $M_2 = 0.0024$

Finally for $M_3 = \min_{1\le i\le n} \{ s(\lambda_i)^{-1} \ |\mathrm{Re}\{\lambda_i\}| \}$ we have

$$s(\lambda_1 = -1 + i)^{-1} \times |-1 + i| = 0.0376$$

$$s(\lambda_2 = -1 - i)^{-1} \times |-1 + i| = 0.0376$$

$$s(\lambda_3 = -1)^{-1} \times |-1| = 0.0581$$

$$s(\lambda_4 = -0.5)^{-1} \times |-0.5| = 0.0245$$

$$s(\lambda_5 = -0.2)^{-1} \times |-0.2| = 0.0273$$

hence $M_3 = 0.0245$

7.2.1.b Robust Performance

The following perturbation is generated randomly using MATLAB is:

$$\Delta A = \begin{bmatrix} 0.0665 & 0.0674 & 0.0549 & 0.0701 & 0.0634 \\ 0.0365 & 0.0999 & 0.0262 & 0.0962 & 0.0803 \\ 0.0140 & 0.0962 & 0.0597 & 0.0751 & 0.0084 \\ 0.0567 & 0.0059 & 0.0049 & 0.0740 & 0.0945 \\ 0.0823 & 0.0360 & 0.0571 & 0.0432 & 0.0916 \end{bmatrix}$$

with $\|\Delta A\| = 0.3004$

The new closed-loop matrix after perturbation is:

$$(A - BK + \Delta A)diagonal = \begin{bmatrix} -0.0429 & 0.1302 & 0.0549 & 0.0701 & 0.0634 \\ 1.3425 & -2.0321 & 1.0069 & 0.0962 & 0.0803 \\ 0.0140 & 1.6912 & -3.0893 & 1.6221 & 0.0084 \\ 0.0567 & 0.0414 & 2.6369 & -4.1830 & 1.9495 \\ 0.0823 & 0.0383 & 0.0571 & 0.2068 & -0.0709 \end{bmatrix}$$

its eigenvalues are: -1.4375 + 1.5326i, -1.4375 - 1.5326i, 0.3693,-0.0529, -0.7496

Computing the relative change of the eigenvalues of the closed-loop matrix due to the perturbation is $r_i = \left| \frac{\lambda_i - \lambda'_i}{\lambda_i} \right|$ where λ_i is the eigenvalue of the closed-loop matrix and λ'_i the eigenvalue of the perturbed closed-loop matrix.

This leads: $r_1 = 0.4874$, $r_2 = 0.4874$, $r_3 = \ \ 2.8465$, $r_4 = 0.8942$, $r_5 = 0.2504$.

7.2.2 State Feedback using Block Poles in the Controllable Form

The desired block poles constructed in controller form

$$R_1 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.7 \end{bmatrix} \qquad R_2 = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$$

The corresponding 2×2 desired right denominator matrix polynomial of degree 2 is

$$D_f(s) = Is^2 + D_{f1}s + D_{f2}$$

where

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = -\begin{bmatrix} R_1^2 & R_2^2 \end{bmatrix} V_R^{-1}$$

this gives

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = \begin{bmatrix} 0.2632 & -0.1053 & 1.9474 & 1.6316 \\ 0.0053 & 0.7579 & -0.6211 & 0.7526 \end{bmatrix}$$

The remaining closed-loop pole is to be assigned at -1.

The computation of 2×4 state feedback gain matrix K_{c1} that places the block poles of $(A_{c1} - B_{c1}K_{c1})$ at D_{f1} and D_{f2}.

$$K_{c1} = \begin{bmatrix} 0.3155 & -0.4970 & 1.2624 & -0.9723 \\ 0.0525 & 0.4580 & -0.4342 & -2.3902 \end{bmatrix}$$

Computing the 1×4 matrix L by solving the Lyapunov equation

$$L(A_{c1} - B_{c1}K_{c1}) - PL = B_{c2}K_{c1}$$

This yields

$$L = \begin{bmatrix} 0.0013 & -0.0027 & 0.0079 & 0.0104 \end{bmatrix}$$

A 2×1 state feedback gain matrix K_{c2} to place the eigenvalue of $P - (B_{c2} + LB_{c1})K_{c2}$ at the desired closed-loop pole -1.

$$K_{c2} = \begin{bmatrix} -162.0316 \\ 0 \end{bmatrix}$$

Using $K_c = \begin{bmatrix} K_{c1} + K_{c2}L & K_{c2} \end{bmatrix}$

This yields

$$K_c = \begin{bmatrix} 0.0988 & -0.0646 & -0.0250 & -2.6588 & -162.0316 \\ 0.0525 & 0.4580 & -0.4342 & -2.3902 & 0 \end{bmatrix}$$

The required state feedback gain matrix in the original coordinate system is given by

$$K = \begin{bmatrix} 58.0567 & -67.3316 & 118.7618 & -94.8833 & 23.8164 \\ 93.9378 & -26.1520 & 6.4780 & 8.0414 & -18.3953 \end{bmatrix}$$

The norm of the feedback gain matrix is $\|K\|_2 = 183.1118$

The closed loop matrix using solvents in controller form will be as follows:

$$(A-BK)_{controllable} = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ -2.3980 & 2.1638 & -6.5963 & 6.0536 & -1.5195 \\ 9.1879 & 3.3250 & -12.1321 & 10.7012 & -4.7478 \\ 13.5223 & 1.4083 & -7.9572 & 6.9258 & -4.3256 \\ 0.8366 & 0.0917 & -0.6653 & 0.8643 & -0.5480 \end{bmatrix}$$

The time response for this choice is summarized in the following table:

Inputs	Transient steady state specifications	M_p	POS	T_s	T_r	SSV
U_1	y_1	0.0868	0.1153%	23.2s	6.55s	0.0867
	y_2	0.19	25.8278%	22.2s	0.876s	0.151
	y_3	0.329	11.9048%	20.6s	3.59s	0.294
	y_4	0.403	0.2488%	14s	4.66s	0.402
	y_5	0.448	0.2237%	25s	8.46s	0.447
U_2	y_1	-0.0331	- 0.3012%	22.5s	5.7s	- 0.0332
	y_2	- 0.0729	26.1246%	21.5s	0.674s	- 0.0578
	y_3	- 0.176	20.5479%	18.5s	0.409s	- 0.146
	y_4	-0.212	- 0.4695%	12.2s	0.513s	- 0.213
	y_5	-0.239	- 0.8299%	22.5s	5.34s	- 0.241

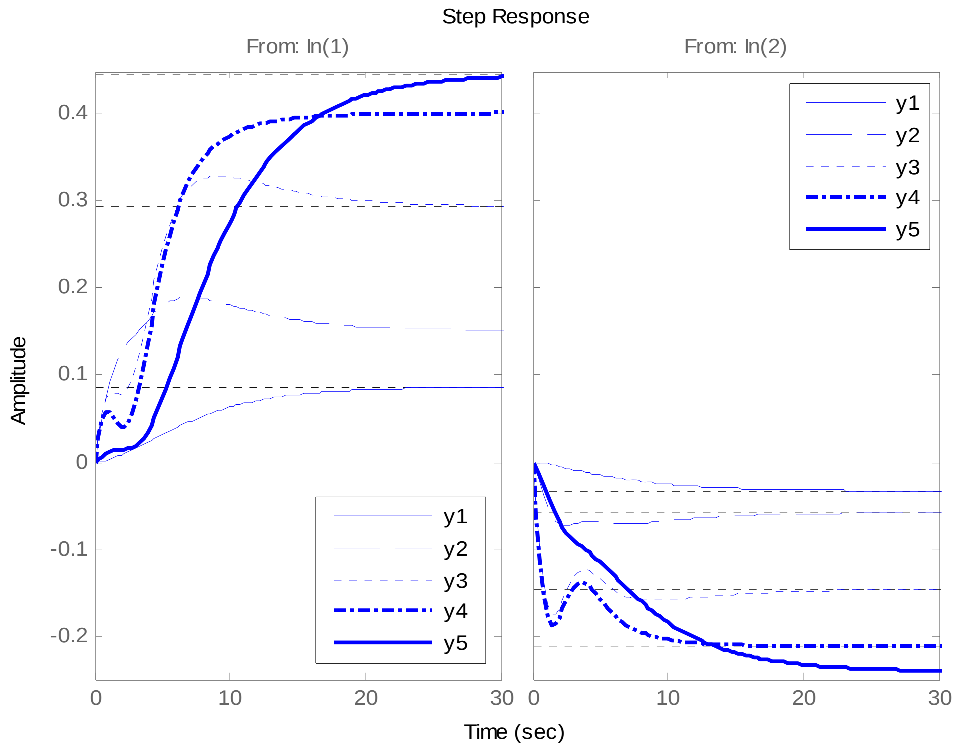


Figure 9: Time response for controller form

Response to Initial Condition in Controller Form

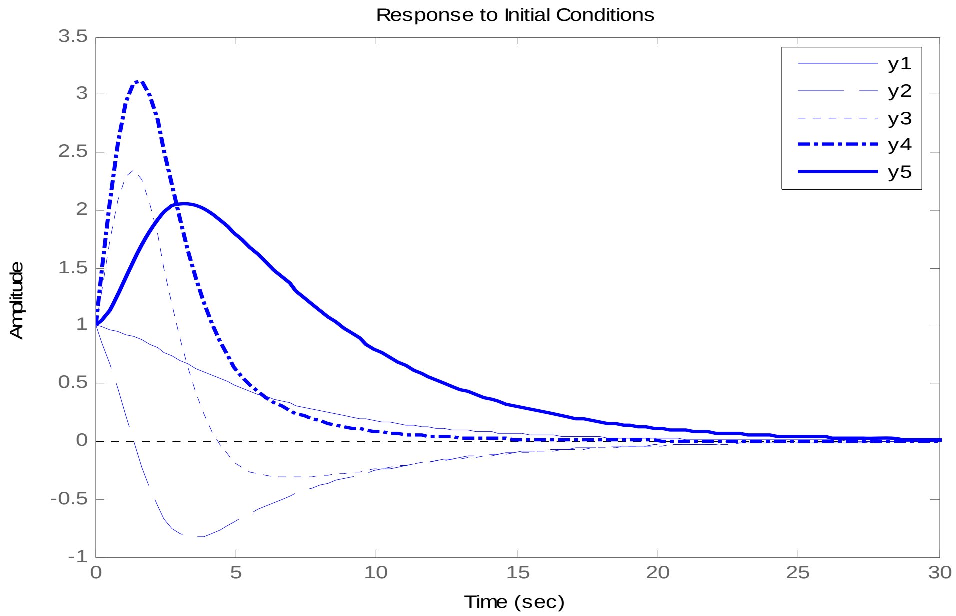


Figure10: Response to initial condition x_0= [1;1;1;1;1]

Transient steady state specifications	M_p	POS	T_s	SSV
y_1	1.	0%	25.1s	0.00596
y_2	1.54	35.1299%	19.4s	-0.00884
y_3	9.11	5.2909%	17.1s	-0.0125
y_4	10.8	0%	11.4s	0.00362
y_5	5.66	0%	24.5s	0.0446

7.2.2.a Robust Stability

Computation of the right and the left eigenvector of the closed-loop matrix

$$V = \begin{bmatrix} 0.0090+0.0042i & 0.0090-0.0042i & -0.1269 & -0.0154 & -0.0405 \\ -0.1947+0.0824i & -0.1947-0.0824i & 0.1831 & 0.2190 & 0.2519 \\ -0.7026 & -0.7026 & 0.2599 & 0.7266 & 0.6942 \\ -0.6684-0.0457i & -0.6684+0.0457i & -0.0737 & 0.6322 & 0.5233 \\ 0.0217+0.1109i & 0.0217-0.1109i & -0.9367 & -0.1553 & -0.4233 \end{bmatrix}$$

its norm is $\|V\|_2 = 1.9946$

the norms of $v_i, i = 1,2,3,4,5$ are equal to 1

The norm of the left eigenvector is $\|T\|_2 = 122.7915$

The norm of $t_i, i = 1,2,3,4,5$

$\|t_1\|_2 = 37.3812$, $\|t_2\|_2 = 37.3812$, $\|t_3\|_2 = 19.1699$, $\|t_4\|_2 = 104.6121$, $\|t_5\|_2 = 61.647$.

The sensitivity of all the eigenvalues is

$$s(\Lambda) = \|V\|_2 \|T\|_2 = 244.9139$$

its inverse is given by $s(\Lambda)^{-1} = 0.0041$

The sensitivity of every eigenvalue is computed as follows:

$$s(\lambda_i) = \|v_i\|_2 \|t_i\|_2, i = 1,2,3,4,5$$

yields

$$s(\lambda_1 = -1+i) = \|v_1\|_2 \|t_1\|_2 = 37.3812$$
$$s(\lambda_2 = -1-i) = \|v_2\|_2 \|t_2\|_2 = 37.3812$$
$$s(\lambda_3 = -1) = \|v_3\|_2 \|t_3\|_2 = 19.1699$$
$$s(\lambda_4 = -0.5) = \|v_4\|_2 \|t_4\|_2 = 104.6121$$
$$s(\lambda_5 = -0.2) = \|v_5\|_2 \|t_5\|_2 = 61.6470$$

Now we can compute the stability robustness measures

$$M_1 = 0.0390$$

$$M_2 = 8.1661 \times 10^{-4}$$

We have

$$s(\lambda_1 = -1+i)^{-1} \times |-1+i| = 0.0268$$
$$s(\lambda_2 = -1-i)^{-1} \times |-1+i| = 0.0268$$
$$s(\lambda_3 = -1)^{-1} \times |-1| = 0.0096$$
$$s(\lambda_4 = -0.5)^{-1} \times |-0.5| = 0.0081$$
$$s(\lambda_5 = -0.2)^{-1} \times |-0.2| = 0.0104$$

hence $M_3 = 0.0081$

7.2.2.b Robust Performance

The new closed-loop matrix after perturbation is:

$$(A - BK + \Delta A)_{controllable} = \begin{bmatrix} -0.0429 & 0.1302 & 0.0549 & 0.0701 & 0.0634 \\ -2.3615 & 2.2637 & -6.5701 & 6.1498 & -1.4392 \\ 9.2019 & 3.4212 & -12.0724 & 10.7763 & -4.7394 \\ 13.5790 & 1.4142 & -7.9523 & 6.9998 & -4.2311 \\ 0.9189 & 0.1277 & -0.6082 & 0.9075 & -0.4564 \end{bmatrix}$$

its eigenvalues are: -1.6749 + 1.6971i, -1.6749 - 1.6971i, 1.0336, -0.4961 + 0.2808i
-0.4961 - 0.2808i

The relative change of the eigenvalues the closed-loop matrix due to the perturbation is given by:

$$r_1 = 0.6861, r_2 = 0.6861, r_3 = 6.1679, \; r_4 = 0.5769, r_5 = 0.5616.$$

7.2.3 State Feedback using Block Poles in the Observable Form

The desired block poles constructed in observer form

$$R_1 = \begin{bmatrix} 0 & -0.1 \\ 1 & -0.7 \end{bmatrix} \qquad R_2 = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$$

This yields

$$\begin{bmatrix} D_{f2} & D_{f1} \end{bmatrix} = \begin{bmatrix} 1.0632 & -0.4947 & 2.4947 & -0.9632 \\ 0.4947 & -0.0421 & 2.0421 & 0.2053 \end{bmatrix}$$

The remaining closed-loop pole is to be assigned at -1.

The computation of 2×4 state feedback gain matrix K_{c1} that places the block poles of $(A_{c1} - B_{c1}K_{c1})$ at D_{f1} and D_{f2}.

$$K_{c1} = \begin{bmatrix} 1.1155 & -0.8865 & 1.8098 & -3.5671 \\ 0.5420 & -0.3420 & 2.2290 & -2.9375 \end{bmatrix}$$

Computing 1×4 matrix L by solving the Lyapunov equation

$$L(A_{c1} - B_{c1}K_{c1}) - PL = B_{c2}K_{c1}$$

This yields

$$L = \begin{bmatrix} 0.0011 & -0.0015 & -0.0048 & -0.0000 \end{bmatrix}$$

A 2×1 state feedback gain matrix K_{c2} to place the eigenvalue of $P - (B_{c2} + LB_{c1})K_{c2}$ at the desired closed-loop pole -1.

$$K_{c2} = \begin{bmatrix} 0 \\ 194.9810 \end{bmatrix}$$

Using $K_c = \begin{bmatrix} K_{c1} + K_{c2}L & K_{c2} \end{bmatrix}$

This yields

$$K_c = \begin{bmatrix} 1.1155 & -0.8865 & 1.8098 & -3.5671 & 0 \\ 0.7532 & -0.6389 & 1.2918 & -2.9461 & 194.9810 \end{bmatrix}$$

The required state feedback gain matrix in the original coordinate system is given by

$$K = \begin{bmatrix} 189.4934 & -0.9360 & 12.0040 & 8.2826 & 5.0825 \\ 115.7060 & 50.6889 & -124.3284 & 131.0557 & -37.3606 \end{bmatrix}$$

The norm of the feedback gain matrix is $\|K\|_2 = 255.8213$

The closed loop matrix using solvents in observer form will be as follows:

$$(A-BK)_{observable} = \begin{bmatrix} -0.1094 & 0.0628 & 0 & 0 & 0 \\ -10.7837 & -2.0723 & 0.2148 & -0.5284 & -0.3243 \\ 1.4301 & 9.2565 & -22.7545 & 20.4589 & -6.0151 \\ 4.8103 & 10.5714 & -24.1848 & 21.9089 & -6.3516 \\ 0.2872 & 0.6570 & -1.6670 & 1.7889 & -0.6728 \end{bmatrix}$$

The time response for the observer choice is summarized in the following table as follows:

Inputs	Transient steady state specifications	M_p	POS	T_s	T_r	SSV
U_1	y_1	-0.0381	- 0.5222%	21.7s	5.61s	- 0.0383
	y_2	- 0.103	54.1916%	20s	1.47s	- 0.0668
	y_3	0.491	18.6%	13.5s	0.856s	0.414
	y_4	0.764	0.1311%	11.8s	1.23s	0.763
	y_5	0.927	0.1075%	20.5s	4.43s	0.93
U_2	y_1	0.0632	0.1585%	21.6s	5.41s	0.0631
	y_2	0.163	48.6%	20.6s	1.16s	0.11
	y_3	- 0.676	11.7355%	13.8s	0.942s	- 0.605
	y_4	-1.11	- 0.8929%	12.3s	1.35s	- 1.12
	y_5	-1.36	0.7299%	20.9s	4.69s	- 1.37

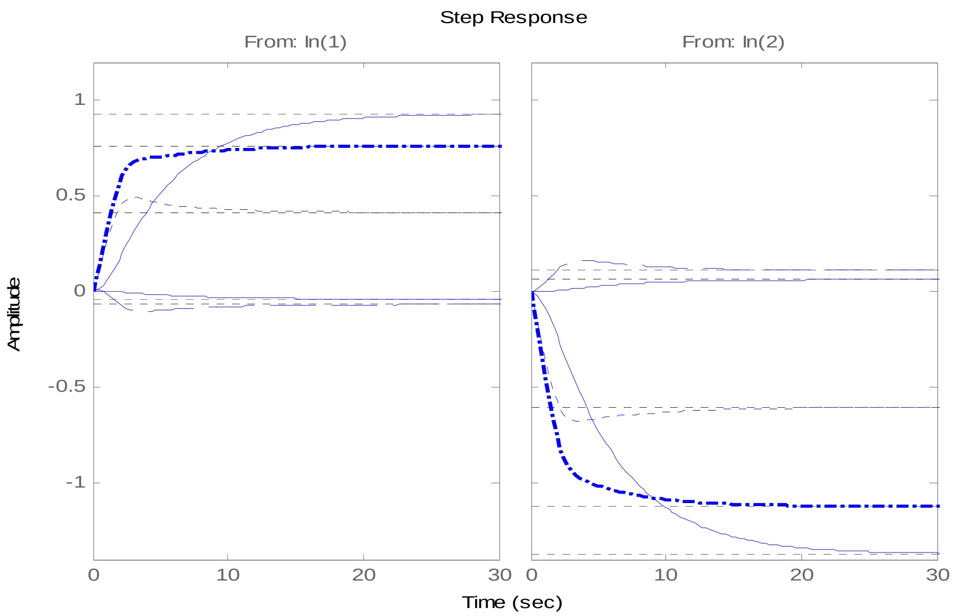


Figure11: Time response for observer form

Response to Initial Condition in Observer Form

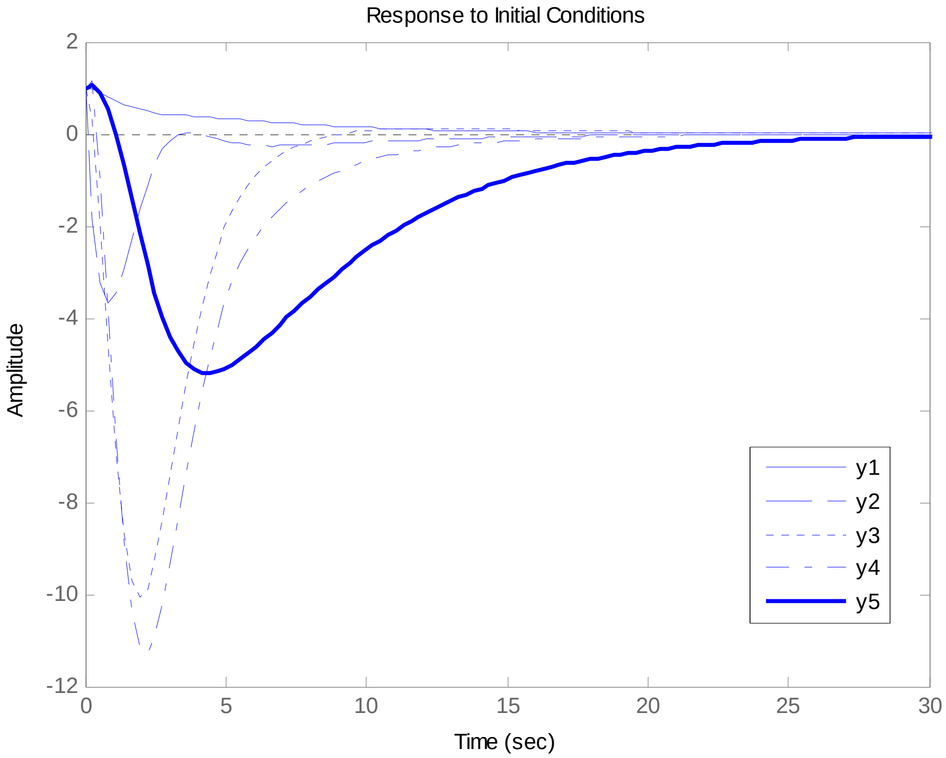


Figure12: Response to initial condition x_0= [1;1;1;1;1]

Transient steady state specifications	M_p	*POS*	T_s	*SSV*
y_1	1	0%	21.1s	0.00275
y_2	4.66	366%	11.1s	-0.0039
y_3	11.1	1010%	8.42s	0.00545
y_4	12.3	1130%	11.2s	-0.00743
y_5	6.2	520%	20.9s	-0.0524

7.2.3.a Robust Stability

Computation of the right and the left eigenvector of the closed-loop matrix

$$V = \begin{bmatrix} 0.0112 - 0.0012i & 0.0112 + 0.0012i & 0.0514 & -0.0125 & -0.0399 \\ -0.1400 + 0.1957i & -0.1400 - 0.1957i & -0.0741 & 0.0775 & 0.5664 \\ 0.6349 + 0.0867i & 0.6349 - 0.0867i & 0.1065 & -0.6278 & -0.4416 \\ 0.7173 & 0.7173 & -0.1408 & -0.5599 & -0.6553 \\ -0.0549 - 0.1181i & -0.0549 + 0.1181i & -0.9802 & 0.5350 & 0.2305 \end{bmatrix}$$

With $\|V\|_2 = 1.8964$

The norms of $v_i, i = 1,2,3,4,5$ are equal to 1

The norm of the left eigenvector is $\|T\|_2 = 63.5080$

The norm of $t_i, i = 1,2,3,4,5$

$\|t_1\|_2 = 35.6008$, $\|t_2\|_2 = 35.6008$, $\|t_3\|_2 = 23.4548$, $\|t_4\|_2 = 49.5561$, $\|t_5\|_2 = 23.1346$.

The sensitivity of all the eigenvalues is

$s(\Lambda) = \|V\|_2 \|T\|_2 = 120.4346$ its inverse is given by $s(\Lambda)^{-1} = 0.0083$

The sensitivity of every eigenvalue is computed as follows:

$$s(\lambda_i) = \|v_i\|_2 \|t_i\|_2, i = 1,2,3,4,5$$

yields

$$s(\lambda_1 = -1 + i) = \|v_1\|_2 \|t_1\|_2 = 35.6008$$
$$s(\lambda_2 = -1 - i) = \|v_2\|_2 \|t_2\|_2 = 35.6008$$
$$s(\lambda_3 = -1) = \|v_3\|_2 \|t_3\|_2 = 23.4548$$
$$s(\lambda_4 = -0.5) = \|v_4\|_2 \|t_4\|_2 = 49.5561$$
$$s(\lambda_5 = -0.2) = \|v_5\|_2 \|t_5\|_2 = 23.1346$$

Now we can compute the stability robustness measures

$$M_1 = 0.0291$$
$$M_2 = 0.0017$$

We have

$$s(\lambda_1 = -1 + i)^{-1} \times |-1 + i| = 0.0281$$
$$s(\lambda_2 = -1 - i)^{-1} \times |-1 + i| = 0.0281$$
$$s(\lambda_3 = -1)^{-1} \times |-1| = 0.0432$$
$$s(\lambda_4 = -0.5)^{-1} \times |-0.5| = 0.0101$$
$$s(\lambda_5 = -0.2)^{-1} \times |-0.2| = 0.0085$$

hence $M_3 = 0.0085$.

7.2.3.b Robust Performance

The new closed-loop matrix after perturbation is:

$$(A - BK + \Delta A)_{observable} = \begin{bmatrix} -0.0429 & 0.1302 & 0.0549 & 0.0701 & 0.0634 \\ -10.7472 & -1.9724 & 0.2410 & -0.4322 & -0.2440 \\ 1.4441 & 9.3527 & -22.6948 & 20.5340 & -6.0067 \\ 4.8670 & 10.5773 & -24.1799 & 21.9829 & -6.2571 \\ 0.3695 & 0.6930 & -1.6099 & 1.8321 & -0.5812 \end{bmatrix}$$

with eigenvalues: -0.2914 + 1.7174i, -0.2914 - 1.7174i, -2.0117 , -0.8401, 0.1262.

The relative change of the eigenvalues the closed-loop matrix due to the perturbation is given by.

$$r_1 = 0.7130, r_2 = 0.7130, r_3 = 9.0584, r_4 = 0.6803, r_5 = 1.1262.$$

7.2.4 Comparison of Results

7.2.4.1 Time Response:

U1		Diagonal Form	Controllable Form	Observable Form
	y1_ MP	-0.0269	0.0868	-0.0381
	y1_ POS	- 0.7435%	- 0.1153%	- 0.5222%
	y1_Ts	21.9s	23.2s	21.7s
	y1_Tr	5.96s	6.55s	5.61s
	y1-SSV	- 0.0269	0.0867	- 0.0383
	y2_ MP	- 0.0736	0.19	- 0.103
	y2_ POS	56.9%	25.8278%	54.1916%
	y2_Ts	19.5s	22.2s	20s
	y2_Tr	1.77s	0.876s	1.47s
	y2-SSV	- 0.0469	0.151	- 0.0668
	y3_MP	0.202	0.329	0.491
	y3_POS	677%	11.9048%	18.6%
	y3_Ts	14.7s	20.6s	13.5s
	y3_Tr	0.0937s	3.59s	0.856s
	y3_SSV	0.0261	0.294	0.414
	y4_MP	0.257	0.403	0.764
	y4_POS	230%	0.2488%	0.1311%
	y4_Ts	10.4s	14s	11.8s
	y4_Tr	0.244s	4.66s	1.23s
	y4_SSV	0.078	0.402	0.763
	y5_MP	0.151	0.448	0.927
	y5_POS	44.3%	0.2237%	0.1075%

	y5_Ts	23s	25s	20.5s
	y5_Tr	1.3s	8.46s	4.43s
	y5_SSV	0.104	0.447	0.93
U2	y1_ MP	0.0685	-0.0331	0.0632
	y1_ POS	0.1460%	- 0.3012%	0.1585%
	y1_Ts	21.4s	22.5s	21.6s
	y1_Tr	5.3s	5.7s	5.41s
	y1-SSV	0.0685	- 0.0332	0.0631
	y2_ MP	0.181	- 0.0729	0.163
	y2_ POS	51.3%	26.1246%	48.6%
	y2_Ts	20.4s	21.5s	20.6s
	y2_Tr	1.13s	0.674s	1.16s
	y2-SSV	0.119	- 0.0578	0.11
	y3_MP	- 0.223	- 0.176	- 0.676
	y3_POS	- 621%	20.5479%	11.7355%
	y3_Ts	16.9s	18.5s	13.8s
	y3_Tr	4.36s	0.409s	0.942s
	y3_SSV	0.0428	- 0.146	- 0.605
	y4_MP	- 0.313	-0.212	-1.11
	y4_POS	$3.1\text{x}10^{3}$%	- 0.4695%	- 0.8929%
	y4_Ts	10.6s	12.2s	12.3s
	y4_Tr	0.0188s	0.513s	1.35s
	y4_SSV	- 0.00981	- 0.213	- 1.12
	y5_MP	- 0.172	-0.239	-1.36
	y5_POS	332%	- 0.8299%	0.7299%
	y5_Ts	25.8s	22.5s	20.9s
	y5_Tr	0.541s	5.34s	4.69s
	y5_SSV	- 0.0398	- 0.241	- 1.37

7.2.4.2 Robust Stability

		Diagonal Form	Controllable Form	Observable Form
M1	M1	0.0611	0.0390	0.0291
M2	M2	0.0024	$8.1661\text{x}10^{-4}$	0.0017
M3	M13	0.0376	0.0268	0.0281
	M23	0.0376	0.0268	0.0281
	M33	0.0245	0.0096	0.0432
	M34	0.0581	0.0081	0.0101
	M35	0.0273	0.0104	0.0085
	M3	0.0245	0.0081	0.0085

7.2.4.3 Robust Performance

	$A-BK$	$(A-BK)+\Delta A$	Relative Change
Diagonal Form	-1.0000 + 1.0000i	-1.4376 + 1.5329i	0.4874
	-1.0000 - 1.0000i	-1.4376 - 1.5329i	0.4874
	-0.2000	0.3693	2.8465
	-0.5000	-0.0529	0.8943
	-1.0000	-0.7496	0.2504
Controllable Form	-1.0000 + 1.0000i	-1.6749 + 1.6971i	0.6861
	-1.0000 - 1.0000i	-1.6749 - 1.6971i	0.6861
	-0.2000	1.0336	6.1679
	-1.0000	-0.4961 + 0.2808i	0.5769
	-0.5000	-0.4961 - 0.2808i	0.5616
Observable Form	-1.0000 + 1.0000i	-0.2914 + 1.7174i	0.7130
	-1.0000 - 1.0000i	-0.2914 - 1.7174i	0.7130
	-0.2000	-2.0117	9.0584
	-0.5000	-0.8401	0.6803
	-1.0000	0.1262	1.1262

In this example and following the tables given above the form of the block poles in controller form yield smaller percent overshoot. The smallest relative change and smallest norm of the feedback gain matrix are given by the block poles in diagonal form. The block form giving the likelihood margin for the dominant eigenvalue and for every eigenvalues of the closed-loop matrix to become unstable is the diagonal form.

7.3 The Case of Compensator Design using Block Poles Placement

Case Study 3

Consider the unity feedback shown in figure 4.4 in the chapter 4, the plant is described by the following 2-input strictly proper rational matrix

$$H(s)=N(s)D^{-1}(s)=\begin{bmatrix}5s-3 & 0.2s+1\\ -0.3s-0.7 & s-5\end{bmatrix}\begin{bmatrix}s^2+4s+5 & 0\\ 0 & s^2+4s+5\end{bmatrix}^{-1}$$

where $D(s)$ and $N(s)$ are assumed to be right coprime polynomial matrices.

The coefficient matrices are

$$D_0 = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad , \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$N_0 = \begin{bmatrix} -3 & 1 \\ -0.7 & -5 \end{bmatrix} \quad , \quad N_1 = \begin{bmatrix} 5 & 0.2 \\ -0.3 & 1 \end{bmatrix} \quad , \quad N_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This yields

$$D(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} s + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

and

$$N(s) = \begin{bmatrix} 5 & 0.2 \\ -0.3 & 1 \end{bmatrix} s + \begin{bmatrix} -3 & 1 \\ -0.7 & -5 \end{bmatrix}$$

7.3.1 Block Poles Constructed in Diagonal Form

We need to find the minimal degree compensator $C(s) = D_c^{-1}(s)N_c(s)$ that achieves the following closed-loop right block poles in diagonal form

$$R_1 = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}, \quad R_2 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.5 \end{bmatrix}$$

The desired matrix polynomial corresponding to the desired set of right solvents is

$$D_f(s) = D_{f3}s^3 + D_{f2}s^2 + D_{f1}s + D_{f0}$$

where $D_{f3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D_{f2} = \begin{bmatrix} 5.4961 & -0.9162 \\ 1.1872 & 6.2039 \end{bmatrix}$, $D_{f1} = \begin{bmatrix} 6.2436 & -5.0391 \\ 4.9860 & 6.3715 \end{bmatrix}$ and

$$D_{f0} = \begin{bmatrix} 1.0369 & -2.2905 \\ 0.9497 & 1.7598 \end{bmatrix}$$

To obtain the row index v of $H(s)$ the modified recursive algorithm is applied to the Sylvester' matrix to get $v = 2$ which means that 3 is the number of block rows of $\hat{S}_2$ sufficient to solve the compensator equation $D_f(s) = D_C(s)D(s) + N_C(s)N(s)$,given in chapter 4.

Applying the row searching algorithm to $\hat{S}_2$, we obtain the following linearly dependent rows: 9,10,13, therefore, the primary dependent rows are 9,10.

This yields:

$$\begin{bmatrix} 0.6797 & -4.9349 & 0.4901 & -2.7910 & 4.1366 & -0.4288 & 4.1366 & -0.4288 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.6711 & 2.0276 & 0.6416 & 1.4750 & -0.6332 & 1.1105 & -0.6332 & 1.1105 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}\hat{S}_2 = 0$$

where C given by

$$C = \begin{bmatrix} 4.1366 & -0.4288 \\ -0.6332 & 1.1105 \end{bmatrix} \text{ is nonsingular}$$

The computation of the minimal degree compensator yields:

$$D_c(s) = \begin{bmatrix} s+0.2412 & -1.067 \\ 0.7418 & s+1.218 \end{bmatrix} \text{ and } N_c(s) = \begin{bmatrix} 0.2569s+0.1896 & 0.0992s-0.5708 \\ 0.1465s+0.6858 & 0.957s+1.093 \end{bmatrix}$$

Finally the minimal degree 2×2 compensator is given by

$$C(s) = D_c^{-1}(s)N_c(s)$$

The closed-loop system is given by

$$H_{cl}(s) = N(s)D_f^{-1}(s)N_c(s) \quad \text{where } D_f(s) = D_c(s)D(s) + N_c(s)N(s)$$

The closed-loop transfer function is proper since $H(\infty)$ is equal to 0.

Time response of the closed-loop transfer function for this choice is summarized in the following table:

Inputs	Transient steady state specifications	M_p	POS	T_s	T_r	SSV
U_1	y_1	<-1.3	- 7.6923%	21.3s	5.95s	- 1.3
	y_2	<-0.997	- 0.1003%	14.4s	2.32s	- 0.997
U_2	y_1	<-0.574	- 0.1742%	21.9s	7.91s	- 0.574
	y_2	<-2.2	- 4.5455%	10.2s	1.82s	- 2.2

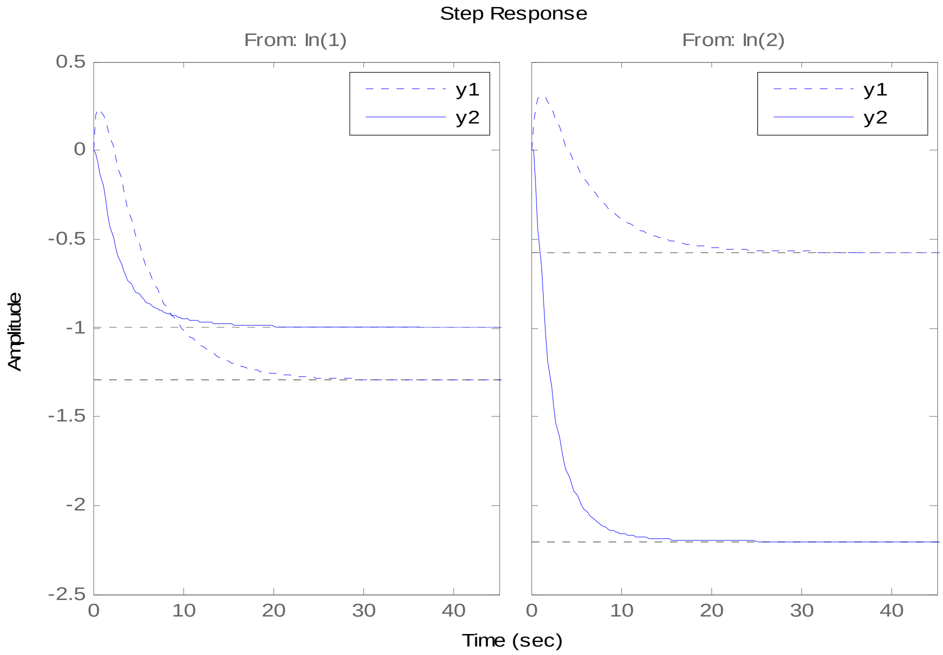


Figure 13: Time response for diagonal form

To assess the robustness of the closed-loop transfer function, we compute first the sensitivity function given by $S = [I + H(s)C(s)]^{-1}$

The smallest and the largest singular values of the closed-loop transfer function are computed as :

- $\sigma_m(H_{cl}(s)) = 0.4435 + 0.0015\text{i}$
- $\sigma_M(H_{cl}(s)) = 0 + 2.2961\text{i}$

The condition number of the closed-loop transfer function is given by

$$K(H_{cl}(s)) = 0.0178 + 5.1774\text{i}$$

The infinity norm of the closed-loop transfer function is computed as: $\|H_{cl}(s)\|_\infty = 2.6695$

and the infinity norm of the sensitivity function is $\|S\|_\infty = 3.6649$

7.3.2 Block Poles Constructed in Controllable Form

We need to find the minimal degree compensator $C(s) = D_c^{-1}(s)N_c(s)$ that achieves the following closed-loop right block poles in controller form

$$R_1 = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}, \quad R_2 = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.7 \end{bmatrix}$$

The desired matrix polynomial corresponding to the desired set of right solvents is

$$D_f(s)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}s^3+\begin{bmatrix}2 & -1\\ 6.8 & 9.7\end{bmatrix}s^2+\begin{bmatrix}2 & -2\\ 8.9 & 19.6\end{bmatrix}s+\begin{bmatrix}0 & -2\\ 2 & 6\end{bmatrix}$$

To obtain the row index v of $H(s)$ the modified recursive algorithm is applied to the Sylvester' matrix, we obtain $v=2$ means that 3 block rows of $\hat{S}_2$ are sufficient to solve the compensator equation $D_f(s)=D_C(s)D(s)+N_C(s)N(s)$.

Applying the row searching algorithm to $\hat{S}_2$, we obtain the following linearly dependent rows: 9, 10, 13, the primary dependent rows are 9, 10.

Then the corresponding coefficient of linear combinations:

$$\begin{bmatrix}0.1810 & -2.1810 & 2.3423 & -3.0668 & -9.3484 & -1.9876 & -9.3484 & -1.9876 & 1 & 0 & 0 & 0 & 0 & 0\\ 1.1456 & 4.9282 & -0.8468 & 4.9727 & 7.7156 & 2.3936 & 7.7156 & 2.3936 & 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}\hat{S}_2=0$$

where C is given by

$$C=\begin{bmatrix}-9.3484 & -1.9876\\ 7.7156 & 2.3936\end{bmatrix}$$

Since C is nonsingular, the solution is given by

$$\begin{bmatrix}0.3849 & -0.6497 & -0.5572 & -0.3612 & 1 & 0 & 1 & 0 & -0.3400 & -0.2823 & 0 & 0 & 0 & 0\\ 1.7193 & 4.1531 & 1.4424 & 3.2416 & 0 & 1 & 0 & 1 & 1.0958 & 1.3277 & 0 & 0 & 0 & 0\end{bmatrix}$$

The computation of the minimal degree compensator yields:

$$D_c(s)=\begin{bmatrix}s-0.3849 & -0.6497\\ 1.719 & s+4.153\end{bmatrix} \text{ and } D_c(s)=\begin{bmatrix}s-0.3849 & -0.6497\\ 1.719 & s+4.153\end{bmatrix}$$

Finally the minimal degree 2×2 compensator is given by

$$C(s)=D_c^{-1}(s)N_c(s)$$

The closed-loop system is given by

$$H_{cl}(s)=N(s)D_f^{-1}(s)N_c(s) \text{ where } D_f(s)=D_c(s)D(s)+N_c(s)N(s)$$

To check the properness of the closed-loop feedback we compute the following matrix

$I+C(\infty)H(\infty)$ must be nonsingular

In our case $H(\infty)=0$, hence $H_{cl}(s)$ is proper transfer matrix.

The following table summarizes the time response of the closed-loop transfer function for this choice:

Inputs	Transient steady state specifications	M_p	POS	T_s	T_r	SSV
U_1	y_1	1.04	67.4%	23.2s	1.86s	0.623
	y_2	- 1.33	1.21%	5.27s	1.4s	- 1.31
U_2	y_1	<-3.06	- 0.3268%	24.3s	8.96s	- 3.06
	y_2	- 1.81	8.91%	16.4s	1.25s	- 1.66

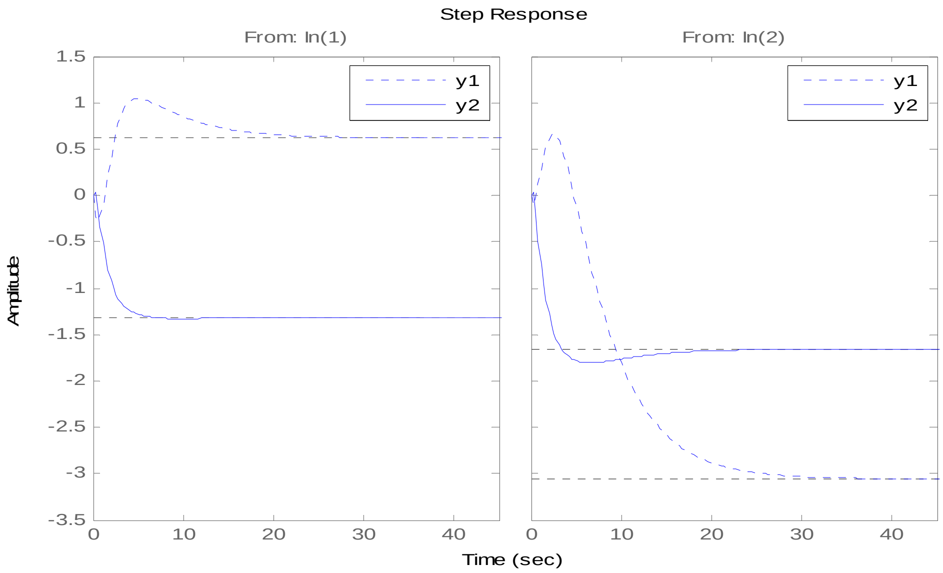


Figure 14: Time response for controller form

To assess the robustness of the closed-loop transfer function, we compute first the sensitivity function given by $S = [I + H(s)C(s)]^{-1}$

: The smallest and the largest singular values of the closed-loop transfer function are given by

- $\underline{\sigma}(H_{cl}(s)) = 0 + 0.4433\text{i}$
- $\overline{\sigma}(H_{cl}(s)) = 2.3112 + 0.0264\text{i}$

The condition number of the closed-loop transfer function is given by

$$K(H_{cl}(s)) = 0.0596 - 5.2139\text{i}$$

The infinity norm of the closed-loop transfer function is computed as: $\|H_{cl}(s)\|_\infty = 3.4785$ and the infinity norm of the sensitivity function is $\|S\|_\infty = 4.2149$

7.3.3 Block Poles Constructed in Observable Form

We need to find the minimal degree compensator $C(s) = D_c^{-1}(s)N_c(s)$ that achieves the following closed-loop right block poles in observer form

$$R_1 = \begin{bmatrix} 0 & -20 \\ 1 & -9 \end{bmatrix}, \quad R_2 = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & -0.1 \\ 1 & -0.7 \end{bmatrix}$$

The desired matrix polynomial corresponding to the desired set of right solvents is

$$D_f(s) = D_{f3}s^3 + D_{f2}s^2 + D_{f1}s + D_{f0}$$

where $D_{f3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D_{f2} = \begin{bmatrix} 8.2331 & 1.9386 \\ 1.5997 & 3.4669 \end{bmatrix}$, $D_{f1} = \begin{bmatrix} 21.3047 & -5.4261 \\ 6.0517 & 1.0534 \end{bmatrix}$ and

$$D_{f0} = \begin{bmatrix} 7.5364 & -2.9612 \\ 1.1433 & 0.0815 \end{bmatrix}$$

To obtain the row index v of $H(s)$ we apply the modified recursive algorithm to the Sylvester' matrix and we get $v = 2$ which means that 3 is the number of block rows of $\hat{S}_2$ sufficient to solve the compensator equation

Applying the row searching algorithm to $\hat{S}_2$, we obtain the following linearly dependent rows: 9,10,13, therefore, the primary dependent rows are 9,10.

This yields

$$\begin{bmatrix} 2.3011 & -4.6289 & 3.9913 & -4.2837 & -0.5959 & 6.1406 & -0.5959 & 6.1406 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1.0524 & 1.2949 & 0.0180 & 1.7738 & 0.7600 & -1.5408 & 0.7600 & -1.5408 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \hat{S}_2 = 0$$

where C given by

$$C = \begin{bmatrix} -0.5959 & 6.1406 \\ 0.7600 & -1.5408 \end{bmatrix}$$ is nonsingular.

The computation of the minimal degree compensator yields:

$$D_c(s) = \begin{bmatrix} s + 2.67 & 0.2185 \\ 0.6338 & s - 0.7326 \end{bmatrix} \text{ and } N_c(s) = \begin{bmatrix} 0.411s + 1.67 & 1.638s + 1.145 \\ 0.2027s + 0.812 & 0.1589s - 0.5865 \end{bmatrix}$$

Finally the minimal degree 2×2 compensator is given by

$$C(s) = D_c^{-1}(s)N_c(s)$$

The closed-loop system is given by

$$H_{cl}(s) = N(s)D_f^{-1}(s)N_c(s) \text{ where } D_f(s) = D_c(s)D(s) + N_c(s)N(s)$$

The closed-loop transfer function is proper since $H(\infty)$ is equal to 0

Time response of the closed-loop transfer function for this choice is summarized in the following table:

Inputs	Transient steady state specifications	M_p	POS	T_s	T_r	SSV
U_1	y_1	<-0.853	- 0.1172%	21.1s	6.6s	- 0.853
	y_2	<-5.71	- 0.1751%	21.6s	5.16s	- 5.71
U_2	y_1	- 1.08	441.0%	21.4s	0.806s	- 0.2
	y_2	>7.45	0.1342%	19.1s	3.63s	7.45

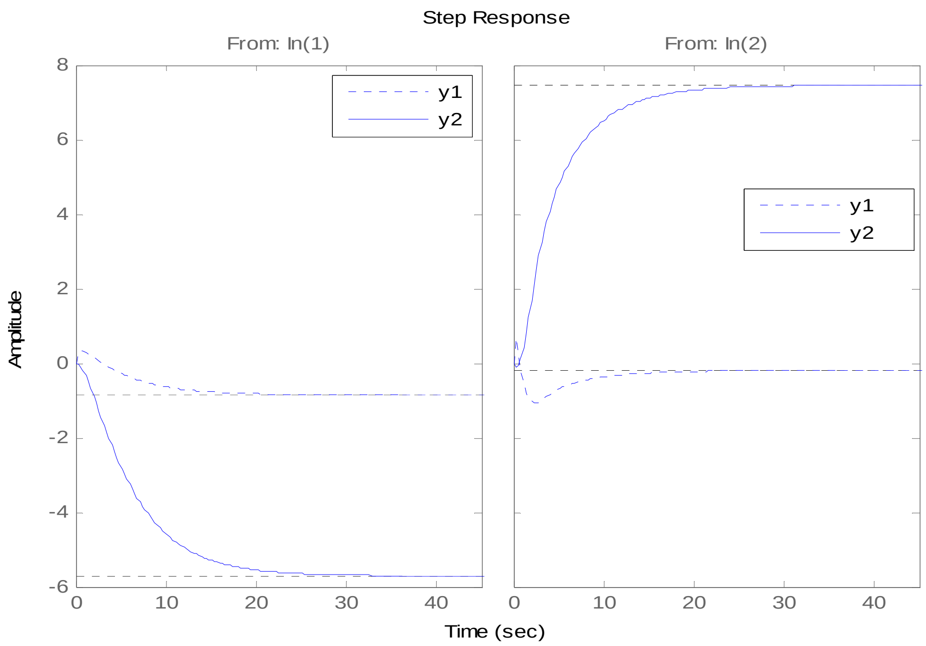


Figure 15: Time response for observer form

The smallest and the largest singular values of the closed-loop transfer function:

- $\underline{\sigma}(H_{cl}(s)) = 0 + 0.4440i$

- $\overline{\sigma}(H_{cl}(s)) = 0.0270 + 2.3126\text{i}$

The condition number of the closed-loop transfer function is given by

$$K(H_{cl}(s)) = 5.2091 - 0.0609\text{i}$$

The infinity norm of the closed-loop transfer function is computed as: $\|H_{cl}(s)\|_{\infty} = 9.3912$

and the infinity norm of the sensitivity function is $\|S\|_{\infty} = 8.6814$

7.3.4 Comparison of the Results

7.3.4.1 Time Response:

		Diagonal Form	Controllable Form	Observable Form
U1	y1_MP	<-1.3	1.04	<-0.853
	y1_POS	- 7.6923%	67.4%	- 0.1172%
	y1_Ts	21.3s	23.2s	21.1s
	y1_Tr	11.1s	0.875s	11s
	y1_SSV	- 1.3	0.623	- 0.853
	y2_MP	<-0.997	- 1.33	<-5.71
	y2_POS	- 0.1003%	1.21%	- 0.1751%
	y2_Ts	14.4s	5.27s	21.6s
	y2_Tr	6.64s	2.79s	12s
	y2_SSV	- 0.997	- 1.31	- 5.71
U2	y1_MP	<-0.574	<-3.06	- 1.08
	y1_POS	- 0.1742%	- 0.3268%	441.0%
	y1_Ts	21.9s	24.3s	21.4s
	y1_Tr	11.3s	11.7s	0.1s
	y1_SSV	- 0.574	- 3.06	- 0.2
	y2_MP	<-2.2	- 1.81	>7.45
	y2_POS	- 4.5455%	8.91%	0.1342%
	y2_Ts	10.2s	16.4s	19.1s
	y2_Tr	4.91s	2.09s	9.76s
	y2_SSV	-2.2	- 1.66	7.45

7.3.4.2 Robust Transfer Function

	Diagonal Form	Controllable Form	Observable Form
the norm of sensitivity function S	3.4759	4.9436	7.7616
the norm of complementary sensitivity function T	2.4790	5.0184	6.9796
the norm of closed loop function H_{cl}	2.4790	5.0184	6.9796
the largest singular value of the closed loop function	2.3089+0.0259i	0+2.3085i	0+2.2902i
the smallest singular value of the closed loop function	0.4435+0.00151i	0.4432	0.0015+0.4436i
the condition number of the closed loop function	5.2065+0.0406i	0+5.2082i	5.1624+0.0170i

In this example the form of the block pole in the diagonal form yields smaller percent overshoot as well as smaller sensitivity function norm and smaller norm of the closed-loop function. The smallest rise time is given in block pole using controller form.

7.4 Comment and Analysis

Large case studies are implemented with block pole placement using both state feedback and compensator design to compare the different solvents forms (diagonal, controllable and observable form). The norm of the feedback gain matrix, the sensitivity of the eigenvalues, condition number of the closed-loop transfer function and others are computed so that the system meet a set of criteria:

i. Better time response characteristics.
ii. Smaller feedback gain norm.
iii. Good robustness.

. The step response of the closed-loop system is plotted and its characteristics (settling time, percent overshoot, rise time, steady state value) are computed.

Comparison the results in the case of the block poles using state feedback

Case Studies	The form of the block which gives the smallest gain matrix norm	the form of the block which gives the smallest left eigenvector norm of every eigenvalues	the form of the block which gives the smallest left eigenvector norm of all eigenvalues	the form of the block which gives the shortest settling time	The form of the block which gives the smallest percent overshoot	The form of the block which gives the smallest peak
Case study 1	controllable	diagonal	observable	/	controllable	observable
Case study 2	observable	controllable	observable	diagonal	diagonal	diag/con
Case study 3	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 4	diagonal	diagonal	diagonal	diag/con/obs	observable	diagonal
Case study 5	diagonal	diagonal	diagonal	controllable	controllable	observable
Case study 6	diagonal	diagonal	diagonal	diagonal	controllable	diagonal
Case study 7	diagonal	diagonal	diagonal	diagonal	observable	controllable
Case study 8	diagonal	diagonal	diagonal	diagonal	diagonal	controllable
Case study 9	diagonal	diagonal	diagonal	diagonal	diagonal	diag/con
Case study 10	diagonal	diagonal	diagonal	diagonal	diag/con	diagonal
Case study 11	diagonal	diagonal	diagonal	diagonal	diag/con/obs	diagonal
Case study 12	diagonal	diagonal	diagonal	diag/obs	controllable	diagonal
Case study 13	diagonal	diagonal	diagonal	diag/obs	diagonal	diagonal
Case study 14	diagonal	diagonal	diagonal	diagonal	con/obs	controllable
Case study 15	diagonal	diagonal	diagonal	diagonal	observable	diag/obs
Case study 16	observable	observable	observable	observable	controllable	observable
Case study 17	diagonal	diagonal	diagonal	diagonal	controllable	diag/con
Case study 18	diagonal	diagonal	diagonal	observable	controllable	observable
Case study 19	controllable	controllable	controllable	con/obs	observable	observable
Case study 20	diagonal	diagonal	diagonal	diagonal	controllable	diagonal
Case study 21	observable	observable	diagonal	diagonal	controllable	observable
Case study 22	controllable	controllable	controllable	controllable	observable	controllable
Case study 23	controllable	controllable	controllable	diagonal	observable	controllable
Case study 24	observable	diagonal	controllable	diagonal	diag/con	diagonal
Case study 25	diagonal	diagonal	diagonal	diagonal	controllable	diag/con
Case study 26	diagonal	diagonal	diagonal	controllable	diagonal	diag/con
Case study 27	observable	diagonal	diagonal	/	observable	observable
Case study 28	observable	observable	observable	con/obs	diag/con/obs	diag/con/obs
Case study 29	controllable	observable	observable	diag/obs	controllable	controllable
Case study 30	diagonal	controllable	controllable	diagonal	observable	diagonal
Case study 31	controllable	diagonal	diagonal	controllable	diag/con/obs	controllable

Case studies	the block which has the smallest sensitivity of all eigenvalues	the block which gives the smallest sensitivity of each eigenvalues	the block which gives the smallest possible matrix variation norm for the closed loop matrix to have an unstable and pure imaginary eigenvalues	the block which gives the smallest likelihood margin for eigenvalue which is close to the imaginary axis to be unstable	the block which gives the smallest likelihood margin for every eigenvalues to become unstable	The form of the block pole which gives the smallest Relative Change
Case study 1	diagonal	observable	diagonal	observable	observable	controllable
Case study 2	observable	controllable	controllable	observable	observable	diagonal
Case study 3	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 4	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 5	diagonal	diagonal	controllable	diagonal	diagonal	controllable
Case study 6	diagonal	diagonal	diagonal	diagonal	diagonal	obs/diag
Case study 7	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 8	diagonal	diagonal	controllable	diagonal	diagonal	diagonal
Case study 9	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 10	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 11	diagonal	diagonal	diagonal	diagonal	diagonal	controllable
Case study 12	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 13	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 14	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 15	diagonal	diagonal	controllable	diagonal	diagonal	controllable
Case study 16	observable	observable	observable	observable	observable	observable
Case study 17	diagonal	diagonal	diagonal	diagonal	diagonal	controllable
Case study 18	diagonal	diagonal	diagonal	diagonal	observable	diagonal
Case study 19	controllable	controllable	controllable	controllable	controllable	controllable
Case study 20	diagonal	diagonal	diagonal	diagonal	diagonal	diagonal
Case study 21	diagonal	observable	observable	diagonal	observable	observable
Case study 22	controllable	controllable	controllable	controllable	controllable	controllable
Case study 23	controllable	controllable	controllable	controllable	controllable	diagonal
Case study 24	controllable	diagonal	observable	controllable	diagonal	observable
Case study 25	diagonal	diagonal	diagonal	diagonal	diagonal	observable
Case study 26	diagonal	diagonal	diagonal	diagonal	diagonal	observable
Case study 27	diagonal	diagonal	diagonal	diagonal	diagonal	observable
Case study 28	observable	controllable	observable	observable	observable	obs/con

Case study 29	observable	controllable	observable	observable	controllable	controllable
Case study 30	diagonal	diagonal	diagonal	diagonal	diagonal	Diagonal
Case study 31	diagonal	diagonal	controllable	diagonal	diagonal	con/diag

Comparison in the case of the block poles using compensator design

Case studies	the block which gives the shortest settling time	the block which gives the smallest percent overshoot	the block which gives the smallest peak	the block which gives the smallest norm of the closed loop function	the block which gives the smallest condition number	the block which gives the smallest norm of sensitivity function	the block which gives the smallest singular value larger
Case study A	diagonal	obs/con	diagonal	diagonal	diagonal	Diagonal	diagonal
Case study B	observable	diag/con	controllable	controllable	diagonal	Controllable	controllable
Case study C	diag/con	observable	diagonal	diagonal	diagonal	Controllable	diagonal
Case study D	controllable	controllable	controllable	controllable	observable	Controllable	observable
Case study E	observable	diagonal	diagonal	diagonal	Controllable	Diagonal	diagonal
Case study F	observable	obs/con	diag/con	diagonal	controllable	Diagonal	Observable
Case study G	diagonal	observable	obs/diag	observable	diagonal	Observable	Controllable
Case study H	diagonal	obs/diag	diagonal	diagonal	diagonal	Diagonal	Diagonal
Case study I	controllable	controllable	diagonal	observable	diagonal	Observable	Diagonal
Case study J	diagonal	diagonal	observable	observable	controllable	Diagonal	Controllable
Case study K	observable	diagonal	diagonal	diagonal	diagonal	Observable	Observable
Case study L	observable	controllable	diagonal	diagonal	controllable	/	Observable
Case study M	con/obs	observable	diag/con	controllable	controllable	Controllable	Controllable

Now we are in a position to analyze and comment the results:

i. The diagonal form for the block poles yields the smallest norm feedback gain matrix.

ii. The diagonal form for block poles yields smallest norm left eigenvectors, hence a better robustness (lower eigenvalue sensitivity).

iii. The diagonal form yields shorter settling time

iv. On the other hand, controller forms for block poles yield smaller percentage overshoot

v. The diagonal form yields smaller sensitivity of all eigenvalues (all eigenvalues are insensitive to uncertainty model or parameters variation).

vi. The diagonal form yields smaller sensitivity of every eigenvalues (every eigenvalues has low sensitivity).

vii. The block pole using diagonal form yields smaller matrix variation norm for the closed loop matrix to have an unstable and pure imaginary eigenvalues.

viii. The diagonal form for block poles yields smaller likelihood margin for eigenvalues which are close to the imaginary axis to be unstable.

ix. The block poles in diagonal form yields smaller likelihood margin for every eigenvalues to become unstable.

As concluding remark; using the block poles in diagonal form to assign the desired eigenvalues makes the system robustly stable; since the three robust stability measures are maximized and all eigenvalues has the smallest likelihood margin to become unstable this means that the eigenvalues stay stable under model uncertainty or parameter variations. The block poles assigned using diagonal form yields smaller feedback gain matrix which is crucial for the system and the diagonal form improves the quickness of the system transient response.

Compensator design case

The proposed method using the design of compensators for block pole placement allows the computation of the proper and minimal degree compensator.
With same set of poles we construct different block poles using different forms (diagonal, controllable and observable). To choose the best block pole we studied their effect on the degree of the compensator and time transient response and the robustness of the closed loop system.
After comparison, we will have:

i. The block pole in observer form yields shorter settling time.
ii. The block pole in controller form yields smallest percent overshoot.
iii. The block pole in diagonal form yields smaller higher peak.

iv. The block pole in diagonal form yields smaller norm of closed loop transfer function.

v. The block pole in diagonal form yields smaller condition number.

vi. The block pole in diagonal form yields larger smallest singular value.

vii. The block pole in diagonal form yields smaller norm of the sensitivity function

General Conclusion

In multivariable system, state feedback design and compensator design may be achieved using block pole assignment. The construction of these block poles is not unique for a given set of desired poles. This nonuniqueness is used in our work by constructing three different canonical forms (diagonal, controller and observer) for the solvents to achieve stability and better performance of the system. The solvents determine the behavior of the multivariable system as shown in our thesis.

The purpose of our work is to choose a block pole form, constructed using the desired poles, that achieves small settling time, small percent overshoot *i.e.*, better time response, and less sensitive to parameter variations and maintains the stability under perturbation which always exist in the system and are inevitable.

Through the comparative study that we have made, block pole constructed using diagonal form gives the smallest norm of feedback gain matrix which is crucial for the system.

The faster the transient response, the better (higher) is the performance of the closed-loop system. Comparative study shows that smallest settling time and smallest time for the system to reach 50% of its final value are given by the block poles in diagonal form.

Because the eigenvalues of the closed-loop matrix determine directly the stability of the system, it is obvious that the sensitivities of these eigenvalues most directly determine a system's robust stability. Our work is based on a result of numerical linear algebra that the sensitivity of the eigenvalues is determined by their corresponding eigenvectors. Using block poles in diagonal form yields less sensitive eigenvalue. We used the condition number of eigenvector of the closed-loop matrix to measure the sensitivity of all eigenvalues; the smallest condition number is given for block pole in diagonal form.

The norm of the left eigenvector plays a role in the sensitivity of the corresponding eigenvalue as it is shown in this thesis.

Robust stability measures are applied in our case studies to evaluate the sensitivity of the eigenvalues used to guarantee both stability and performance of the system. Using solvents in diagonal form the closed-loop system is low sensitive to parameter variations.

In the case of block pole placement using compensator design, the infinity norm is used to assess the robustness of the unity feedback design. The infinity norm used is related to the robustness improvement and sensitivity reduction. In our work the smallest infinity norm of the closed-loop transfer function is given by block poles in diagonal form.

The sensitivity function and complementary sensitivity function express important properties of a feedback design as response of the output to disturbances and response to noise, the block pole in diagonal form yields smaller infinity norm of the sensitivity function and complementary sensitivity function.

In light of the results obtained and illustrated in the simulation study, it is observed that the block poles in diagonal form constructed from a set of desired poles yield robust closed-loop system with low sensitivity to parameter variations, better closed-loop time response and small state feedback gain.

Using the diagonal form improves the system’s performance and robustness of the system.

As further studies we may suggest the following problems:

I. More investigations of other additional robust stability measures in order to improve the results obtained in this thesis.

II. Profound investigations of stability robustness and performance robustness with respect to structured or unstructured uncertainties and additive or multiplicative perturbations in the case of unity feedback design.

III. Study the sensitivity of the zeros of a closed-loop system.

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Appendix A

The Recursive Algorithm

Given a set of n-dimensional rows $T_1, T_2, ..., T_m$, an $n \times n$ matrix $P(k)$ is determined recursively for $k = 1,2,...,m$

1. initialize $P(0) = I_n$ ($n \times n$ identity matrix)
2. for $k = 1,2,...m$ do

if $T_K P(k-1) T_K^T \neq 0$, then

$$P(k) = P(k-1) - \frac{\left[P(k-1)T_K^T\right]\left[P(k-1)T_K^T\right]^T}{T_K P(k-1) T_K^T}$$

and T_k is linearly independent of the previous rows

else $P(k) = P(k-1)$

and T_k is linearly dependent.

Proof: see Yaissi [75]

The coefficients of combination of the $j-th$ linearly dependent row on its previous $j-1$ rows can be computed by solving an equation of the type $xA = b$.

Appendix B

Computing the Coefficient of the Combination using Row-Searching Algorithm [2]

In the row searching algorithm the idea is to search for linearly independent rows using elementary operations.

Consider the $n \times n$ matrix $A = (a_{ij})$

1- Choose a pivot as a nonzero element in the first row of *A*, say a_{1k}

2 – Construct the matrix K_1 as

$$K_1 = \begin{bmatrix} 1 & 0 & 0 & . & 0 \\ e_{21} & 1 & 0 & . & 0 \\ e_{31} & 0 & 1 & . & 0 \\ . & . & . & . & . \\ e_{n1} & 0 & 0 & . & 1 \end{bmatrix}$$

with $e_{i1} = -a_{ik} / a_{1k} \quad i = 1,2,...,n$ then the $k-th$ column , except the first element of $K_1 A = (a^1_{ij})$ is a zero column , where $a^1_{ij} = a_{ij} + e_{i1} a_{1j}$

3- Let a^1_{2j} be any nonzero element in the second row of $K_1 A$ .Let K_2 be of the form

$$K_2 = \begin{bmatrix} 1 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & . & 0 \\ 0 & e_{32} & 1 & . & 0 \\ . & . & . & . & . \\ 0 & e_{n2} & 0 & . & 1 \end{bmatrix}$$

with $e_{i2} = -a^1{}_{ij} / a^1{}_{2j} \quad i = 1,2,...,n$, then the $j-th$ column , except the first element of $K_2 K_1 A = (a^2_{ij})$ is a zero column , where $a^2_{ij} = a^1{}_{ij} + e_{i2} a^1{}_{2j}$

4- If there is no nonzero element in a row , we assign K_i as a unit matrix and then proceed to the next row .

5- The process is carried to the last row , and finally we obtain

$$K_{n-1} K_{n-2} \ldots K_2 K_1 A = K A = \tilde{A}$$

The number of nonzero rows in $\tilde{A}$ gives the rank of A .If the $j-th$ row of $\tilde{A}$ is a zero row, then the $j-th$ row of A is linearly dependent of its previous rows. The coefficients of the combination

$$\begin{bmatrix} b_{j1} & b_{j2} & . & . & . & b_{j(j-1)} & b_{jj} & 0 & . & . & . & 0 \end{bmatrix} A = 0$$

with $b_{jj} = 1$, is just the $j-th$ row of K .

The matrix K can be computed using the following procedure:

1- We store the *i-th* column of K_i in the *i-th* column of

$$F = \begin{bmatrix} 1 & 0 & 0 & . & 0 \\ e_{21} & 1 & 0 & . & 0 \\ e_{31} & e_{32} & 1 & . & 0 \\ . & . & . & . & . \\ e_{n1} & e_{n2} & e_{n3} & . & 1 \end{bmatrix}$$

2- The *j-th* row of K is computed using the first *j*-rows of F as follows:

$$b_{jj} = 1$$

$$b_{jk} = \begin{bmatrix} b_{j(k+1)} & b_{j(k+2)} & . & . & . & b_{jj} \end{bmatrix} \begin{bmatrix} e_{(k+1)k} \\ e_{(k+2)k} \\ . \\ . \\ . \\ e_{jk} \end{bmatrix}$$

$$= \sum_{p=k+1}^{j} b_{jp} e_{pk} \qquad k = j-1, j-2, \ldots, 1$$