Analysis of Elliptical Cracks in Static and in Fatigue by Hybridization of Green's Functions

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Abstract A hybrid weight function technique is presented. It consists of dividing an elliptical crack into two zones, then using the appropriate weight function in the area where it is more efficient. The proportion between zones is determined by optimizing two crack parameters (axis ratio and curvature radius). Stress intensity factors are hence computed by a self developed computer code. Static and fatigue loadings are considered. The results found by the present approach are in good correlation with the analytical and experimental solutions (when available) as well as with those obtained numerically by other researchers.

Keywords: Hybridization, Weight Function, SIF, Elliptical Crack, Fatigue-Crack-Growth.

1. Introduction

The principle of the weight function (called the Green's function) technique consists of employing one or more known solutions (known as reference solutions) of a particular case in order to find the solution for the general case. The reference solution generally comes from the analytical results (exact). But in some cases, the absence of such results obliges researchers to use approximate solutions which could be already existing weight functions. In this paper, a method improving the calculation of SIF in mode I for elliptical and semi-elliptical cracks is developed by means of hybridization of two weight functions and coupling to the Point Weight Function Method (PWFM). In the fatigue problems, two crack propagation laws have been incorporated. Crack propagation life and crack profile are investigated for various applications. The development of weight functions in fracture mechanics started with the work of Bueckner [2], based on the formulation by the Green's function, for a semi-infinite crack in an infinite medium. The investigation of the weight functions on the one hand and the evaluation of the energy balance formula of Rice [14] on the other hand, allowed the extension of the use of the weight functions by several authors such as Oore and Burns [10] and Bortmann

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and Banks-Sills [1]. In 1986, Gao and Rice [4] introduced the study of the stability of the rectilinear form of a semi-infinite crack front during its coplanar propagation from which result the values of stress intensity factor (SIF) along the crack front. Recently, Sun and Wang [16] gave in-depth interpretations of the energy release rate of the crack front. Other investigations followed related especially to the crack shape (ellipse, half of ellipse, quarter of ellipse, rectangle, …) as well as to the fracture mode (I, II, III or mixed) and to the application domain (elastoplastic, elastodynamic, …). This paper is structured as follow. Detailed presentation of the hybridization approach is presented in Section 2. Fatigue crack propagation models are then implemented into the hybridization technique, subject of Section 3. In the next section, two industrial applications are discussed, one of them is in static and the other one is in fatigue loading. We end this paper by drawing some conclusions. This work is an extension of already published studies of crack modeling by the hybridization technique [5, 6, 8].

2. Presentation of the hybridization technique

The solution of the SIF in mode I using the weight function technique is given by the general form [10]:

$$K_{IQ'} = \int_S W_{QQ'} q(Q).dS \tag{1}$$

where $K_{IQ'}$ is the stress intensity factor in mode I at the Q' point of the crack front. $W_{QQ'}$ is the weight function related to the problem and $q(Q)$ symmetrical loading applied to the arbitrary Q point of the crack area S.

This study is based on the hybridization of two weight functions deduced from a Green's function formulation.

The first one is developed by Oore and Burns [10] to model any closed shape of a crack in an infinite body, including elliptical cracks. Its expression is as follows:

$$W_{QQ'} = \sqrt{2} \Bigg/ \left(\pi l_{QQ'}^2 \sqrt{\int_\Gamma \frac{d\Gamma}{\rho_Q{}^2}} \right) \tag{2}$$

The second one is developed by Krasowsky et al. [9] to model elliptical cracks in an infinite body. Its expression is as follows:

$$W_{QQ'} = 2\prod^{1/4}(\theta) \Bigg/ \sqrt{\pi a \left(1 - \frac{r^2(\varphi)}{R^2(\varphi)}\right) l_{QQ'}^2 \int_\Gamma \frac{d\Gamma}{\rho_Q{}^2}} \tag{3}$$

In expressions (2) and (3), r and φ are the polar coordinates of an arbitrary point Q. $l_{QQ'}$ is the distance between the Q' point and the arbitrary Q point. Γ is the curve of the ellipse (the crack front), and ρ_Q is the distance between the Q point and the elementary segment $d\Gamma$, $\prod$ is a function such as $\prod(\theta) = (\sin^2\theta + \alpha^4\cos^2\theta)/(\sin^2\theta + \alpha^2\cos^2\theta)$ and $\alpha = a/b$.

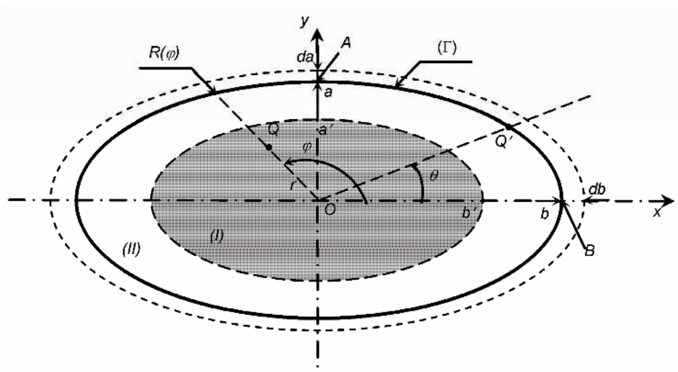

Fig. 1. Subdivision of the elliptical crack in two zones and its geometrical parameters

The principle of hybridization is to divide, as shown in Fig. 1, the elliptical crack into two zones, an internal zone I (ellipse in grey) and an external zone II (in white), then to use each of the two weight functions in the area where it is more efficient. The two zones are defined by the following relations:

$$\begin{cases} zone\ I: (x/b')^2 + (y/a')^2 \leq 1 \\ zone\ II: (x/b')^2 + (y/a')^2 > 1 \end{cases} \tag{4}$$

where a' and b' are such as $a'/a = b'/b = \beta$ and $\beta \in [0, 1]$, β being the proportion between the two zones and a, b are the axes of ellipse (e.g. Fig. 1).

The weight function of Eq. (3) is intended exclusively for cracks of elliptical form. Nevertheless, it presents an additional singularity $(1 - r/R)^{-1/2}$ compared to Eq. (2). This makes Eq. (3) less efficient in the vicinity of the crack front $(r \to R)$. This argument leads us to make the following choice:

$$\left. \begin{array}{l} W_{QQ'} = W_{QQ'}\ of\ Eq.(3)\ if\ Q \in Zone\ I \\ W_{QQ'} = W_{QQ'}\ of\ Eq.(2)\ if\ Q \in Zone\ II \end{array} \right\} \tag{5}$$

It remains to determine the appropriate proportion β between two zones I and II. By construction of each weight function (see details in [6, 8]), the function (2) is preferable to the function (3) in the two following cases:

- When the crack front is close to a circle ($\alpha \to 1$)
- When the crack front is close to a straight line (low values of α with value of θ far from zero)

In fact, these two cases correspond to situations where the variation of the curvature radius R_c of the crack front is weak, excluding a very narrow zone corresponding of the smallest values of R_c (θ close to zero with low values of α). In this case and for a relatively high variation of the curvature radius of ellipse, the weight function (3) of Krasowsky et al. is more adapted. This is confirmed by the presence via the function $\Pi(\theta)$ of curvature radius $R_c = (a/\alpha)\Pi^{3/2}(\theta)$ in the expression of the function (3).

Consequently, more the radius of curvature is relatively weak or its derivative (spatial gradient of the radius of curvature) is high, more the zone I should increase with respect to the zone II and vice versa.

Taking into account all these considerations, we propose the relative parameter expressed by:

$$\beta_1 = \left(b - \min\left(R_c, b\right)\right)/b \tag{6}$$

For the representation of the influence of the curvature radius compared to the large axis of the ellipse b, and the following one:

$$\beta_2 = \left[\left(\frac{\partial R_c}{\partial \theta}\right) - \left(\frac{\partial R_c}{\partial \theta}\right)_{\min}\right] \Bigg/ \left[\left(\frac{\partial R_c}{\partial \theta}\right)_{\max} - \left(\frac{\partial R_c}{\partial \theta}\right)_{\min}\right] \tag{7}$$

To represent the influence of the gradient of the curvature radius for a given α. In this relation $\partial R_c / \partial \theta$ is the partial derivative with respect to the angular position of the point Q', its maximum and minimum values are calculated by “sweeping” completely the contour of the ellipse for a given α. The computation of the partial derivative is achieved numerically.

The proportion parameter β takes the value:

$$\beta = \max\left(\beta_1, \beta_2\right) \tag{8}$$

Details of the method regarding its numerical implementation, treatment of singularities, meshing can be found in references [5, 6, 8].

To extend the use of this hybrid approach for the semi-elliptical crack modelling, it’s coupling with the point weight function method [11] (PWFM) is considered in order to take into account the free edge effect. More details of the coupling between the present hybrid technique and the PWFM method can be found in [7].

3. The hybrid method in fatigue

The fatigue crack growth prediction is classically based on the SIF approach, known as the Paris law [12]. This last one is given as follows:

$$da/dN_c = C(\Delta K_I)^m \tag{9}$$

C and m are material parameters related to Paris law usage.

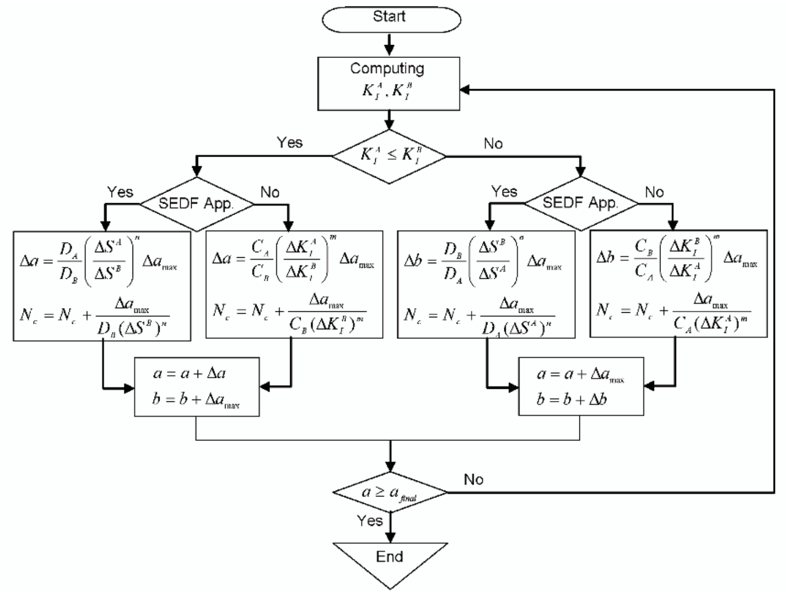

Fig. 2. Algorithmic schemes of the computing procedure

As previously mentioned, the fatigue crack growth prediction can also be based on the SEDF approach. Its expression has first been given in 1979 by Sih [15], in which the strain energy density factor range replaces the stress intensity factor range in classical laws (Paris law for example). The Sih's law (based on SEDF approach) has the following form:

$$da/dN_c = D(\Delta S_{\min})^n \tag{10}$$

D and n are material parameters related to Sih's law usage.

In the above equation $S_{\min}$ is the necessary strain energy density factor for a crack to propagate, and satisfying the condition $dS = 0$. This factor may be obtained in mode I according to the SIF value [15].

Written for two points A ($\theta = 90°$) and B ($\theta = 0°$) of the ellipse contour (e.g. Fig. 1), one can obtain from Eqs. (9) and (10), respectively:

$$da/db = (C_A/C_B)\left(\Delta K_I^A/\Delta K_I^B\right)^m \tag{11}$$

And:

$$da/db = (D_A/D_B)\left(\Delta S^A/\Delta S^B\right)^n \tag{12}$$

From Eqs. (11) and (12), one can say that for two different values of SIF range ΔK^A and ΔK^B or SEDF range ΔS^A and ΔS^B, two different values of the crack growth segments *da* and *db* can be obtained. Consequently, a change in the crack profile is expected. On the algorithmic scheme presented in Fig. 2, are illustrated the different steps for the computation of the fatigue life and the evolution of the crack shape.

4. Numerical tests, results and discussions

In this section, practical applications are numerically treated using the computer operational software with graphic interface using the C++ object-oriented language named HWFun which we develop for this purpose.

4.1. Internal semi-elliptical surface crack in a pressurized tube

The theory of thick tubes (Lamé's theory) shows that longitudinal cracks located on an internal face of the tube are the most dangerous ones. For this kind of application, the efficiency of hybridization approach coupled with the PWFM [11] is evaluated. For values of $\alpha = 1.0$ and $\alpha = 0.4$, *n*umerical tests are carried out on tubes of $t/R_{int} = 0.1$, where t is the thickness of tube and R_{int} is its internal radius. The loading inside the crack has the form $p = (y/a)^i$ with $i \in \{1,2,3\}$.

In accordance with the PWFM method ad for the sake of comparison, we choose a reference solution in the form $p = \sigma_0$. The present results are for two characteristics angles $\theta = 0°$ and $\theta = 90°$, and for linear (i = 1), quadratic (i = 2) and cubic (i = 3) loadings. From Fig. 3, the present results are in good agreement with those found by Raju and Newman [13] using finite element method and those of Krasowsky et al. [9], Vainshtok [17] et Orynyak et al. [11] using weight function methods such as $\overline{K}_I = K_I E(k) \big/ (\sigma_0 \sqrt{\pi a}\, \Pi^{1/4}(\theta))$, where $E(k)$ is the elliptic integral of second kind and $k = \sqrt{1-\alpha^2}$. According to those graphs, loading mode has a significant effect on the stress intensity factor (SIF). In fact:

- The SIF values increase with the decrease of α
- The maximum value of SIF is for a uniform loading
- Only uniform loading induces a more important value of SIF at a surface point ($\theta = 0°$) than at a depth point of ($\theta = 90°$)

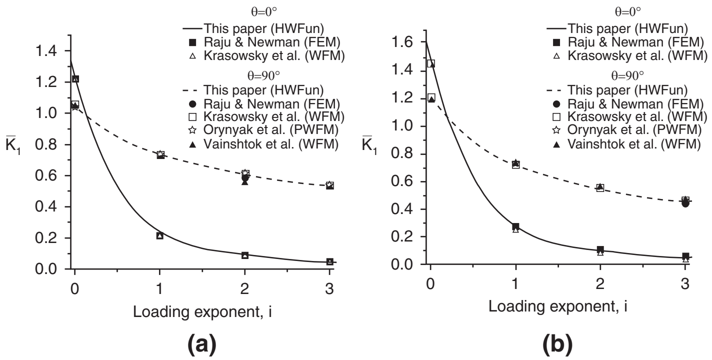

Fig. 3. Adimensional SIF of a semi elliptical crack at the internal surface of a tube

4.2. Out-of-plane gusset welded joint

To this mechanical component (see Fig. 4) a tensile cyclic load is applied. Due to stress concentration, a surface crack can initiate either at the junction between portions 2 and 1, or between portions 2 and 3. Two cases have been considered: $\rho = 0$ mm and $\rho = 30$ mm (ρ is the radius of curvature of the welded joint) as shown on Fig. 4. The numerical computations in [3] used a finite element code known as "LUSAS". The mechanical properties of the treated component are those of the steel called POSTEN 80. They are given in [3] as $\mu = 77GPa$, and $\nu = 0.3$.

In the numerical computations, and in order to take into account of the stress concentration, the coefficient F_g was given in a format of curves [3]. It is introduced by modifying the value of the tensile stress by the term $F_g\sigma_t$ at both points A and B and at each crack growth da or db. To facilitate the numerical implementation, this coefficient F_g has been substituted by a fifth order polynomial obtained by fitting the above mentioned curves.

The values of the crack growth material parameters *D, n, C, m* are those used by [3]. It should be pointed out that the parameter D has been modified by multiplying it by a factor $(\pi)^n$. This difference comes from the fact that in the case of plane stress, the SEDF equals to $K_I^2(1-2\nu)/4\pi\mu$.

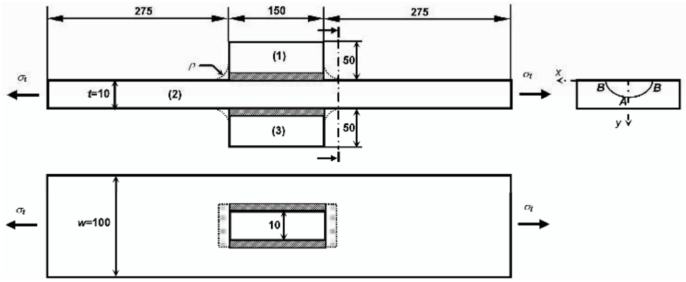

Fig. 4. Configurations and dimensions of the mechanical component

The first numerical tests deal with those treated in [3], and concern the following configurations:

(a) $\rho = 0$ mm, and $R = 0.1$

- $a_0 = 0.4\,\text{mm}$, $\alpha_0 = 0.4$, $\Delta a_{\max} = 0.2 a_0$, $\Delta\sigma_t = 93\,\text{MPa}$
- $a_0 = 0.4\,\text{mm}$, $\alpha_0 = 0.4$, $\Delta a_{\max} = 0.2 a_0$, $\Delta\sigma_t = 124\,\text{MPa}$
- $a_0 = 0.4\,\text{mm}$, $\alpha_0 = 0.4$, $\Delta a_{\max} = 0.2 a_0$, $\Delta\sigma_t = 155\,\text{MPa}$

(b) $\rho = 30$ mm, and $R = 0.1$

- $a_0 = 0.3\,\text{mm}$, $\alpha_0 = 0.1$, $\Delta a_{\max} = 0.2 a_0$, $\Delta\sigma_t = 124\,\text{MPa}$
- $a_0 = 0.3\,\text{mm}$, $\alpha_0 = 0.1$, $\Delta a_{\max} = 0.2 a_0$, $\Delta\sigma_t = 155\,\text{MPa}$
- $a_0 = 0.3\,\text{mm}$, $\alpha_0 = 0.1$, $\Delta a_{\max} = 0.2 a_0$, $\Delta\sigma_t = 207\,\text{MPa}$

Figure 5a, b shows the fatigue crack growth life numerically estimated via different approaches, and experimentally measured as well. At the first stage, one can make the following physical observations:

1. The fatigue life is much affected by the radius of curvature ρ. Indeed, as the radius of curvature decreases ($\rho = 0$), the fatigue life decreases.
2. The fatigue life is also affected by the stress range $\Delta\sigma$, i.e. as $\Delta\sigma$ increases, the fatigue life decreases.

The second observations deal with the numerical comparisons from which one can state the following:

1. The hybrid approach (HWFM) is in correlation with the experimental data, for both radius of curvature and for the various values of $\Delta\sigma$.
2. For the lower value of the radius curvature ($\rho = 0$), the SEDF approach gives better predictions than the SIF one.

In the following numerical computations, the influence of the physical stress ratio parameter R on the fatigue crack growth is discussed. Figure 6 shows in a log-log scale, the number of cycles to failure N_c, versus the stress range $\Delta\sigma$ for the curvature radius $\rho = 0$ and $\rho = 30$, and for various values of the stress ratio R.

On the same figure, the experimental data given by [3], which are only available for R = 0.1 are inserted. One can observe that a good correlation exists between the previously mentioned data and the numerical results obtained by the HWFM when using the SEDF approach. However, the usage of the SIF approach gives less accuracy in the fatigue life prediction. From Fig. 6a, b, it is also observed that the fatigue life decreases as the stress ratio increases. Overall, it is noticed that the curve relative to the number of cycles versus stress range obtained by the usage of the SIF approach lies in between the curves $R = 0.1$ and $R = 0.3$, obtained when using the SEDF approach. This interval can be considered as the interval of validity of the SIF approach for such treated problem.

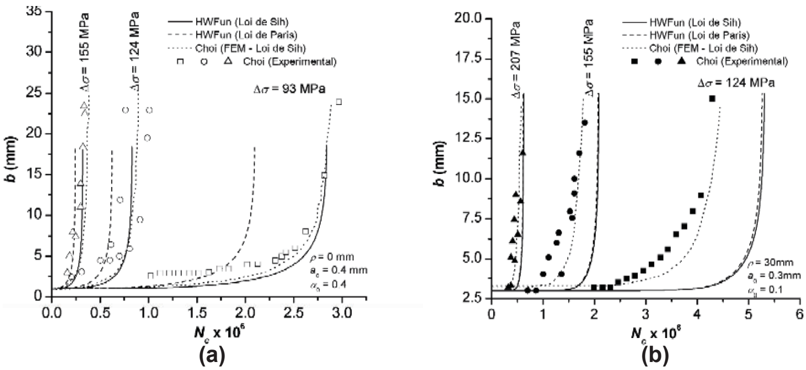

Fig. 5. Fatigue life prediction with different approaches: (**a**) ρ = 0, (**b**) ρ = 30 mm

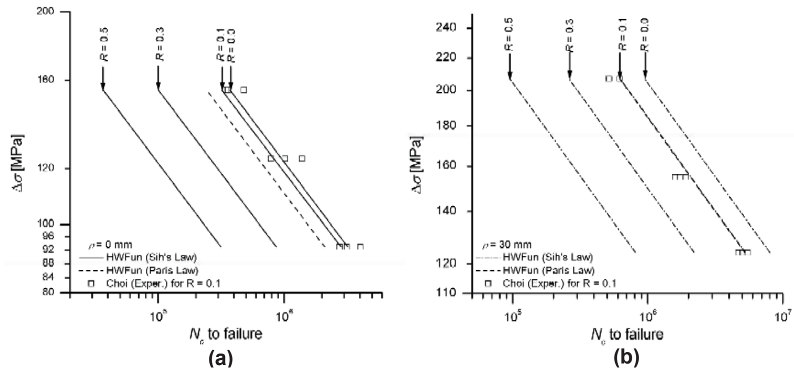

Fig. 6. Influence of the parameter R on the fatigue crack growth: (**a**) ρ = 0 mm, (**b**) ρ = 30 mm

5. Conclusions

In this work, a hybridization weight function approach based on Green's function formulation is developed. The method has been applied to static and fatigue loads. The idea of hybridization leads us to an optimization problem of two geometrical parameters (the ratio axes and curvature radius of ellipse). A computer code named *HWFun* has been developed and tested on various practical applications under static and fatigue loadings. In this modeling, a computation dealing with the stress intensity factors in mode I is first performed. The results obtained show a clear reduction in the error. The present approach was also tested on fatigue-crack-growth problems. The predicted crack shape evolution and the fatigue crack growth life are in perfect concordance with the results obtained by other researchers. The idea of hybridization undoubtedly opens horizons for the treatment of other complex problems in fracture mechanics such as mixed mode and interaction among cracks.

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