



## ALMOST AND WEAKLY NSR, NSM AND NSH SPACES

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**ABSTRACT.** We introduce and study some new types of star-selection principles (almost and weakly neighborhood star-Menger, neighborhood star-Rothberger, and neighborhood star-Hurewicz). We establish some properties of these selection principles and their relations with other selection properties of topological spaces. The behavior of these classes of spaces under certain kinds of mappings is also considered.

### 1. INTRODUCTION AND PRELIMINARIES

We give firstly definitions of notions, which are used in this paper.

Throughout the paper,  $\mathbb{N}$  and  $\mathbb{R}$  denote the set of positive integers and the set of real numbers, respectively. Let  $X$  be a topological space, let  $\mathcal{U}$  be a collection of subsets of  $X$ , and let  $A \subset X$ . Then the set  $\text{St}(A, \mathcal{U}) := \bigcup \{P \in \mathcal{U} : P \cap A \neq \emptyset\}$  is called the *star* of  $A$  with respect to  $\mathcal{U}$ ; we write  $\text{St}(x, \mathcal{U})$  instead of  $\text{St}(\{x\}, \mathcal{U})$ . For more information about star covering properties, see [5, 18].

Let  $\mathcal{A}$  and  $\mathcal{B}$  be collections of covers of a space  $X$ . Then the symbol  $S_1(\mathcal{A}, \mathcal{B})$  denotes the selection hypothesis that, for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$ , there exists a sequence  $(U_n : n \in \mathbb{N})$  such that for each  $n \in \mathbb{N}$ ,  $U_n \in \mathcal{U}_n$  and  $\{U_n : n \in \mathbb{N}\}$  is an element of  $\mathcal{B}$ . The symbol  $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$  denotes the selection hypothesis that for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$ , there exists a sequence  $(\mathcal{V}_n : n \in \mathbb{N})$  such that for each  $n \in \mathbb{N}$ ,  $\mathcal{V}_n$  is a finite subset of  $\mathcal{U}_n$  and  $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$  is an element of  $\mathcal{B}$  (see [13, 22]).

Kočinac [10] introduced the star selection hypothesis similar to the previous ones. Let  $\mathcal{A}$  and  $\mathcal{B}$  be collections of covers of a space  $X$ .

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(A) The symbol  $S_{\text{fin}}^*(\mathcal{A}, \mathcal{B})$  denotes the selection hypothesis that for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$ , there exists a sequence  $(\mathcal{V}_n : n \in \mathbb{N})$  such that for each  $n \in \mathbb{N}$ ,  $\mathcal{V}_n$  is a finite subset of  $\mathcal{U}_n$  and  $\bigcup_{n \in \mathbb{N}} \{\text{St}(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$  is an element of  $\mathcal{B}$ .

(B) The symbol  $SS_{\text{comp}}^*(\mathcal{A}, \mathcal{B})$  (resp.  $SS_{\text{fin}}^*(A, B)$ ) denotes the selection hypothesis that for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$ , there exists a sequence  $(K_n : n \in \mathbb{N})$  of compact (resp. finite) subsets of  $X$  such that  $\{\text{St}(K_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$ .

Let  $\mathcal{O}$  denote the collection of all open covers of a space  $X$ .

**Definition 1.1** (see [10, 14]). A space  $X$  is said to be *star-Menger* (resp. *star-Rothberger*) if it satisfies the selection hypothesis  $S_{\text{fin}}^*(\mathcal{O}, \mathcal{O})$  (resp.  $S_1^*(\mathcal{O}, \mathcal{O})$ ).

**Definition 1.2** (see [3, 10, 14]). A space  $X$  is said to be *star-Hurewicz* if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , one can choose finite  $\mathcal{V}_n \subset \mathcal{U}_n$ ,  $n \in \mathbb{N}$ , such that for every  $x \in X$ , we have  $x \in \text{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$  for all but finitely many  $n$ .

The following three generalizations of star selection properties were introduced (in a general form and under different names) in [11] and studied in details in [4].

**Definition 1.3** (see [4]). A space  $X$  is said to be *neighborhood star-Menger* (NSM) if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , one can choose finite  $F_n \subset X$ ,  $n \in \mathbb{N}$ , such that  $\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n) = X$  for every open  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ .

**Definition 1.4** (see [4]). A space  $X$  is said to be *neighborhood star-Rothberger* (NSR) if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , one can choose  $x_n \in X$ ,  $n \in \mathbb{N}$ , such that  $\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n) = X$  for every open  $O_n \ni x_n$ ,  $n \in \mathbb{N}$ .

**Definition 1.5** (see [4]). A space  $X$  is said to be *neighborhood star-Hurewicz* (NSH) if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , one can choose finite  $F_n \subset X$ ,  $n \in \mathbb{N}$ , such that for every open  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , each  $x \in X$  belongs to  $\text{St}(O_n, \mathcal{U}_n)$  for all but finitely many  $n$ .

In the last few years, weaker forms of selection and star selection properties became an important area of investigation (see, for example, [1, 2, 8, 9, 12, 15, 19–21, 23–26]). In this paper, we apply the same scenario to neighborhood star selection properties.

## 2. ABOUT wNSL SPACES

In this section, we give some facts about weaker forms of neighborhood star-Lindelöf spaces. Recall that a space  $X$  is said to be *neighborhood star-Lindelöf* (NSL) if for every open cover  $\mathcal{U}$  of  $X$ , one can choose countable  $A \subset X$  such that for every neighborhood  $O$  of  $A$ , we have  $\text{St}(O, \mathcal{U}) = X$  [18].

**Definition 2.1.** A space  $X$  is said to be *weakly neighborhood star-Lindelöf* (wNSL) if for every open cover  $\mathcal{U}$  of  $X$ , one can choose countable  $A \subset X$  such that for every neighborhood  $O$  of  $A$ , we have  $\overline{\text{St}(O, \mathcal{U})} = X$ .

**Theorem 2.2.** *An open  $F_\sigma$ -subset of a wNSL space is wNSL.*

*Proof.* Let  $(X, \tau)$  be a **wNSL** space and let  $Y = \bigcup\{H_n : n \in \mathbb{N}\}$  be an open  $F_\sigma$ -subset of  $X$ , where the set  $H_n$  is closed in  $X$  for each  $n \in \mathbb{N}$ . We show that  $Y$  is **wNSL**. Let  $\mathcal{U}$  be an open cover of  $(Y, \tau_Y)$ . We have to find a countable subset  $F$  of  $Y$  such that for each  $\tau_Y$ -open  $O \supseteq F$ ,  $Y \subseteq \overline{\text{St}(O, \mathcal{U})}$ .

For each  $n \in \mathbb{N}$ , consider the open cover  $\mathcal{U}_n = \mathcal{U} \cup \{X \setminus H_n\}$  of  $X$ . Since  $X$  is **wNSL**, there exists a countable subset  $F_n$  of  $X$  such that for each (in  $X$ ) open  $O' \supseteq F_n$ ,  $X = \overline{\text{St}(O', \mathcal{U}_n)}$ . For each  $n \in \mathbb{N}$ , let  $M_n = F_n \cap Y$ . [ $F_n$  must meet  $Y$ , because otherwise for the neighborhood  $X \setminus H_n$  of  $F_n$  we have  $\text{St}(X \setminus H_n, \mathcal{U}_n) \subset X \setminus H_n$ .] Then  $M_n$  is a countable subset of  $Y$  such that for each open  $O \supseteq M_n$ ,  $H_n \subseteq \overline{\text{St}(O, \mathcal{U})}$ . If we put  $F = \bigcup\{M_n : n \in \mathbb{N}\}$ , then  $F$  is a countable subset of  $Y$  such that for each open  $O \supseteq F$ ,  $\overline{\text{St}(O, \mathcal{U})} \supseteq Y$ , which shows that  $Y$  is **wNSL**.  $\square$

A *cozero-set* in a space  $X$  is a set of the form  $f^{-1}(\mathbb{R} \setminus \{0\})$  for some real-valued continuous function  $f$  on  $X$  [6].

Since a cozero-set is an open  $F_\sigma$ -set, we have the following corollary of Theorem 2.2.

**Corollary 2.3.** *A cozero-set in a **wNSL** space is **wNSL**.*

**Definition 2.4** (see [4]). Let  $Y$  be a subspace of a space  $X$ . Then

- (1)  $Y$  is *relatively NSL* in  $X$  if for each open cover  $\mathcal{U}$  of  $X$ , one can choose a countable subset  $A$  of  $X$  such that for each open  $O \supset A$ ,  $\text{St}(O, \mathcal{U}) \supset Y$ ;
- (2)  $Y$  is said to be a *relatively closed NSL space* if it is closed and relatively NSL in  $X$ .

Of course, every **NSL** space is relatively closed **NSL**.

**Theorem 2.5.** (1) *If  $X = \bigcup\{Y_k : k \in \mathbb{N}\}$  and each  $Y_k$  is relatively **NSL** in  $X$ , then  $X$  is **NSL**.*

- (2) *If  $X = \bigcup_{k \in \mathbb{N}} \overline{Y_k}$  and each  $Y_k$  is relatively **NSL** in  $X$ , then  $X$  is **wNSL**.*

*Proof.* (1) Let  $\mathcal{U}$  be an open cover of  $X$ . Then for each  $k \in \mathbb{N}$ ,  $\mathcal{U}$  covers  $Y_k$ , and since  $Y_k$  is relatively **NSL**, there is a countable set  $F_k \subset X$ , such that for each open set  $O \supset F_k$ , we have  $\text{St}(O, \mathcal{U}) \supset Y_k$ . Put  $F = \bigcup_{k \in \mathbb{N}} F_k$ . Then  $F$  is a countable subset of  $X$ . Let  $O$  be any open set containing  $F$ . Using the fact that  $O$  contains all  $F_k$ ,  $k \in \mathbb{N}$ , we conclude that  $\text{St}(O, \mathcal{U}) \supset \bigcup_{k \in \mathbb{N}} Y_k = X$ , which means that  $X$  is **NSL**.

(2) Let  $\mathcal{U}$  be an open cover of  $X$ . Each  $Y_k$  is covered by  $\mathcal{U}$ . As  $Y_k$  is relatively **NSL** in  $X$ , for each  $k \in \mathbb{N}$ , there is countable  $F_k \subset X$  such that for each open  $O \subset X$  containing  $F_k$ ,  $Y_k \subset \text{St}(O, \mathcal{U})$ . Let  $F = \bigcup_{k \in \mathbb{N}} F_k$  and let  $G$  be an open set containing  $F$ . Then

$$X = \bigcup_{k \in \mathbb{N}} \overline{Y_k} \subset \overline{\text{St}(G, \mathcal{U})},$$

that is,  $X$  is **wNSL**.  $\square$

**Theorem 2.6.** *Let  $X$  be a **wNSL** topological space and let  $Y$  be a topological space. If  $f : X \rightarrow Y$  is a continuous surjection, then  $Y$  is a **wNSL**.*

*Proof.* Let  $\mathcal{V}$  be an open cover of  $Y$ . Then  $\mathcal{U} = f^{-1}(\mathcal{V}) = \{f^{-1}(V) : V \in \mathcal{V}\}$  is an open cover of  $X$ . Since  $X$  is **wNSL**, there is countable  $F \subset X$  such that for each open  $O$  containing  $F$  we have  $X = \overline{\text{St}(O, \mathcal{U})}$ . Let  $K = f(F)$  and let  $G$  be an open neighborhood of  $K$ . Then  $f^{-1}(G)$  is an open neighborhood of  $F$  such that  $X = \overline{\text{St}(f^{-1}(G), \mathcal{U})}$ . We prove  $Y = \overline{\text{St}(G, \mathcal{V})}$ .

Let  $y \in Y$  and let  $x \in X$  be such that  $y = f(x)$ . Then  $x \in \overline{\text{St}(f^{-1}(G), \mathcal{U})}$ . It follows,  $y = f(x) \in f(\overline{\text{St}(f^{-1}(G), \mathcal{U})}) \subset \overline{\text{St}(G, \mathcal{V})}$ . Therefore,  $K$  and  $G$  witness for  $\mathcal{V}$  that  $Y$  is **wNSL**.  $\square$

Recall that a space  $X$  is *para-Lindelöf* if every open cover  $\mathcal{U}$  of  $X$  has a locally countable open refinement.

**Theorem 2.7.** *Every para-Lindelöf wNSL space  $X$  is weakly Lindelöf.*

*Proof.* Let  $\mathcal{U}$  be an open cover of  $X$ . There exists a locally countable open refinement  $\mathcal{V}$  of  $\mathcal{U}$ . For each  $x \in X$ , there exists an open neighborhood  $W_x$  of  $x$  such that  $W_x \subset V$  for some  $V \in \mathcal{V}$  and  $\{V \in \mathcal{V} : W_x \cap V \neq \emptyset\}$  is countable. Let  $\mathcal{W} = \{W_x : x \in X\}$ . Then  $\mathcal{W}$  is an open refinement of  $\mathcal{V}$ . Since  $X$  is **wNSL**, there exists a countable subset  $A$  of  $X$  such that for every open  $O \supset A$ ,  $X = \overline{\text{St}(O, \mathcal{V})}$ . Especially, it is true for  $O_A = \bigcup \{W_x \in \mathcal{W} : x \in A\} \supset A$ , that is,  $\overline{\text{St}(O_A, \mathcal{V})} = X$ . Set  $\tilde{\mathcal{V}} = \{V \in \mathcal{V} : V \cap O_A \neq \emptyset\}$ . Then  $\tilde{\mathcal{V}}$  is a countable cover of  $X$ . For each  $V \in \tilde{\mathcal{V}}$ , choose  $U_V \in \mathcal{U}$  with  $V \subseteq U_V$ . Then  $\{U_V : V \in \tilde{\mathcal{V}}\}$  is a countable subcover of  $\mathcal{U}$ , and  $X = \overline{\bigcup_{V \in \tilde{\mathcal{V}}} U_V}$ , which shows that  $X$  is weakly Lindelöf.  $\square$

### 3. NEW SELECTION PRINCIPLES

In this section, we introduce weaker versions of NSM, NSR, and NSH spaces.

**Definition 3.1.** A space  $X$  is said to be:

- (1) *almost neighborhood star-Menger (aNSM)* (resp. *weakly neighborhood star-Menger (wNSM)*); *faintly neighborhood star-Menger (fNSM)*) if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , one can choose finite sets  $F_n \subset X$ ,  $n \in \mathbb{N}$ , such that for every open  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , we have  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(O_n, \mathcal{U}_n)} = X$  (resp.  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(O_n, \mathcal{U})} = X$ ;  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(O_n, \mathcal{U}_n)} = X$ ).
- (2) *almost neighborhood star-Rothberger (aNSR)* (resp. *weakly neighborhood star-Rothberger (wNSR)*); *faintly neighborhood star-Rothberger (fNSR)*) if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , one can choose the sequence  $(x_n : n \in \mathbb{N})$  of elements of  $X$  such that for every open  $O_n \ni x_n$ ,  $n \in \mathbb{N}$ , we have  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(O_n, \mathcal{U}_n)} = X$  (resp.  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(O_n, \mathcal{U}_n)} = X$ ;  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(O_n, \mathcal{U}_n)} = X$ ).
- (3) *almost neighborhood star-Hurewicz (aNSH)* (resp. *faintly neighborhood star-Hurewicz (fNSH)*) if for every sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , one can choose finite  $F_n \subset X$ ,  $n \in \mathbb{N}$ , such that for every open  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , each  $x \in X$  belongs to  $\overline{\text{St}(O_n, \mathcal{U}_n)}$  (resp. to  $\overline{\text{St}(O_n, \mathcal{U}_n)}$ ) for all but finitely many  $n$ .

*Remark 3.2.* Of course, every NSM space is both aNSM and fNSM, and every aNSM space is wNSM, and similarly, for Rothberger-type properties. Also, every

NSH space is aNSH and fNSH. A regular space  $X$  is fNSM if and only if it is aNSM, and similarly for Rothberger and Hurewicz cases.

$$\begin{array}{ccccccccc}
R & \Longrightarrow & SSR & \Longrightarrow & NSR & \Longrightarrow & \text{aNSR} & \Longrightarrow & \text{wNSR} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
M & \Longrightarrow & SSM & \Longrightarrow & NSM & \Longrightarrow & \text{aNSM} & \Longrightarrow & \text{wNSM} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \\
H & \Longrightarrow & SSH & \Longrightarrow & NSH & \Longrightarrow & \text{aNSH} & & 
\end{array}$$

Diagram 1

Observe also that every wNSM space is wNSL.

$$\begin{array}{ccccc}
\text{NSM} & \Longrightarrow & \text{aNSM} & \Longrightarrow & \text{wNSM} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{NSL} & \Longrightarrow & \text{aNSL} & \Longrightarrow & \text{wNSL}
\end{array}$$

Diagram 2

Diagrams 1 and 2 give relations among classes of spaces defined above.

**Example 3.3.** (1) There is a wNSM space that is not NSM.

Let  $\mathbb{P}$  be the space of irrational numbers with the usual metric topology, let  $[0, \omega]$  be the ordinal space, let  $\mathbb{Q}$  be the set of rational numbers, let  $A = \pi + \mathbb{Q} = \{\pi + q : q \in \mathbb{Q}\}$ , and let  $X = (\mathbb{P} \times [0, \omega]) \setminus (\mathbb{P} \setminus A) \times \{\omega\}$  equipped with the following topology  $\tau$ : A basic neighborhood of a point  $\langle x, n \rangle$ ,  $x \in \mathbb{P}$ ,  $n < \omega$ , is of the form  $U \times \{n\}$ , where  $U$  is a neighborhood of  $x \in \mathbb{P}$ , while a basic neighborhood of a point  $\langle y, \omega \rangle$ ,  $y \in A$ , is of the form  $U \times (n, \omega) \cup \{\langle y, \omega \rangle\}$ .

Notice that the space  $X$  is separable. Let  $D = \{d_1, d_2, \dots\}$  be a countable dense subset of  $X$ .

**Claim 1.**  $X$  is wNSM.

Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$ . Consider the sequence  $(D_n : n \in \mathbb{N})$ , where  $D_n = \{d_i : i \leq n\}$ , of finite subsets of  $X$  and a sequence  $(O_n : n \in \mathbb{N})$  of neighborhoods of  $D_n$ ,  $n \in \mathbb{N}$ . Then  $\overline{\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n)} \supset \overline{D} = X$ , that is,  $X$  is wNSM.

**Claim 2.**  $X$  is not NSM.

The space  $X$  contains a copy of the space  $\mathbb{P}$  as a closed subspace. Thus  $X$  cannot be Menger, because  $\mathbb{P}$  is not Menger, and the Menger property is preserved by closed subspaces. On the other hand, in the class of paracompact spaces, the Menger property coincides with the NSM property.

(2) The Sorgenfrey line [6] is an aNSR space that is not Rothberger (it is not Menger).

**Definition 3.4** (see [9]). We say that a topological space  $X$  is  $d$ -paracompact if every dense family of subsets of  $X$  has a locally finite refinement.

**Theorem 3.5.** *If a topological space  $X$  is wNSM and  $d$ -paracompact, then  $X$  is aNSM.*

*Proof.* Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$ . Since  $X$  is wNSM, one can choose finite  $F_n \subset X$ , such that for every open  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , we have  $\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n)$  is dense in  $X$ . By the assumption,  $\{\text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$  has a locally finite refinement  $\mathcal{W}$ . Then  $\bigcup \mathcal{W} = (\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n))$  and therefore  $\overline{\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n)} = \overline{\bigcup \mathcal{W}}$ , that is,  $\bigcup \mathcal{W}$  is dense in  $X$ . As  $\mathcal{W}$  is a locally finite family, we have that  $\overline{\bigcup \mathcal{W}} = \bigcup_{W \in \mathcal{W}} \overline{W}$ .

Since for every  $W \in \mathcal{W}$ , there is  $k = k(W) \in \mathbb{N}$  such that  $W \subset \text{St}(O_k, \mathcal{U}_k)$ . Therefore, it follows  $X = \bigcup_{n \in \mathbb{N}} \overline{\text{St}(O_n, \mathcal{U}_n)}$ , that is,  $X$  is aNSM.  $\square$

**Theorem 3.6.** *Let  $X$  be an aNSM space and let  $Y$  be a topological space. If  $f : X \rightarrow Y$  is a continuous surjection, then  $Y$  is also an aNSM space.*

*Proof.* Let  $(\mathcal{V}_n : n \in \mathbb{N})$  be a sequence of open covers of  $Y$ . For each  $n \in \mathbb{N}$ , the set  $\mathcal{U}_n := \{f^{-1}(V) : V \in \mathcal{V}_n\}$  is an open cover of  $X$ . Since  $X$  is aNSM, there are finite sets  $F_n \subset X$ ,  $n \in \mathbb{N}$ , such that for every open  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , we have  $\{\overline{\text{St}(O_n, \mathcal{U}_n)} : n \in \mathbb{N}\}$  is a cover of  $X$ . The sets  $f(F_n)$ ,  $n \in \mathbb{N}$ , are finite in  $Y$ . For each  $n$ , let  $G_n$  be an open neighborhood of  $f(F_n)$ . Then  $f^{-1}(G_n) = H_n$  is an open subset of  $X$  for each  $n \in \mathbb{N}$  and  $H_n \supset F_n$ . Thus  $X = \bigcup_{n \in \mathbb{N}} \overline{\text{St}(H_n, \mathcal{U}_n)}$ . We prove that  $Y = \bigcup_{n \in \mathbb{N}} \overline{\text{St}(G_n, \mathcal{V}_n)}$ .

Let  $y \in Y$  and let  $x \in X$  be such that  $y = f(x)$ . Then there is  $k \in \mathbb{N}$  such that  $x \in \overline{\text{St}(H_k, \mathcal{U}_k)}$ . Then  $y = f(x) \in \overline{f(\text{St}(H_k, \mathcal{U}_k))}$ . Because  $f(\text{St}(H_k, \mathcal{U}_k)) \subset \overline{f(\text{St}(f^{-1}(G_k), \mathcal{U}_k))} \subset \overline{\text{St}(G_k, \mathcal{V}_k)}$ , we have  $y \in \overline{\text{St}(G_k, \mathcal{V}_k)}$ . Hence,  $Y = \bigcup_{n \in \mathbb{N}} \overline{\text{St}(G_n, \mathcal{V}_n)}$ , that is,  $Y$  is aNSM.  $\square$

**Definition 3.7** (see [6]). We say that a subset  $U$  of a space  $X$  is *regular open* (regular closed) if  $U = \text{int}(\overline{U})$  ( $U = \overline{\text{int}(U)}$ ).

**Theorem 3.8.** *A space  $X$  is a wNSM if and only if for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of covers of  $X$  by regular open sets, there exist finite sets  $F_n \subset X$ ,  $n \in \mathbb{N}$ , such that for every open  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , it holds  $\overline{\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n)} = X$ .*

*Proof.* ( $\Rightarrow$ ): It is obvious.

( $\Leftarrow$ ): Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$ . Putting  $\mathcal{V}_n := \{\text{int}(\overline{U}) : U \in \mathcal{U}_n\}$ ,  $n \in \mathbb{N}$ , we obtain a sequence  $(\mathcal{V}_n : n \in \mathbb{N})$  of covers of  $X$  by regular open sets. Then there exist finite  $F_n \subset X$ ,  $n \in \mathbb{N}$ , such that for every open  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , we have  $\overline{\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{V}_n)} = X$ . For every  $n \in \mathbb{N}$  and every  $V \in \mathcal{V}_n$ , there exists  $U_V \in \mathcal{U}_n$  such that  $V = \text{int}(\overline{U_V})$ . Consider the sequence  $(\mathcal{W}_n : n \in \mathbb{N})$ , where  $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_n\}$ . We claim that  $\overline{\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n)} = X$ .

Let  $x \in X$  and let  $G$  be a neighborhood of  $x$ . There exist  $k \in \mathbb{N}$  and  $V \in \mathcal{V}_k$  such that  $G \cap V \neq \emptyset$  and  $V \cap O_k \neq \emptyset$ , that is, there is  $U = U_V \in \mathcal{U}_k$  such that  $G \cap \text{int}(\overline{U}) \neq \emptyset$  and  $O_k \cap \text{int}(\overline{U}) \neq \emptyset$ . Then  $G \cap U \neq \emptyset$  and  $O_k \cap U \neq \emptyset$ . [Let us prove  $G \cap U \neq \emptyset$ . Let  $y \in G \cap \text{int}(\overline{U}) \neq \emptyset$ . There is an open set  $H$  containing  $y$  such that  $H \subset \overline{U}$ , hence the neighborhood  $G \cap H$  of  $y$  intersects  $U$ . It follows  $G \cap U \neq \emptyset$ .] Therefore,  $x \in \overline{\bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n)}$ , that is,  $X$  is wNSM.  $\square$

**Theorem 3.9.** *If  $X$  is an aNSM space, then every clopen subset of  $X$  is aNSM.*

*Proof.* Let  $(Y, \tau_Y)$  be a clopen subset of an aNSM space  $(X, \tau)$  and let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $(Y, \tau_Y)$ . As  $Y$  is clopen,  $\mathcal{V}_n = \mathcal{U}_n \cup (X \setminus Y)$  is an

open cover of  $X$  for every  $n \in \mathbb{N}$ . Since  $X$  is aNSM, one can choose finite  $F_n \subset X$ , such that for every open (in  $X$ )  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , we have  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(O_n, \mathcal{V}_n)} = X$ .

Define now  $H_n = Y \cap F_n$  if  $Y \cap F_n \neq \emptyset$ ; and  $H_n$  any finite subset of  $Y$ , otherwise. [Note that not all  $Y \cap F_n$  can be empty, because otherwise  $X \setminus Y$  is a neighborhood of all  $F_n$  intersecting only  $X \setminus Y$  from  $\mathcal{V}_n$ .]

We claim that  $(H_n : n \in \mathbb{N})$  witnesses for  $(\mathcal{U}_n : n \in \mathbb{N})$  that  $Y$  is aNSM.

Let  $G_n$  be an open set in  $(Y, \tau_Y)$  containing  $H_n$ ,  $n \in \mathbb{N}$ . Then  $W_n = G_n \cup (X \setminus Y)$  is an open set in  $X$  containing  $F_n$ ,  $n \in \mathbb{N}$  and thus  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(W_n, \mathcal{V}_n)} = X$ . Because  $Y$  is closed in  $X$  and  $H_n \cap (X \setminus Y) = \emptyset$ , we conclude  $\bigcup_{n \in \mathbb{N}} \overline{\text{St}(H_n, \mathcal{U}_n)} = Y$ , which means that  $(Y, \tau_Y)$  is aNSM.  $\square$

**Definition 3.10** (see [4]). Let  $Y$  be a subset of a space  $X$ . Then

- (1)  $Y$  is *relatively NSM* (resp. *relatively NSH*) in  $X$  if for each  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , one can choose a sequence  $(A_n : n \in \mathbb{N})$  of finite subsets of  $X$ , such that for every open  $O_n \supset A_n$ ,  $n \in \mathbb{N}$ , we have  $\bigcup \{ \text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N} \} \supset Y$  (resp. for each  $y \in Y$ ,  $y \in \text{St}(O_n, \mathcal{U}_n)$  for all but finitely many  $n$ ).
- (2)  $Y$  is *relatively NSR* in  $X$  if for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , there are  $x_n \in X$ ,  $n \in \mathbb{N}$ , such that for all open  $O_n \ni x_n$ ,  $n \in \mathbb{N}$ , we have  $\bigcup \{ \text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N} \} \supset Y$ .

**Proposition 3.11.** *The following statements hold:*

- (1) *If  $X = \bigcup \{ Y_k : k \in \mathbb{N} \}$ , and each  $Y_k$  is relatively NSM (resp. relatively NSH, relatively NSR) in  $X$ , then  $X$  is NSM (resp. NSH, NSR);*
- (2) *If  $X = \bigcup \{ \overline{Y_k} : k \in \mathbb{N} \}$ , and each  $Y_k$  is relatively NSM (resp. relatively NSH, relatively NSR) in  $X$ , then  $X$  is aNSM (resp. aNSH, aNSR).*

*Proof.* (1) Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$ . Then for each  $k, n \in \mathbb{N}$ ,  $\mathcal{U}_n$  covers  $Y_k$ , and since  $Y_k$  is relatively NSM, there are countable sets  $F_{k,n} \subset X$ ,  $n \in \mathbb{N}$ , such that for each open set  $O_{k,n} \supset F_{k,n}$ , we have  $\text{St}(O_{k,n}, \mathcal{U}_n) \supset Y_k$ . Consider the sequences  $(F_{k,n} : k, n \in \mathbb{N})$  and  $(G_{k,n} : k, n \in \mathbb{N})$  of neighborhoods of  $F'_{k,n}$ s. It is easy to conclude that  $\bigcup_{k \in \mathbb{N}} \text{St}(G_{k,n}, \mathcal{U}_n) \supset \bigcup_{k \in \mathbb{N}} Y_k = X$ , which means that  $X$  is NSM.

(2) Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$ . Each  $Y_k$  is covered by  $\mathcal{U}_n$ . As  $Y_k$  is relatively NSM in  $X$ , for each  $k \in \mathbb{N}$ , there is a sequence  $(F_{k,n} : n \in \mathbb{N})$  of finite subsets of  $X$  such that for all open  $O_{k,n} \supset F_{k,n}$ ,  $Y_k \subset \bigcup_{n \in \mathbb{N}} \text{St}(O_{k,n}, \mathcal{U}_n)$ . Then

$$X = \bigcup_{k \in \mathbb{N}} \overline{Y_k} \subset \overline{\bigcup_{k \in \mathbb{N}} \text{St}(O_{k,n}, \mathcal{U}_n)},$$

that is,  $X$  is wNSM.  $\square$

**Proposition 3.12.** *If  $X = \bigcup \{ Y_k : k \in \mathbb{N} \}$  and each  $Y_k$  is wNSM (wNSR) in  $X$ , then  $X$  is aNSM (aNSR).*

*Proof.* We shall prove the NSM case. Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$ . Rearrange this sequence to  $(\mathcal{U}_{k,m} : k, m \in \mathbb{N})$ . For each  $k \in \mathbb{N}$ ,  $(\mathcal{U}_{k,m} : m \in \mathbb{N})$  is a sequence of covers of  $Y_k$  by sets open in  $X$ . For each  $k$ ,



$Y_k$  is wNSM, and thus there are finite sets  $F_{k,m} \subset X$ ,  $m \in \mathbb{N}$ , such that for every open  $O_{k,m} \supset F_{k,m}$ ,  $m \in \mathbb{N}$ , we have  $\bigcup_{m \in \mathbb{N}} \overline{\text{St}(O_{k,m}, \mathcal{U}_{k,m})} \supset Y_k$ . By the assumption,  $X = \bigcup_{k \in \mathbb{N}} Y_k$ . It follows  $X = \bigcup_{k \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \overline{\text{St}(O_{k,m}, \mathcal{U}_{k,m})}$ , that is,  $X$  is aNSM.  $\square$

**Lemma 3.13** (see [26]). *Let  $A$  be a regular closed subset of a space  $X$  and let  $B$  be a dense subset of  $X$ . Then  $B \cap A$  is a dense subset of  $A$ .*

**Lemma 3.14** (see [2]). (1) *Let  $X$  be a topological space. If  $Y \subset X$  is dense in  $X$ , and  $D \subset Y$  is dense in  $Y$ , then  $D$  is dense in  $X$ ;*  
 (2) *If  $D \subset X$  is dense in  $X$  and  $E \subset Y$  is dense in  $Y$ , then  $D \times E$  is dense in  $X \times Y$ .*

**Proposition 3.15.** *The following properties hold:*

1. *Let  $F$  be a regular closed subset of a space  $X$  and let  $Y$  be a wNSM (wNSR) in  $X$ . Then  $Y \cap F$  is wNSM (wNSR) in  $F$ .*
2. *Let  $A \subset X$  be wNSM (wNSR) in  $X$  and let  $B \subset A$  be wNSM (wNSR) in  $A$ . Then  $B$  is wNSM (wNSR) in  $X$ .*
3. *If  $D \subset X$  is wNSM (wNSR) in  $X$  and  $E \subset Y$  is wNSM (wNSR) in  $Y$ , then  $D \times E$  is wNSM (wNSR) in  $X \times Y$ .*

*Proof.* By Lemmas 3.13 and 3.14, the proof is evident.  $\square$

**Lemma 3.16** (see [10]). *If  $\mathcal{U}$  is an  $\omega$ -cover of a space  $X$ , then  $\{U^2 : U \in \mathcal{U}\}$  is an  $\omega$ -cover of  $X^2$ .*

An open cover  $\mathcal{U}$  of a space  $X$  is said to be an *almost  $\omega$ -cover* if each finite  $F \subset X$  is contained in  $\overline{U}$  for some  $U \in \mathcal{U}$ .

Denote by  $\text{aNSM}(\mathcal{O}, \Omega)$  the following statement on a space  $X$ : For each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ , there are finite sets  $F_n \subset X$ ,  $n \in \mathbb{N}$ , such that for each sequence  $(O_n : n \in \mathbb{N})$  of open sets with  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ ,  $\{\text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$  is an almost  $\omega$ -cover of  $X$ .

**Theorem 3.17.** *If all finite powers of a space  $X$  are aNSM, then  $X$  satisfies  $\text{aNSM}(\mathcal{O}, \Omega)$ .*

*Proof.* Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$  and let  $\mathbb{N} = N_1 \cup N_2 \cup \dots$  be a partition of  $\mathbb{N}$  into infinite pairwise disjoint sets. For every  $k \in \mathbb{N}$  and every  $m \in N_k$ , let  $\mathcal{W}_m = (\mathcal{U}_m)^k = \{U^k : U \in \mathcal{U}_m\}$ . Then  $(\mathcal{W}_m : m \in N_k)$  is a sequence of open covers of  $X^k$ . Applying to this sequence the fact that  $X^k$  is aNSM, we find a sequence  $(A_m : m \in N_k)$  of finite subsets of  $X^k$  such that for every open sequence  $(O_m : m \in N_k)$  of neighborhoods of  $A_m$ ,  $m \in N_k$ ,  $X^k = \bigcup_{m \in N_k} \overline{\text{St}(O_m, \mathcal{W}_m)}$ . For every  $m \in N_k$ , let  $B_m$  be a finite subset of  $X$  such that  $B_m^k \supset A_m$ . Consider the sequence of all  $B_m, m \in N_k, k \in \mathbb{N}$ , chosen in this way; denote it as  $(B_n : n \in \mathbb{N})$ . Let  $(H_n : n \in \mathbb{N})$  be a sequence of neighborhoods of  $B_n$ ,  $n \in \mathbb{N}$ . We claim that  $\{\overline{\text{St}(H_n, \mathcal{U}_n)} : n \in \mathbb{N}\}$  is an  $\omega$ -cover of  $X$ . Let  $F = \{x_1, \dots, x_p\}$  be a finite subset of  $X$ . Then  $\langle x_1, \dots, x_p \rangle \in X^p$ . There exists  $n \in N_p$  such that  $(H_n^p : n \in \mathbb{N})$  is a sequence of neighborhoods of  $A_n$ . Hence, there exists  $n \in \mathbb{N}$  such that  $\langle x_1, \dots, x_p \rangle \in \overline{\text{St}(H_n^p, \mathcal{W}_n)}$ . Therefore we have  $F \subset \overline{\text{St}(H_n, \mathcal{U}_n)}$ , that is,  $X$  satisfies  $\text{aNSM}(\mathcal{O}, \Omega)$ .  $\square$



The symbol  $\mathcal{O}^{\text{awgp}}$  denotes the collection of almost weakly groupable covers of a space. A countable open cover  $\mathcal{U}$  of a space  $X$  is said to be *almost weakly groupable* if there is a partition  $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$  of  $\mathcal{U}$  into finite, pairwise disjoint subcollections, such that for each finite subset  $F$  of  $X$  there is  $n \in \mathbb{N}$  with  $F \subset \overline{\bigcup \mathcal{U}_n}$ .

**Theorem 3.18.** *For a space  $X$ , the following statements are equivalent:*

- (1)  $X$  satisfies  $\text{aNSM}(\mathcal{O}, \Omega)$ ;
- (2)  $X$  satisfies  $\text{NSM}(\mathcal{O}, \mathcal{O}^{\text{awgp}})$ .

*Proof.* (1)  $\Rightarrow$  (2) This implication directly follows from the facts that the selection principle  $\text{aNSM}$  is monotone in the second coordinate and each countable almost  $\omega$ -cover is almost weakly groupable.

(2)  $\Rightarrow$  (1) Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $X$ . For each  $n \in \mathbb{N}$ ,  $\mathcal{V}_n = \bigwedge_{i \leq n} \mathcal{U}_i$ . For each  $n$ ,  $\mathcal{V}_n$  is an open cover of  $X$  that refines  $\mathcal{U}_i$  for all  $i \leq n$ . By applying (2) to the sequence  $(\mathcal{V}_n : n \in \mathbb{N})$ , there is a sequence  $A_n, n \in \mathbb{N}$ , of finite subsets of  $X$  such that for every sequence  $(O_n : n \in \mathbb{N})$  of open neighborhoods of  $A_n, n \in \mathbb{N}$ , the cover  $\{\text{St}(O_n, \mathcal{V}_n)\}$  is an almost weakly groupable cover of  $X$ . In other words, there is an increasing sequence  $n_1 < n_2 < \dots < n_k < \dots$  of natural numbers such that for each finite set  $F$  in  $X$ , one has

$$F \subset \overline{\bigcup \{\text{St}(O_i, \mathcal{V}_i) : n_k < i \leq n_{k+1}\}}$$

for some  $k$ . Define now

$$B_n = \bigcup_{i < n} A_i, \text{ for } n < n_1,$$

$$B_n = \bigcup_{n_k < i \leq n_{k+1}} A_i, \text{ for } n_k < n \leq n_{k+1}.$$

Each  $B_n$  is a finite subset of  $X$ . For each  $n$ , take a neighborhood  $H_n$  of  $B_n$ . Then, by the construction of sets  $B_n$  and covers  $\mathcal{V}_n$ , we easily conclude that  $\{\text{St}(H_n, \mathcal{U}_n) : n \in \mathbb{N}\}$  is an almost  $\omega$ -cover of  $X$ , that is,  $X$  satisfies  $\text{aNSM}(\mathcal{O}, \Omega)$ .  $\square$

#### 4. MORE ON SPACES AND MAPPINGS

In this section, we investigate the preservation of the properties, which we consider it in this article under some kinds of mappings.

**Theorem 4.1.** *Let  $X$  be an  $\text{aNSH}$  space and let  $Y$  be a topological space. If  $f : X \rightarrow Y$  is a continuous mapping from  $X$  onto  $Y$ , then  $Y$  is also an  $\text{aNSH}$  space.*

*Proof.* Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of open covers of  $Y$ . For each  $n \in \mathbb{N}$ , the set  $\mathcal{U}'_n := \{f^{-1}(U) : U \in \mathcal{U}_n\}$  is an open cover of  $X$ . Since  $X$  is  $\text{aNSH}$ , there are finite sets  $(F_n \subset X), n \in \mathbb{N}$ , such that for every open set  $O_n \supset F_n, n \in \mathbb{N}$ , and each  $x \in X$ , we have  $x \in \overline{\text{St}(O_n, \mathcal{U}'_n)}$  for all but finitely many  $n$ . The sets  $f(F_n), n \in \mathbb{N}$ , are finite in  $Y$ . Let  $G_n \supset f(F_n)$  for each  $n$  be an open set in  $Y$ . Then  $f^{-1}(G_n) = H_n$  is an open subset of  $X$  for each  $n \in \mathbb{N}$  and  $H_n \supset F_n$ . Thus for

each  $x \in X$ , we have  $x \in \overline{\text{St}(H_n, \mathcal{U}_n)}$  for all but finitely many  $n$ . We prove that for each  $y \in Y$ ,  $y \in \overline{\text{St}(G_n, \mathcal{U}_n)}$  for all but finitely many  $n$ .

Let  $y \in Y$  and let  $x \in X$  be such that  $y = f(x)$ . Then there is  $k_0 \in \mathbb{N}$  such that  $x \in \overline{\text{St}(H_k, \mathcal{U}'_k)}$  for each  $k \geq k_0$ . Then  $y = f(x) \in \overline{f(\text{St}(H_k, \mathcal{U}'_k))}$ . Because  $f(\text{St}(H_k, \mathcal{U}'_k)) \subset f(\text{St}(f^{-1}(G_k), \mathcal{U}'_k)) \subset \text{St}(G_k, \mathcal{U}_k)$ , we have  $y \in \overline{\text{St}(G_k, \mathcal{U}_k)}$  for all  $k \geq k_0$ , that is,  $Y$  is aNSH.  $\square$

A mapping  $f : X \rightarrow Y$  is *weakly continuous* [16] (resp.  $\theta$ -*continuous* [7], *strongly  $\theta$ -continuous* [17]) if for each  $x \in X$  and each open neighborhood  $V$  of  $f(x)$ , there is an open neighborhood  $U$  of  $x$  such that  $f(U) \subset \overline{V}$  (resp.  $f(\overline{U}) \subset \overline{V}$ ,  $f(\overline{U}) \subset V$ ).

**Theorem 4.2.** *A  $\theta$ -continuous image  $Y = f(X)$  of an fNSR space  $X$  is also fNSR.*

*Proof.* Let  $(\mathcal{V}_n : n \in \mathbb{N})$  be a sequence of open covers of  $Y$ . Fix  $x \in X$ . For each  $n \in \mathbb{N}$ , pick a set  $V(x, n) \in \mathcal{V}_n$  with  $f(x) \in V(x, n)$ . Using the fact that  $f$  is  $\theta$ -continuous, take an open set  $U(x, n) \subset X$  such that  $x \in U(x, n)$  and  $f(\overline{U(x, n)}) \subset \overline{V(x, n)}$ . So, for each  $n$ , the set  $\mathcal{U}_n := \{U(x, n) : x \in X\}$  is an open cover of  $X$ . Since  $X$  is fNSR, there is a sequence  $(a_n : n \in \mathbb{N})$  of points in  $X$  such that for any sequence  $(S_n : n \in \mathbb{N})$  of neighborhoods of  $a_n$ ,  $\{\text{St}(\overline{S_n}, \mathcal{U}_n) : n \in \mathbb{N}\}$  is an open cover of  $X$ .

Consider the sequence  $(b_n = f(a_n) : n \in \mathbb{N})$  of points in  $Y$  and a sequence  $(T_n : n \in \mathbb{N})$  of neighborhoods of  $b_n$ ,  $n \in \mathbb{N}$ . For each  $n$ , there exists an open set  $O_n \ni a_n$  such that  $f(\overline{O_n}) \subset \overline{T_n}$ . Then  $X = \bigcup_{n \in \mathbb{N}} \text{St}(\overline{O_n}, \mathcal{U}_n)$  implies  $f(X) = Y = \bigcup_{n \in \mathbb{N}} \text{St}(\overline{T_n}, \mathcal{V}_n)$ ; it is not hard to check. It follows that  $Y$  is an fNSR space.  $\square$

Similarly, we can prove the following.

**Theorem 4.3.** *A  $\theta$ -continuous image  $Y = f(X)$  of an fNSM (resp. fNSH) space  $X$  is also fNSM (resp. fNSH).*

The following results show the relationships between NSM (resp. NSH, NSR) spaces and fNSM (resp. fNSH, fNSR) spaces.

**Theorem 4.4.** *If a space  $Y$  is a weakly continuous image of an NSM space  $X$ , then  $Y$  is fNSM.*

*Proof.* Let  $f : X \rightarrow Y$  be a weakly continuous mapping and let  $(\mathcal{V}_n : n \in \mathbb{N})$  be a sequence of open covers of  $Y$ . For each  $x \in X$  and each  $n \in \mathbb{N}$ , there is  $V(f(x), n) \in \mathcal{V}_n$  containing  $f(x)$ . Because  $f$  is weakly continuous, pick an open set  $U(x, n) \subset X$  that contains  $x$  with  $f(U(x, n)) \subset \overline{V(f(x), n)}$ . In this way, one obtains for each  $n \in \mathbb{N}$ , an open cover  $\mathcal{U}_n = \{U(x, n) : x \in X\}$  of  $X$ . As  $X$  is NSM, there is a sequence  $(A_n : n \in \mathbb{N})$  of finite subsets of  $X$  such that for any open sets  $G_n \supset A_n$ ,  $n \in \mathbb{N}$ ,  $X = \bigcup_{n \in \mathbb{N}} \text{St}(G_n, \mathcal{U}_n)$ . Put  $B_n = f(A_n)$ ,  $n \in \mathbb{N}$ . We have the sequence  $(B_n : n \in \mathbb{N})$  of finite subsets of  $Y$ . We prove that this sequence witnesses for  $(\mathcal{V}_n : n \in \mathbb{N})$  that  $Y$  is fNSM.

For each  $n \in \mathbb{N}$ , take an arbitrary neighborhood  $H_n$  of  $B_n$ . Since  $A_n \subset X$  is finite and  $f$  is weakly continuous, there is a neighborhood  $O_n$  of  $A_n$  such that

$f(O_n) \subset \overline{H_n}$ . It is easy now to prove that from the construction of the sequences  $\mathcal{U}_n$  and  $X = \bigcup_{n \in \mathbb{N}} \text{St}(O_n, \mathcal{U}_n)$ , it follows  $Y = \bigcup_{n \in \mathbb{N}} \text{St}(\overline{H_n}, \mathcal{V}_n)$ . This shows that  $Y$  is an fNSM space.  $\square$

Quite similarly one can prove the following.

**Theorem 4.5.** *If  $Y = f(X)$  is a weakly continuous image of an NSH (resp. NSR) space  $X$ , then  $Y$  is fNSH (resp. fNSR).*

**Theorem 4.6.** *A strongly  $\theta$ -continuous image  $Y = f(X)$  of an fNSM (resp. fNSH, fNSR) space  $X$  is an NSM (resp. NSH, NSR) space.*

*Proof.* (for the Rothberger case; other two cases are proved similarly) Let  $(\mathcal{V}_n : n \in \mathbb{N})$  be a sequence of open covers of  $Y$ . Since each  $\mathcal{V}_n$  covers  $X$ , for each  $x \in X$  and each  $n \in \mathbb{N}$ , one can find a set  $V_x^n$  that contains  $f(x)$ . The strong  $\theta$ -continuity of  $f$  implies the existence of an open neighborhood  $U_x^n$  of  $x$  with  $f(\overline{U_x^n}) \subset V_x^n$ . Letting  $\mathcal{U}_n := \{U_x^n : x \in X\}$ , we get a sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of open covers of  $X$ . There are points  $p_1, p_2, \dots$  in  $X$  such that for arbitrary open sets  $G_1 \ni p_1, G_2 \ni p_2, \dots$ ,  $X = \bigcup_{n \in \mathbb{N}} \text{St}(\overline{G_n}, \mathcal{U}_n)$ .

Set  $q_n = f(p_n)$ ,  $n \in \mathbb{N}$ , and take for each  $n$ , an open set  $H_n \ni q_n$ . Next, for each  $n$ , take an open set  $O_n \ni p_n$  such that  $f(\overline{O_n}) \subset H_n$ . Then  $X = \bigcup_{n \in \mathbb{N}} \text{St}(\overline{O_n}, \mathcal{U}_n)$ . Let  $y \in Y$  and let  $x \in X$  be such that  $y = f(x)$ . There is  $k \in \mathbb{N}$  such that  $x \in \text{St}(\overline{O_k}, \mathcal{U}_k)$ . It is easily verified that  $y \in \text{St}(H_k, \mathcal{V}_k)$ . In other words,  $Y = \bigcup_{n \in \mathbb{N}} \text{St}(H_n, \mathcal{V}_n)$ , that is,  $Y$  is an NSR space.  $\square$

## 5. CONCLUSION

New classes of spaces related to star versions of the classical Menger, Hurewicz, and Rothberger covering properties have been considered. This idea may be further applied to bitopological spaces and we hope it can open a new research direction. Here is a short explanation for the NSM case. A bitopological space  $(X, \tau_1, \tau_2)$  is a  $(\tau_i, \tau_j)$ -NSM-space,  $i, j = 1, 2$ , if for each sequence  $(\mathcal{U}_n : n \in \mathbb{N})$  of  $\tau_i$ -open covers of  $X$ , there is a sequence  $(F_n : n \in \mathbb{N})$  of finite subsets of  $X$  such that for any  $\tau_j$ -open sets  $O_n \supset F_n$ ,  $n \in \mathbb{N}$ , the set  $\{\text{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$  is a cover of  $X$ . It is worth also to study the infinitely long two-person game associated to NSM. Similarly, we define bitopological versions of other properties studied in this article.

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