J. Appl. Comput. Mech., xx(x) (2021) 1-8
DOI: 10.22055/JACM.2021.35369.2642

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

Solving Duffing-Van der Pol Oscillator Equations of Fractional Order by an Accurate Technique

Nourhane Attia[1], Djamila Seba[2], Ali Akgül[3], Abdelkader Nour[4]

[1] Dynamic of Engines and Vibroacoustic Laboratory, University M'hamed Bougara of Boumerdes, Boumerdes, 35000, Algeria, Email: n.attia@univ-boumerdes.dz

[2] Dynamic of Engines and Vibroacoustic Laboratory, University M'hamed Bougara of Boumerdes, Boumerdes, 35000, Algeria, Email: seba@univ-boumerdes.dz

[3] Art and Science Faculty, Department of Mathematics, Siirt University, Siirt, TR-56100, Turkey, Email: aliakgul00727@gmail.com

[4] Dynamic of Engines and Vibroacoustic Laboratory, University M'hamed Bougara of Boumerdes, Boumerdes, 35000, Algeria, Email: abdelkader_nour@hotmail.fr

Received October 10 2020; Revised February 09 2021; Accepted for publication February 09 2021.
Corresponding author: A. Akgül (aliakgul00727@gmail.com)

Abstract. In this paper, an accurate technique is used to find an approximate solution to the fractional-order Duffing-Van der Pol (DVP, for short) oscillators equation which is reproducing kernel Hilbert space (RKHS, for short) method. The numerical results show that the n-term approximation is a rapidly convergent series representation and they present also the high accuracy and effectiveness of this method. The efficiency of the proposed method has been proved by the theoretical predictions and confirmed by the numerical experiments.

Keywords: Duffing-Van der Pol oscillator equations of fractional order; Method of reproducing kernel Hilbert space; Caputo derivative; Convergence; Numerical Solutions.

1. Introduction

The theory of integrations and derivations of arbitrary order is well known, nowadays, as fractional calculus (FC, for short). The FC concept aims to generalize the notion of integer order differentiation and n-fold integration to the non-integer order and we call it generalized differintegrals. Due to its wide applications in physics, dynamics, viscoelasticity, engineering, control theory, economics, and many other areas, fractional calculus attracted many researchers [23, 1, 16, 8].

Since finding exact solutions to problems described by fractional-order equations are sometimes too complicated, many analytical and numerical methods have been developed for this purpose, including spectral Tau method [6], finite difference method [24], finite element method [27], fractional-order Wavelet method [18]. For further applications of numerical methods, we refer to other recent studies [12, 20, 25, 19, 14, 15, 4, 7, 11].

The DVP oscillator equation of fractional order is considered among the most intensively used in the design of various systems, from dynamic, biology, seismology, and so on. This equation, in the classical derivative case, can be considered as the generalization of the classic van der Pol oscillator equation [17, 26, 13]. As is well-known, it is somehow difficult to obtain a closed-form numerical solution to the fractional DVP oscillator equation due to the complexities that occur in the fractional derivatives. Various techniques have been used to solve this kind of oscillator equations like the decomposition method of Adomian [2], homotopy perturbation [22], and Runge-Kutta methods. Despite the development of the analytical and numerical methods to handle such type of equations, the challenge is still on-going to solve them using new numerical methods that can be easily implemented. In this orientation, the basic motivation of this paper is to extend the application of the RKHS method to conduct a numerical investigation for the emerging problems modeled from nonlinear fractional oscillation phenomena.

In this research, the reproducing kernel Hilbert space method will be used to find a numerical approach of the DVP oscillator equation in which we will discuss both theoretical predictions and numerical results in Hilbert space. Applying the RKHS method is mainly based on the orthogonal basis system and the structures of the used reproducing kernel functions. This technique has been developed by many researchers to find the exact or approximate solutions to a linear or nonlinear oscillatory problem. RKHS method has many advantages compared with the classical methods. It provides rapidly convergent solutions and gives approximate solutions with higher precision [3, 28, 21].

We organize the paper as follows: The second section of this paper presents some basic concepts and definitions. The third section gives the DVP oscillator equation. In this section, the RKHS method has been applied to the DVP oscillator equations too. The fourth section provides two illustrative examples to show the method's effectiveness and the accuracy of the obtained solutions. The conclusion has been presented in the last section.

Published online: February 18 2021

2. Some Preliminary Results

Here, we recall basic definitions from fractional calculus. We have provided some important concepts, properties, and theorems of the general theory of reproducing kernel that we will use in this study.

Definition 2.1. *A function* $K:\mathcal{X}\times\mathcal{X}\to\mathbb{C}$ *is called a reproducing kernel for* H *provided* [28]

1. $K_\eta(\cdot)\in H,\ \forall\eta\in\mathcal{X}.$
2. $\langle\phi,K_\eta(\cdot)\rangle=\phi(\eta)$ for all $\phi\in H,\eta\in\mathcal{X}.$

In which H *is a Hilbert space consists of real functions defined on a nonempty set* $\mathcal{X}$.

Remark 1. Let H be a Hilbert space which has a reproducing kernel function. We then call H a reproducing kernel Hilbert space. Condition 2 in Definition 2.1 is "the reproducing kernel property".

Let us here give some RKHSs,

Definition 2.2. *Denote* $W_2^3[0,1]$ *the space of functions such as* [10]

$$W_2^3[0,1]=\left\{\varphi\mid\varphi^{(j)}(\eta)\ \text{are absolutely continuous functions on}\ [0,1],\varphi^{(3)}(\eta)\in L^2[0,1],j=0,1,2,\ \text{and}\ \varphi(0)=\varphi'(0)=0\right\}.$$

We define its inner product and norm as:

$$\langle\varphi,\tau\rangle_{W_2^3}=\sum_{j=0}^{2}\varphi^{(j)}(0)\tau^{(j)}(0)+\int_0^1\varphi^{(3)}(\eta)\tau^{(3)}(\eta)d\eta,$$

and $\|\varphi\|_{W_2^3}=\langle\varphi,\varphi\rangle_{W_2^3}^{1/2}$ *in which* $\varphi,\tau\in W_2^3[0,1].$

Theorem 2.1. (see [3]) *For each fixed* $\eta\in[0,1]$ *and for any* $\varphi(\zeta)$ *in the reproducing kernel space* $W_2^3[0,1]$, $\exists K_\eta(\zeta)\in W_2^3[0,1]$ *such that* $\langle\varphi(\zeta),K_\eta(\zeta)\rangle_{W_2^3}=\varphi(\eta)$ *and* $K_\eta(\zeta)$ *is called the reproducing kernel function in the space* $W_2^3[0,1]$.

Theorem 2.2. (see [5]) $K_\eta(\zeta)$ *is the reproducing kernel function associated to the space* $W_2^3[0,1]$, *given by*

$$K_\eta(\zeta)=\begin{cases}k(\eta,\zeta), & \zeta\le\eta,\\ k(\zeta,\eta), & \zeta>\eta,\end{cases}\tag{1}$$

where $k(\eta,\zeta)=\left(\zeta^2/120\right)\left(-5\eta\zeta^2+\zeta^3+10\eta^2(3+\zeta)\right).$

Definition 2.3. *Denote* $W_2^1[0,1]$ *the space of functions such as* [9]

$$W_2^1[0,1]=\left\{\varphi\mid\varphi(\eta)\ \text{is absolutely continuous functions on}\ [0,1],\varphi'(\eta)\in L^2[0,1]\right\}.$$

We define its inner product and norm as:

$$\langle\varphi,\tau\rangle_{W_2^1}=\int_0^1\left(\varphi(\eta)\tau(\eta)+\varphi'(\eta)\tau'(\eta)\right)d\eta,$$

and $\|\varphi\|_{W_2^1}=\langle\varphi,\varphi\rangle_{W_2^1}^{1/2}$ *in which* $\varphi,\tau\in W_2^1[0,1].$

Theorem 2.3. *The reproducing kernel function* $R_\eta(\zeta)$ *associated to the complete reproducing kernel space* $W_2^1[0,1]$ *is given by* [9]

$$R_\eta(\zeta)=\frac{1}{2\sinh(1)}\times\left[\cosh(\eta+\zeta-1)+\cosh\left(|\eta-\zeta|-1\right)\right].\tag{2}$$

Definition 2.4. *If a function* $\phi(\eta)$, $\eta>0$ *can be written as* $\phi(\eta)=\eta^p\phi_1(\eta)$ *for some* $p>\mu$, *where* $\mu\in\mathbb{R}$, *and* $\phi_1(\eta)$ *is continuous in* $[0,\infty)$, *then we say that* $\phi(\eta)$ *is in the space* C_μ. *If* $\phi^{(m)}\in C_\mu$, $m\in\mathbb{N}$, *we say that* $\phi\in C_\mu^m$.

Definition 2.5. *The integral operator in Riemann-Liouville sense of order* $\varrho>0$ *is*

$$\begin{aligned}\left(\mathcal{J}^\varrho\phi\right)(\eta)&:=\frac{1}{\Gamma(\varrho)}\int_0^\eta(\eta-\zeta)^{\varrho-1}\phi(\zeta)d\zeta,\ \eta>0,\\ \left(\mathcal{J}^0\phi\right)(\eta)&:=\phi(\eta).\end{aligned}\tag{3}$$

We have :

1. $\mathcal{J}^\varrho\mathcal{J}^\upsilon\phi(\eta)=\mathcal{J}^{\varrho+\upsilon}\phi(\eta),\ \varrho,\upsilon\ge0.$
2. $\mathcal{J}^\varrho\eta^\gamma=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\varrho+1)}\eta^{\gamma+\varrho},\ \varrho>0,\ \gamma>-1,\ \eta>0.$
3. $\left(\mathcal{J}^\varrho D^\varrho\phi\right)(\eta):=\mathcal{J}^m D^m\phi(\eta)=\phi(\eta)-\sum_{k=0}^{m-1}\phi^{(k)}\left(0^+\right)\frac{\eta^k}{k!},\ \eta>0,\ m-1<\varrho\le m.$

Definition 2.6. *The fractional derivative in Caputo sense of* $\phi(\eta)$ *function of order* $\varrho > 0$ *is*

$$\left(D^{\varrho}\phi\right)(\eta) := \left(\mathcal{J}^{m-\varrho}\phi^{(m)}\right)(\eta) = \frac{1}{\Gamma(m-\varrho)}\int_0^{\eta}(\eta-\zeta)^{m-\varrho-1}\,\phi^{(m)}(\zeta)d\zeta, \tag{4}$$

for $\phi(\eta) \in C^{m}_{-1}$, $\eta > 0$, $m-1 < \varrho \le m$, *and* $m \in \mathbb{N}$.

3. Analysis of the Method

Next, we consider the following DVP oscillator equation of fractional order:

$$\begin{cases} D^{\varrho}\varkappa(\eta) - \mu\left(1-\varkappa^{2}(\eta)\right)\varkappa'(\eta) + a\varkappa(\eta) + b\varkappa^{3}(\eta) = \mathrm{g}(\phi,\omega,\eta), \quad 0 \le \eta < 1,\ 1 < \varrho \le 2, \\ \varkappa(0) = \lambda_0,\ \varkappa'(0) = \lambda_1. \end{cases} \tag{5}$$

Here D^{ϱ} is the fractional derivative in Caputo sense of order ϱ, ϕ denotes the forcing strength, $\varkappa$ represents the displacement of its equilibrium position, and the damping parameter of the system is defined by $\mu > 0$. The periodic driving function g depends on ϕ, angular frequency of the driving force ω and the time η will be denoted by $\mathrm{g}(\phi,\omega,\eta)$. We can express the DVP oscillator equation in three different physical situations, if $a > 0$ and $b > 0$ then it is a single-well, if $a < 0$ and $b > 0$ then it is a double-well and if $a > 0$ and $b < 0$ then it is a double-hump.

To apply the proposed algorithm easily, we first homogenize the initial conditions $\varkappa(0) = \lambda_0$ and $\varkappa'(0) = \lambda_1$. To do so, take $\hbar(\eta) = \varkappa(\eta) - (\lambda_1\eta + \lambda_0)$ in eq. (5). we obtain:

$$\begin{cases} D^{\varrho}\hbar(\eta) = \Phi\left(\eta,\hbar(\eta),\hbar'(\eta)\right), \\ \hbar(0) = 0 \text{ and } \hbar'(0) = 0, \end{cases} \tag{6}$$

where $0 \le \eta < 1$, $1 < \varrho \le 2$, and $\Phi\left(\eta,\hbar(\eta),\hbar'(\eta)\right) = \phi\left(\eta,\hbar(\eta) + (\lambda_1\eta + \lambda_0),\hbar'(\eta) + \lambda_1\right)$. Then, we get a new formula eq. (7) by integrating both sides of eq. (6) in Riemann-Liouville sense

$$\hbar(\eta) = G(\eta), \tag{7}$$

where $G(\eta) = \mathcal{J}^{\varrho}\left(\Phi\left(\eta,\hbar(\eta),\hbar'(\eta)\right)\right)$.

Anyhow, define the operator L as $L : W_2^3[0,1] \to W_2^1[0,1]$ such that $L\hbar(\eta) = \hbar(\eta)$. As a result, we can rewrite the formula eq. (7) as follows:

$$L\hbar(\eta) = G(\eta), \quad 0 \le \eta < 1, \tag{8}$$

Applying the RKHS method based on the construction of the orthogonal function system of $W_2^3[0,1]$. To do so, let $\varphi_i(\eta) = R_{\eta_i}(\eta)$ and $\psi_i(\eta) = L^{*}\varphi_i(\eta)$, where the countable set $\{\eta_i\}_{i=1}^{\infty}$ is dense in the interval $[0,1]$, the operator L^{*} is the formal adjoint of L, and $R_{\eta_i}(\eta)$ represents the reproducing kernel function associated to the space $W_2^1[0,1]$.

Next, the orthonormal system $\{\bar{\psi}_i\}_{i=1}^{\infty}$ in $W_2^3[0,1]$ is obtained by applying the process of Gram-Schmidt as follows:

$$\bar{\psi}_i(\eta) = \sum_{k=1}^{i}\vartheta_{ik}\psi_k(\eta), \quad \vartheta_{ii} > 0,\ i = 1,2,\ldots, \tag{9}$$

in which the orthogonalization coefficients ϑ_{ik} are considered by the following cases :

$$\begin{cases} \vartheta_{ij} = \dfrac{1}{\|\psi_1\|}, & \text{for } i = j = 1, \\ \vartheta_{ij} = \dfrac{1}{e_i}, & \text{for } i = j \ne 1, \\ \vartheta_{ij} = -\dfrac{1}{e_i}\displaystyle\sum_{k=j}^{i-1} C_{ij}\vartheta_{kj}, & \text{for } i > j, \end{cases}$$

where $e_i = \sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} C_{ik}^2}$, $C_{ik} = \left\langle \psi_i, \bar{\psi}_k \right\rangle_{W_2^3}$, and $\{\psi_i\}_{i=1}^{\infty}$ denotes the orthonormal system in the space $W_2^3[0,1]$.

Remark 2. It is easy to determine the expression of $\psi_i(\eta)$ function

$$\psi_i(\eta) = L^{*}\varphi_i(\eta) = \left\langle L^{*}\varphi_i(\zeta), K_{\eta}(\zeta)\right\rangle_{W_2^3} = \left\langle \varphi_i(\zeta), LK_{\eta}(\zeta)\right\rangle_{W_2^1} = \left\langle R_{\eta_i}(\zeta), LK_{\eta}(\zeta)\right\rangle_{W_2^1} = L_{\zeta}K_{\eta}(s)|_{\zeta=\eta_i},$$

where the operator L_{ζ} means that the operator L is applied to the variable ζ.

Theorem 3.1. *Let* $\{\eta_i\}_{i=1}^{\infty}$ *be a dense set on* $[0,1]$, *then* $\{\psi_i\}_{i=1}^{\infty}$ *is the complete system of the space* $W_2^3[0,1]$.

Proof. Let us first put $\langle \varkappa(\eta), \psi_\iota(\eta)\rangle_{W_2^3} = 0$, $\iota = 1,2,\ldots$, for each fixed $\varkappa(\eta) \in W_2^3[0,1]$.
Since

$$\langle \varkappa(\eta), \psi_\iota(\eta)\rangle_{W_2^3} = \langle \varkappa(\eta), L^*\varphi_\iota(\eta)\rangle_{W_2^3} = \langle L\varkappa(\eta), \varphi_\iota(\eta)\rangle_{W_2^1} = L\varkappa(\eta_\iota) = 0,$$

and $\{\eta_\iota\}_{\iota=1}^{\infty}$ is a dense set on the interval $[0,1]$, we reach $L\varkappa(\eta)$. From the existence of L^{-1}, we find $\varkappa(\eta) = 0$.

Lemma 3.1. *Assume* $\varkappa(\eta) \in W_2^3[0,1]$, *then we have*

$$\|\varkappa^{(\iota)}(\eta)\|_C \le F\|\varkappa(\eta)\|_{W_2^3}, \quad \iota = 0,1,2,$$

the constant F *is positive and we note that* $\|\varkappa(\eta)\|_C = \max_{0\le\eta\le1}|\varkappa(\eta)|$.

Proof. $\forall \eta, \zeta \in [0,1]$ we have

$$\varkappa^{(\iota)}(\eta) = \langle \varkappa(\zeta), \partial_\eta^\iota K_\eta(\zeta)\rangle_{W_2^3}, \quad \iota = 0,1,2.$$

From the expression of the RK function "$K_\eta(\zeta)$", we find the following estimate

$$\|\partial_\eta^\iota K_\eta(\zeta)\|_{W_2^3} \le M_\iota, \quad \iota = 0,1,2.$$

Consequently,

$$|\varkappa^{(\iota)}(\eta)| = |\langle \varkappa(\eta), \partial_\eta^\iota K_\eta(\eta)\rangle_{W_2^3}| \le \|\partial_\eta^\iota K_\eta(\eta)\|_{W_2^3}\|\varkappa(\eta)\|_{W_2^3} \le F_\iota\|\varkappa(\eta)\|_{W_2^3}, \quad \iota = 0,1,2.$$

Here, suppose $F = \max_{\iota=0,1,2}\{F_\iota\}$, then the last estimate become as

$$\|\varkappa^{(\iota)}(\eta)\|_C \le F\|\varkappa(\eta)\|_{W_2^3}, \quad \iota = 0,1,2.$$

Theorem 3.2. *Assume* $\{\eta_\iota\}_{\iota=1}^{\infty}$ *is a dense set on* $[0,1]$ *and the problem* eq. (8) *has at most one solution on* $W_2^3[0,1]$, *then the solution of the problem* eq. (8) *is obtained as:*

$$\hbar(\eta) = \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota}\vartheta_{\iota k}G(\eta_k)\overline{\psi}(\eta), \tag{10}$$

and we have also the solution of the problem eq. (5) *as:*

$$\varkappa(\eta) = \left(\sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota}\vartheta_{\iota k}G(\eta_k)\overline{\psi}(\eta)\right) + (\lambda_1\eta + \lambda_0). \tag{11}$$

Proof. We first note that from the completeness of the orthonormal basis $\{\overline{\psi}_\iota(\eta)\}_{\iota=1}^{\infty}$ in the space $W_2^3[0,1]$, we get

$$\begin{aligned}
\hbar(\eta) &= \sum_{\iota=1}^{\infty}\langle \hbar(\eta), \overline{\psi}_\iota(\eta)\rangle_{W_2^3}\overline{\psi}_\iota(\eta)\\
&= \sum_{\iota=1}^{\infty}\left\langle \hbar(\eta), \sum_{k=1}^{\iota}\vartheta_{\iota k}\psi_k(\eta)\right\rangle_{W_2^3}\overline{\psi}_\iota(\eta)\\
&= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota}\vartheta_{\iota k}\langle \hbar(\eta), \psi_k(\eta)\rangle_{W_2^3}\overline{\psi}_\iota(\eta)\\
&= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota}\vartheta_{\iota k}\langle \hbar(\eta), L^*\varphi_k(\eta)\rangle_{W_2^3}\overline{\psi}_\iota(\eta)\\
&= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota}\vartheta_{\iota k}\langle L\hbar(\eta), \varphi_k(\eta)\rangle_{W_2^1}\overline{\psi}_\iota(\eta)\\
&= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota}\vartheta_{\iota k}\langle L\hbar(\eta), R_\eta(\eta_k)\rangle_{W_2^1}\overline{\psi}_\iota(\eta)\\
&= \sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota}\vartheta_{\iota k}G(\eta_k)\overline{\psi}_\iota(\eta),
\end{aligned}$$

with $G(\eta_k) = L\hbar(\eta_k)$.
In addition, we mention that $\varkappa(\eta) = \hbar(\eta) + (\lambda_1\eta + \lambda_0)$ and from the expression of $\hbar(\eta)$, it easy to get

$$\varkappa(\eta) = \left(\sum_{\iota=1}^{\infty}\sum_{k=1}^{\iota}\vartheta_{\iota k}G(\eta_k)\overline{\psi}(\eta)\right) + (\lambda_1\eta + \lambda_0).$$

Here, If we take a finite many terms in the series which is represent $\varkappa(\eta)$, then the approximate solution $\varkappa_n(\eta)$ is given as

$$\varkappa_n(\eta)=\left(\sum_{\imath=1}^{n}\sum_{k=1}^{\imath}\vartheta_{\imath k}G(\eta_k)\bar{\psi}(\eta)\right)+(\lambda_1\eta+\lambda_0).$$

It is easy to see that $\sum_{\imath=1}^{\infty}\sum_{k=1}^{\imath}\vartheta_{\imath k}G(\eta_k)\bar{\psi}(\eta)+(\lambda_1\eta+\lambda_0)<\infty$ because of the Hilbert space $W_2^3[0,1]$ properties.

Consequently, the approximate solution $\varkappa_n(\eta)$ converges normally.

Theorem 3.3. *The approximate solution* $\varkappa_n(\eta)$ *converges uniformly and its derivatives* $\varkappa_n^{(\imath)}(\eta)$ *of order* $\imath=1,2,$ *are also uniformly convergent.*

Proof. First, to prove that $\varkappa_n(\eta)$ is uniformly convergent let us give the following estimate for all $\eta\in[0,1]$,

$$\begin{aligned}\left|\varkappa_n(\eta)-\varkappa(\eta)\right|&=\left|\left\langle\varkappa_n(\eta)-\varkappa(\eta),K_\eta(\eta)\right\rangle_{W_2^3}\right|\\&\le\left\|K_\eta(\eta)\right\|_{W_2^3}\left\|\varkappa_n(\eta)-\varkappa(\eta)\right\|_{W_2^3}\\&\le\mathcal{C}_0\left\|\varkappa_n(\eta)-\varkappa(\eta)\right\|_{W_2^3},\end{aligned}$$

where $\mathcal{C}_0$ is a constant.

Next, we prove the uniform convergence of the approximate derivatives as follows.

$$\begin{aligned}\left|\varkappa_n^{(\imath)}(\eta)-\varkappa^{(\imath)}(\eta)\right|&=\left|\left\langle\varkappa_n^{(\imath)}(\eta)-\varkappa^{(\imath)}(\eta),\partial_\eta^{\imath}K_\eta(\eta)\right\rangle_{W_2^3}\right|\\&\le\left\|\partial_\eta^{\imath}K_\eta(\eta)\right\|_{W_2^3}\left\|\varkappa_n^{(\imath)}(\eta)-\varkappa^{(\imath)}(\eta)\right\|_{W_2^3},\end{aligned}$$

from the uniform boundedness of $\partial_\eta^{\imath}K_\eta(\eta)$ about η, we have

$$\left\|\partial_\eta^{\imath}K_\eta(\eta)\right\|_{W_2^3}\le\mathcal{C}_\imath,\quad\imath=1,2,$$

where $\mathcal{C}_\imath$, $\imath=1,2,$ are positive constants. Therefore

$$\left|\varkappa_n^{(\imath)}(\eta)-\varkappa^{(\imath)}(\eta)\right|\le\mathcal{C}_\imath\left\|\varkappa_n(\eta)-\varkappa(\eta)\right\|_{W_2^3}.$$

Thus, we complete the proof of the theorem.

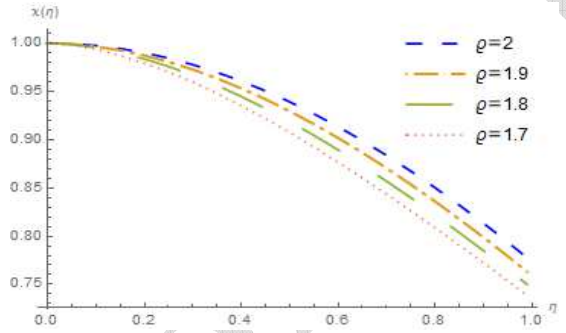

Fig. 1. Approximate solution of Example 4.1 for $n=8$ with $\varrho=2$, $\varrho=1.9$, $\varrho=1.8$, and $\varrho=1.7$.

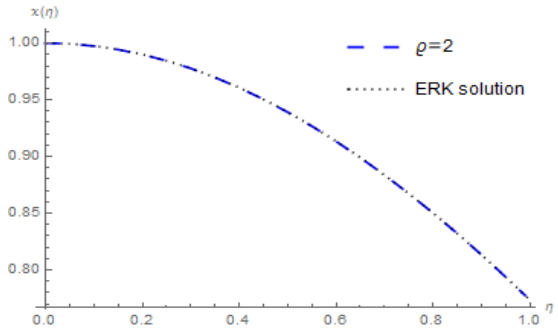

Fig. 2. Comparison of numerical solutions of the RKHS method for Example 4.1 by the ERK method at $\varrho=2$.

Table 1. Comparison between the numerical results of RKHS method at $\varrho=2$ and other methods for Example 4.1.

η	ADM	RK	RKHSM	Absolute error in ADM	Absolute error in RKHSM
0	1.00000000	1.00000000	1.00000000	2.6870728×10^{-12}	0.0
0.1	0.99750285	0.99750285	0.99749662	2.4645763×10^{-10}	6.2324216×10^{-6}
0.2	0.99004513	0.99004516	0.99003779	3.6372929×10^{-8}	7.3700462×10^{-6}
0.3	0.97772508	0.97772579	0.97772970	7.1333925×10^{-7}	3.9070112×10^{-6}
0.4	0.96069642	0.96070247	0.96070229	6.0511724×10^{-6}	1.7672169×10^{-6}
0.5	0.93915076	0.93918309	0.93918437	3.2329971×10^{-5}	1.2767889×10^{-6}
0.6	0.91328632	0.91341506	0.91341602	1.2874056×10^{-4}	9.5529184×10^{-7}
0.7	0.88325529	0.88367350	0.88367672	4.1821118×10^{-4}	3.2145910×10^{-6}

Table 2. Approximate solution of Example 4.1 for some values of order ϱ.

η	$\varrho = 1.9$	$\varrho = 1.8$	$\varrho = 1.7$
0	1	1	1
0.1	0.99657171	0.99532568	0.9936548
0.2	0.9872057	0.98365046	0.97922534
0.3	0.9726684	0.96666744	0.95963955
0.4	0.95324689	0.94474603	0.93518248
0.5	0.92948456	0.91884116	0.90736383
0.6	0.90175879	0.88941418	0.87661035
0.7	0.87048529	0.85699852	0.84354686

4. Numerical Applications and Graphical Results

We present some numerical results in this section. The reproducing kernel Hilbert space approach is implemented for two fractional-order DVP oscillator equations: force and force-free DVP oscillator equation of fractional order. Additionally, we compared the numerical results obtained by RKHS method (RKHSM, for short) with the other results found in the literature.

Example 4.1. Let us consider the fractional order forced DVP oscillator equation given by

$$\begin{cases} D^{\varrho}\varkappa(\eta) - \mu\left(1-\varkappa^2(\eta)\right)\varkappa'(\eta) + a\varkappa(\eta) + b\varkappa^3(\eta) = \phi\cos(\omega\eta), \quad 1 < \varrho \le 2,\ 0 \le \eta < 1, \\ \varkappa(0) = 1,\ \varkappa'(0) = 0. \end{cases}$$

Using the RKHSM, taking $\eta_\imath = \imath/n$, $\imath = 1,2,\ldots n$, and $n = 8$, for the situation single-well. Here, we take $\mu = 0.1$, $a = 0.5$, $b = 0.5$, $\phi = 0.5$ and $\omega = 0.79$. In addition, we fixed the fractional derivative at $\varrho = 2$ then we compared the numerical results obtained via the Adomian decomposition method and RKHSM with the results based on the Explicit Runge-Kutta method (ERK). Also, in order to demonstrate the accuracy of the method and its effectiveness, we give Table 1. In this table, the numerical results and the absolute errors of Example 4.1 have been presented. While, Table 2 shows the obtained solutions by using the RKHSM for various values of ϱ and they are plotted in Fig. 1. The comparison between the numerical results obtained by ERK method and RKHS method are depicted in Fig. 2. It is clear that the approximate solutions when $\varrho = 2$ coincide with those obtained by ADM and ERK. Herein, we can conclude that the solutions depend continuously on the fractional derivative ϱ and the RKHS method is more accurate.

Example 4.2. Let us consider the fractional order forced DVP oscillator equation given by

$$\begin{cases} D^{\varrho}\varkappa(\eta) - \mu\left(1-\varkappa^2(\eta)\right)\varkappa'(\eta) + a\varkappa(\eta) + b\varkappa^3(\eta) = 0, \quad 1 < \varrho \le 2,\ 0 \le \eta < 1, \\ \varkappa(0) = 1,\ \varkappa'(0) = 0. \end{cases}$$

Using the RKHSM, taking $\eta_\imath = \imath/n$, $\imath = 1,2,\ldots n$, and $n = 8$, for the situation single-well. Here, we take $\mu = 0.1$, $a = 1$, and $b = 0.01$. In addition, we fixed the fractional derivative at $\varrho = 2$ then we compared the numerical results obtained via the Adomian decomposition method and RKHSM with the results based on the ERK method. Also, in order to demonstrate the accuracy of the method and its effectiveness, we give Table 3. In this table, the numerical results and the absolute errors of Example 4.2 have been presented. Whilst, Table 4 shows the solutions obtained by using the RKHS method for various values of ϱ and they are plotted in Fig. 3. The comparison between the numerical results obtained by ERK method and RKHS method are depicted in Fig. 3. It is clear that the approximate solutions when $\varrho = 2$ coincide with those obtained by ADM and ERK. Herein, we can conclude that the solutions depend continuously on the fractional derivative ϱ and the RKHS method is more accurate.

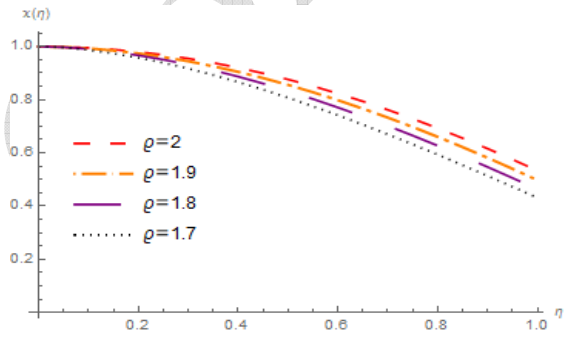

Fig. 3. Approximate solution of Example 4.2 for $n = 8$ with $\varrho = 2$, $\varrho = 1.9$, $\varrho = 1.8$, and $\varrho = 1.7$.

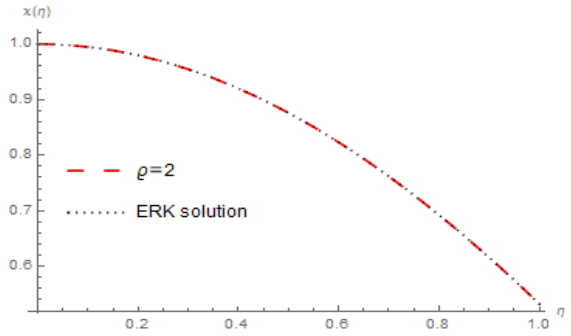

Fig. 4. Comparison of numerical solutions of the RKHS method for Example 4.2 by the ERK method at

Table 3. Comparison between the numerical results of RKHS method at $\varrho = 2$ and other methods for Example 4.2.

η	ADM	RK	RKHSM	Absolute error in ADM	Absolute error in RKHSM
0	1.00000000	1.00000000	1.00000000	0.0	0.0
0.1	0.99495428	0.99495428	0.99495033	1.1436407×10^{-12}	3.9473568×10^{-6}
0.2	0.97986763	0.97986763	0.97985758	2.5632396×10^{-10}	1.0043970×10^{-5}
0.3	0.95488774	0.95488773	0.95489245	7.6843231×10^{-9}	4.7189498×10^{-6}
0.4	0.92025257	0.92025248	0.92025175	8.7366210×10^{-8}	7.3410256×10^{-7}
0.5	0.87628345	0.87628287	0.87628332	5.8091852×10^{-7}	4.5406808×10^{-7}
0.6	0.82337917	0.82337643	0.82337548	2.7431360×10^{-6}	9.4148834×10^{-7}
0.7	0.76201184	0.76200163	0.76200691	1.0210254×10^{-5}	5.2767533×10^{-6}

Table 4. Approximate solution of Example 4.2 for some values of order ϱ .

η	$\varrho = 1.9$	$\varrho = 1.8$	$\varrho = 1.7$
0	1	1	1
0.1	0.9930881	0.99058278	0.98723008
0.2	0.97409728	0.96683317	0.95772855
0.3	0.94449563	0.93205904	0.91731167
0.4	0.90475052	0.886845	0.8663568
0.5	0.85576801	0.83281744	0.80743264
0.6	0.79820158	0.77084442	0.74151353
0.7	0.73279636	0.7019202	0.66980078

5. Conclusion

In this work, we have presented an efficient RKHS method for the fractional-order DVP oscillator equations. The approximate solution $\varkappa_n(\eta)$ and its derivative converge both uniformly. It was found from the obtained results that the RKHS method gives good results for the fractional DVP oscillator equation, which has wide applications in widespread fields of physics and engineering. We also provided some plots that show how the obtained solutions behave. The comparisons have been demonstrated a good agreement between the RKHS method's results and the results obtained via other methods. It is shown that the RKHS method can be easily used to approximate the solution of the considered problem. Additionally, its algorithm is simple, efficient, and can be applied in treating systems with nonlinear oscillations arising in many practical physical and engineering applications.

Acknowledgments

We are thankful to the anonymous reviewers for carefully reading the paper and their helpful comments.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

Funding

The authors received no financial support for the research, authorship, and publication of this article.

References

[1] Ahmad, Z., Arif, M., Ali, F., Khan, Ilyas, Nisar, K.S., A report on COVID-19 epidemic in Pakistan using SEIR fractional model, *Scientific Reports*, 10, 2020, 22268.
[2] Asadi Cordshooli, Gh., Vahidi, A.R., Solutions of Duffing-van der Pol Equation Using Decomposition Method, *Advanced Studies in Theoretical Physics*, 5(3), 2011, 121-129.
[3] Bushnaq, S., Momani, S., Zhou, Y., A Reproducing Kernel Hilbert Space Method for Solving Integro-Differential Equations of Fractional Order, *Journal of Optimization Theory and Applications*, 156, 2013, 96-105.
[4] Bhangale, N., Kachhia, K.B., Gómez-Aguilar, J.F., A new iterative method with ρ-Laplace transform for solving fractional differential equations with Caputo generalized fractional derivative, *Engineering with Computers*, 2020, 1-14. https://doi.org/10.1007/s00366-020-01202-9
[5] Cui, M., Geng:, F., Solving singular two-point boundary value problem in reproducing kernel space, *Journal of Computational and Applied Mathematics*, 205(1), 2007, 6-15.
[6] Doha, E.H., Bhrawy, A.H., Ezz-Eldien, S.S., An Efficient Legendre Spectral Tau Matrix Formulation for Solving Fractional Subdiffusion and Reaction Subdiffusion Equations, *Journal of Computational and Nonlinear Dynamics*, 10(2), 2015, 021019.
[7] Dwivedi, K.D., Rajeev, Das, S., Gómez-Aguilar, J.F., Finite difference/collocation method to solve multi term variable-order fractional reaction-advection-diffusion equation in heterogeneous medium, *Numerical Methods for Partial Differential Equations*, 2020. https://doi.org/10.1002/num.22648
[8] Du, M., Wang, Z., Hu, H., Measuring memory with the order of fractional derivative, *Scientific Reports*, 3, 2013, 3431.
[9] Geng, F., Cui, M., Solving a nonlinear system of second order boundary value problems, *Journal of Mathematical Analysis and Applications*, 327(2), 2007, 1167-1181.
[10] Gao, E., Song, S., Zhang, X., Solving singular second-order initial/boundary value problems in reproducing kernel Hilbert space, *Boundary Value Problems*, 2012, 3, 2012.
[11] Inc, M., Ic, Ü., Inan, i. E., Goméz-Aguilar, J. F., Generalized (G'/G)-expansion method for some soliton wave solutions of Burgers-like and potential KdV equations, *Numerical Methods for Partial Differential Equations*, 2020. https://doi.org/10.1002/num.22637
[12] Akgül, A., A novel method for a fractional derivative with non-local and non-singular kernel, *Chaos, Solitons & Fractals*, 114, 2018, 478-482.
[13] Mall, S., Chakraverty, S., Hermite functional link neural network for solving the van der Pol-Duffing oscillator equation, *Neural Computation*, 28(8), 2016, 1574-1598.

[14] Morales-Delgado, V.F., Gómez-Aguilar, J.F., Y épez-Martínez, H. et. al., Laplace homotopy analysis method for solving linear partial differential equations using a fractional derivative with and without kernel singular, *Advances in Difference Equations*, 2016, 2014, 1-17.
[15] Pandey, P., Kumar, S., Gómez-Aguilar, J.F., Baleanu, D., An efficient technique for solving the space-time fractional reaction-diffusion equation in porous media, *Chinese Journal of Physics*, 68, 2020, 483-492.
[16] Patnaik, S., Semperlotti, F., Variable-order fracture mechanics and its application to dynamic fracture, *npj Computational Materials*, 7, 2021, 27.
[17] Quaranta, G., Monti, G., Marano, G.C., Parameters identification of Van der Pol-Duffing oscillators via particle swarm optimization and differential evolution, *Mechanical Systems and Signal Processing*, 24(7), 2010, 2076-2095.
[18] Rahimkhani, P., Ordokhani, Y., Babolian, E., Fractional-order Bernoulli wavelets and their applications, *Applied Mathematical Modelling*, 40(17), 2016, 8087-8107.
[19] Saad, K.M., Khader, M.M., Gómez-Aguilar, J.F., Baleanu, D., Numerical solutions of the fractional Fisher's type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 29(7), 2019, 023116.
[20] Safdari, H., Aghdam, Y. E., Gómez-Aguilar, J.F., Shifted Chebyshev collocation of the fourth kind with convergence analysis for the space-time fractional advection-diffusion equation, *Engineering with Computers*, 2020. https://doi.org/10.1007/s00366-020-01092-x
[21] Shawagfeh, N., Arqub, O.A., Momani, S., Analytical solution of nonlinear second-order periodic boundary value problem using reproducing kernel method, *Journal of Computational Analysis and Applications*, 16(4), 2014, 750-762.
[22] Sajadi, H., Ganji, D.D., Vazifeshenas, M.R., Application of numerical and semi-analytical approach on van der Pol Duffing oscillators, *Journal of Advanced Research in Mechanical Engineering*, 1, 2010, 136-141.
[23] Sahlol, A.T., Yousri, D., Ewees, A.A., et. al., COVID-19 image classification using deep features and fractional-order marine predators algorithm, *Scientific Reports*, 10, 2020, 15364.
[24] Wang, H., Du, N., Fast alternating-direction finite difference methods for three-dimensional space-fractional diffusion equations, *Journal of Computational Physics*, 258, 2014, 305-318.
[25] Baleanu, D., Fernandez, A., Akgül, A., On a fractional operator combining proportional and classical differintegrals, *Mathematics*, 8(3), 2020, 360.
[26] Yin, K.L., Pu, Y.F., Lu, L., Combination of fractional FLANN filters for solving the Van der Pol-Duffing oscillator, *Neurocomputing*, 399, 2020, 183-192.
[27] Zhuang, P., Liu, F., Turner, I., Gu, Y.T., Finite volume and finite element methods for solving a one-dimensional space-fractional Boussinesq equation, *Applied Mathematical Modelling*, 38(15), 2014, 3860-3870.
[28] Cui, M., Geng:, F., *Nonlinear Numerical Analysis in the Reproducing Kernel Space*, Nova Science Publishers, Inc, New York, 2009.

ORCID iD

Nourhane Attia https://orcid.org/0000-0002-9937-8665
Djamila Seba https://orcid.org/0000-0002-7910-3074
Ali Akgül https://orcid.org/0000-0001-9832-1424

How to cite this article: Attia N., Seba D., Akgül A. Solving Duffing-Van der Pol Oscillator Equations of Fractional Order by an Accurate Technique, *J. Appl. Comput. Mech.*, xx(x), 2021, 1–8. https://doi.org/10.22055/JACM.2021.35369.2642