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Controllability Results for Nondensely Defined Impulsive Fractional-Order Functional Semilinear Differential Inclusions in Abstract Space

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Abstract. In this work, we prove the controllability result of integral solutions defined on a real compact interval for a class of impulsive functional differential inclusions with fractional order and nonlocal conditions, in the case when the linear part is a non-densely defined operator and satisfies the Hille-Yosida condition. The main tool is an appropriate fixed point theorem, integrated semigroup, and the known facts about fractional calculus.
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INTRODUCTION

The topic of fractional differential equations has been of great interest for many researchers in view of its theoretical development and widespread applications in various fields of science and engineering such as physics, biophysics, chemistry, statistics, economics, blood flow phenomena, control theory, porous media, electromagnetic, and other fields, for more details we refer the reader to [2, 7, 9] and the references therein.

The controllability is one of the most powerful concepts used in modern mathematical control theory and applicable in various areas of life. Very recently, the study of controllability problems has always been considered as a hot topic because of their fruitful achievements and huge authors turnout for study in finite or infinite demonsional space with different conditions see [1, 3]. Because controllability problems for impulsive fractional differential inclusions with nondense domain have not yet been studied, our main goal in this paper is to discussed the controllability results of integral solutions on a separable real Banach space $(E, |\cdot|)$ for the following problem:

$$^{c}D^{\alpha}y(t) \in Ay(t) + F(t, y_t) + (\mathfrak{B}u)(t), \quad t \in J := [0, b], \quad t \neq t_k, \tag{1}$$

$$y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), k = 1, \dots, \sigma, \tag{2}$$

$$y(t) - h_t(y) = \phi(t), \qquad t \in [-r, 0], \tag{3}$$

where $b, r \in \mathbb{R}^+$, $^{c}D^{\alpha}$ is the caputo fractional derivative of order $\alpha \in]0, 1[$, and $F : J \times \mathfrak{D} \to \mathcal{P}(E)$ is a bounded multivalued map. Here, $\mathfrak{D}$ is a Banach space equipped by this norm: $\|y\|_{\mathfrak{D}} = \sup_{\theta \in [-r,0]} |y(\theta)|$, $\phi \in \mathfrak{D}$, for any $t \in J, y_t(\cdot)$ represent the history of the state from time $t-r$, up to the present time t defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in [-r, 0]$ and the function $h_t : \mathcal{K} \to \overline{D(A)}$ is continuous and compact, $0 = t_0 < t_1 < \dots < t_\sigma < t_{\sigma+1} = b$. In addition, $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E, $I_k \in C(E, E)$, and $y(t_k^+), y(t_k^-)$ are the right and left limits of $y(t)$ at

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the point $t = t_k$ respectively. Finally the control function $u(\cdot)$ is given in $L^2(J, U)$, a Banach space of admissible control functions with U as a Banach space and $\mathfrak{B}$ is a bounded linear operator from U to $\overline{D(A)}$.

RESULTS

Firstly, we will provide the spaces that we will used in our work.

$$PC(J, \overline{D(A)}) = \Big\{ y : J \to \overline{D(A)}, y(t) \text{ is continuous everywhere except for some } t_k \text{ at which } y(t_k^-) \text{ and } y(t_k^+) \text{ exist and } y(t_k^-) = y(t_k), \text{ for each } k = 1, 2, \ldots, \sigma \Big\},$$

and $\mathcal{K} = \Big\{ y : [-r, b] \to \overline{D(A)},\ y|_{[-r,0]} \in \mathfrak{D},\ y|_J \in PC(J, \overline{D(A)}) \Big\}$,
where $y_{/_J}$ is the restriction of y to J. Clearly, PC and $\mathcal{K}$ are Banach spaces normed by

$$\|y\|_{PC} = \sup\{|y(t)|; t \in J\} \quad \text{and} \quad \|y\|_{\mathcal{K}} = \sup\{|y(t)|; t \in [-r, b]\}.$$

A multivalued map $G : E \to \mathcal{P}(E)$ is convex (closed) values if $G(y)$ is convex (closed) for all $y \in E$. Moreover, we say that G is bounded on bounded sets if $G(\Lambda)$ is bounded in E for each bounded set Λ of E. The multivalued map G is said to be completely continuous if $G(\Lambda)$ is relatively compact for each Λ a bounded set of $\mathcal{P}(E)$.

Corollary 1 *Let G be a set-valued map from E to a compact space Y whose graph is closed. Then G is upper semi-continuous.*

Lemma 1 *Let $G : J \times \mathfrak{D} \to \mathcal{P}_{cp,cv}(E)$ be an L^1−Caratheodory multivalued map and let Γ be a linear continuous mapping from $L^1(J, E)$ into $C(J, E)$, then the operator $\Gamma \circ S_F : C(J, E) \to \mathcal{P}_{cp,cv}(C(J, E))$ defined by $y \mapsto (\Gamma \circ S_F)(y) = \Gamma(S_{F,y})$ is a closed graph operator in $C(J, E) \times C(J, E)$.*

For more details on multivalued maps we refer to the book of Deimling [4].

Now, we define the set of the selection of F at y by $S_{F,y} = \{h \in L^1(J, E); h(t) \in F(t, y_t) \text{ for a.e } t \in J\}$.

Lemma 2 *[4] Let Ω be a bounded and convex set in Banach space X. $F : \Omega \to 2^{\Omega} \backslash \{\emptyset\}$ be an upper semi-continuous and condensing multivalued map. If for every $x \in \Omega$, $F(x)$ is closed and convex set in Ω, then F has a fixed point in Ω.*

Definition 1 *An operator A is called a generator of an integrated semigroup, if there exists $w \in \mathbb{R}$ such that $(w, +\infty) \subset \rho(A)$ ($\rho(A)$ is the resolvent set of A), and there exists a strongly continuous exponentially bounded family $\{T(t)\}_{t \geq 0}$ of linear bounded operators such that,* $T(0) = 0$, *and* $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} T(t)\, dt$ *for all $\lambda > w$.*

Let A be a Hille-Yosida operator, $A_0 = A$ on $D(A_0) = \{y \in D(A); Ay \in \overline{D(A)}\}$, and $\{T(t)\}_{t\geq 0}$ be the locally Lipschitz continuous integrated semigroup generated by A. Thus from [8], the derivative $\{T'(t)\}_{t\geq 0}$ is a C_0−semigroup on $\overline{D(A)}$ generated by A_0. For more information about integrated semigroup, please see [5].

For our study we use the standard definition of Caputo fractional derivative introduced in [6].

Definition 2 *We say that $y \in \mathcal{K}$ is an integral solution of* (1) − (3) *if condition* (3) *satisfies, $\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s)\, ds \in D(A)$ for $t \in J$, and there exist a function $v \in L^1(J, E)$, such that $v(t) \in F(t, y_t)$ a.e. $t \in J$ and y achieves*

$$y(t) = \begin{cases} T'(t)[\phi(0) - h_0(y)] + \sum\limits_{0<t_k<t} T'(t - t_k) I_k(y(t_k^-)) \\ \quad + \lim\limits_{\lambda\to\infty} \dfrac{1}{\Gamma(\alpha)} \sum\limits_{0<t_k<t} \displaystyle\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T'(t_k - s) B_\lambda[v(s) + \mathfrak{B}u(s)]\, ds \\ \qquad + \lim\limits_{\lambda\to\infty} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^{t} (t - s)^{\alpha-1} T'(t - s) B_\lambda[v(s) + \mathfrak{B}u(s)]\, ds, \qquad t \in J_k, \end{cases} \tag{4}$$

where $J_k :=]t_k, t_{k+1}]$, $k = 1, \cdots, \sigma$ and $B_\lambda = \lambda R(\lambda, A) = \lambda(\lambda I - A)^{-1}$ such that for all $x \in \overline{D(A)}$, $B_\lambda x_{\lambda\to\infty} \to x$. Moreover, from the Hille-Yosida condition we have $\lim\limits_{\lambda\to+\infty} |B_\lambda x| \leq M|x|$.

Definition 3 *The fractional inclusions problem* $(1)-(3)$ *is named nonlocally controllable on J, if for any* $\phi \in \mathfrak{D}$ *and any final state* $y_1 \in \overline{D(A)}$*, there exists a control function* $u \in L^2(J;U)$ *such that the integral solution* $y(\cdot)$ *of* $(1)-(3)$ *satisfies* $y(b)+h_b(y)=y_1$.

Theorem 1 *Let* $F: J\times\mathfrak{D}\to\mathcal{P}(E)$ *is a* L^1*-Caratheodory multivalued map with convex and compact values and* $\phi(0)\in\overline{D(A)}$*. Assume the following conditions are satisfied:*

K_1. *A satisfies the Hille-Yosida condition;*

K_2. *the operator* $T'(t)$ *is compact in* $\overline{D(A)}$ *for* $t>0$ *and satisfies* $\sup_{t\in[0,b]}\|T'(t)\| = \widetilde{M} < \infty$ *where* $\widetilde{M}$ *is constant;*

K_3. *there exists two constants* $\xi>0$ *and* $0<k_1<1$ *such that*
   (a) $|h_t(u)|\le\xi$*, for all* $u\in\mathcal{K}$*, and* $t\in[-r,0]$;
   (b) $|h_t(u)-h_t(w)|\le k_1\|u-w\|_{\mathcal{K}}$*, for all* $u,w\in\mathcal{K}$*, and* $t\in[-r,0]$;

K_4. *there exist constants* $c_k>0$ *and* $d_k>0, k=1,\dots,\sigma$ *with* $\widetilde{M}\sum_{k=1}^{\sigma} d_k\le k_1$ *such that*
   1) $|I_k(y)|\le c_k$*, for all* $y\in E$;
   2) $|I_k(x)-I_k(y))\le d_k|x-y|$*, for all* $x,y\in E$;

K_5. *there exist a function* $p\in L^{\frac{1}{q}}(J,\mathbb{R}_+)$*,* $q\in]0,\alpha[$ *such that*

$$\|F(t,u)\|_{\mathcal{P}(E)} := sup\{|v| : v\in F(t,u)\| \le p(t)(\|u\|_{\mathfrak{D}}+1)\ \text{for each}\ (t,u)\in J\times\mathfrak{D};$$

K_6. *the linear operator* $W: L^2(J,U)\longrightarrow\overline{D(A)}$ *defined by*

$$Wu=\lim_{\lambda\to\infty}\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<b}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T'(t_k-s)B_\lambda(\mathfrak{B}u(s))\,ds+\lim_{\lambda\to\infty}\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-s)^{\alpha-1}T'(b-s)B_\lambda(\mathfrak{B}u(s))\,ds$$

has a bounded invertible operator $W^{-1}:\overline{D(A)}\longrightarrow L^2(J,U)\setminus KerW$*, and there exists two positive constants* R_1,R_2 *such that* $\|\mathfrak{B}\|\le R_1$ *and* $\|W^{-1}\|\le R_2$.

Then the problem $(1)-(3)$ *is nonlocally controllable on* $[-r,b]$.

Proof. We transform the problem $(1)-(3)$ into a fixed point problem. Consider the operator $\mathcal{N}:\mathcal{K}\longrightarrow\mathcal{P}(\mathcal{K})$ defined by $\mathcal{N}(y)=\{g\in\mathcal{K}, g(t)$ achieves (4)$\}$. Using hypothesis (K_6) for an arbitrary function y and $t\in J_k, k=1\cdots,\sigma$, we define the control

$$u(t)=W^{-1}\Big[y_1-h_b(y)-T'(b)[\phi(0)-h_0(y)]-\lim_{\lambda\to\infty}\frac{1}{\Gamma(\alpha)}\sum_{0<t_k<b}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}T'(t_k-s)B_\lambda v(s)\,ds$$
$$-\lim_{\lambda\to\infty}\frac{1}{\Gamma(\alpha)}\int_{t_k}^{b}(b-s)^{\alpha-1}T'(b-s)B_\lambda v(s)\,ds-\sum_{0<t_k<b}T'(b-t_k)I_k(y(t_k^-))\Big](t),$$

where $v\in S_{F,y}$. We demonstrate that the operator $\mathcal{N}$ has a fixed point which is an integral solutions of $(1)-(3)$. The proof is devided into the following steps.

First step: The values of $\mathcal{N}$ are convex and closed.

We can easily get it because the values of F is convex and compact.

Second step: $\mathcal{N}$ maps bounded sets into bounded sets in $\mathcal{K}$.

for any $\xi>0$ we need to show that there exists a constant $\eta>0$ such that $\|\mathcal{N}(y)\|_{\mathcal{P}(E)}\le\eta$ for each y belong to the bounded, closed and convex set $\Lambda_0=\{y\in\mathcal{K},\|y\|_{\mathcal{K}}\le\xi\}$ of $\mathcal{K}$.

Let $g\in\mathcal{N}(y)$ and $y\in\Lambda_0$, by applying Hölder inequality, using the fact that $\lim_{\lambda\to+\infty}|B_\lambda|\le M$ and the estimates $(K2),(K4)$ and $(K5)$, we obtain $\|g\|_{\mathcal{K}}\le\eta$.

Third step: $\mathcal{N}$ is upper semi-continuous and condensing.

In this last step we present the decomposition of $\mathcal{N}$ as $\mathcal{N}=\mathcal{N}_1+\mathcal{N}_2$, where the operators $\mathcal{N}_1$ and $\mathcal{N}_2$ are defined by

$$(N_1y)(t):=\begin{cases}\phi(t)-h_t(y), & t\in[-r,0],\\ \sum\limits_{0<t_k<t}T'(t-t_k)I_k(y(t_k^-)), & t\in J_k=]t_k,t_{k+1}], k=1,\dots,\sigma.\end{cases}$$

$$(N_2y)(t) := \left\{ g \in \mathcal{K};\, g(t) = \left\{ \begin{array}{ll} 0 & t \in [-r,0], \\ \lim\limits_{\lambda\to\infty} \dfrac{1}{\Gamma(\alpha)} \sum\limits_{0<t_k<t} \displaystyle\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} T'(t_k - s) B_\lambda [v(s) + \mathfrak{B}u(s)]\, ds & \\ \quad + \lim\limits_{\lambda\to\infty} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1} T'(t-s) B_\lambda [v(s) + \mathfrak{B}u(s)]\, ds, & t \in J_k, k = 1, \cdots, \sigma. \end{array} \right. \right\}$$

and we show that:

1. $\mathcal{N}_1$ is a contraction operator, we make sure through conditions (K_4) and (K_5);
2. $\mathcal{N}_2$ is upper semi-continuous and completely continuous (bounded and equicontinuous)
   From a direct application of corollarly 1 and Lemma 1, we find $\mathcal{N}$ is upper semi-continuous.
   On the other hand, we show that $\mathcal{N}_2$ is completely continuous. Namely we confirme that $\mathcal{N}_2(\Lambda_0)$ is bounded and equicontinuous. We can easily deduce them through step 2 and the above conditions.
   Consequently, all conditions of Lemma 2 are hold, thus the operator $\mathcal{N}$ has an integral solution. We remark that any integral solution achieves $y(b) + h_b(y) = y_1$ which means that problem $(1) - (3)$ is nonlocally controllable on $[-r, b]$.

CONCLUSION

In this manuscript, we have succeeded in presenting outstanding and fruitful results in the field about controlled problems by studying the controlability results of impulsive functional semilinear differential inclusions with fractional order, when the linear part is a non-densely defined operator and satisfies the Hille-Yosida condition. The objective to be obtain here is achieved by using fixed point theorem for condensing multivalued map due to Martelli, the know fact about integrated semiroup and multivalued map, and Caputo fractional derivative under suitable conditions like those are in the above. In the last, I want to leave a field for research in this paper, how can we study problem (1) – (3) with Hadamard fractional derivative and infinite delay.

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