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Mokhtar BOULOUDENE

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NONLINEAR p-LAPLACIAN BOUNDARY VALUE PROBLEMS IN THE FRAME OF CONFORMABLE FRACTIONAL DERIVATIVES

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Committee members:

Chairperson	**Djamila SEBA**	Professor	U. M. B. B.
Examiner	**Maamar BENBACHIR**	Professor	U. S. D. Blida
Examiner	**Toufik LAADJ**	Associate Professor	U. S. T. H .B.
Examiner	**Salima MECHROUK**	Associate Professor	U. M. B. B.
Advisor	**Amar HEMINNA**	Professor	U. S. T. H. B.
Co-Advisor	**Yassine ADJABI**	Associate Professor	U. M. B. B.

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Nonlinear p-Laplacian boundary value problems in the frame of conformable fractional derivatives

Abstract

Turbulent flow in a porous medium is a fundamental mechanics problem. In 1983, L. S. Leibenson introduced the p-Laplacian equation as a model to the mentioned problem. As a result of intensive development of fractional derivative, a natural generalization of the p-Laplacian differential equation was proposed through the replacement of ordinary derivative by a fractional derivative yielding fractional p-Laplacian equation.

The objective of this thesis is to develop a p-Laplacian boundary value problems using fractional calculus. By applying the coincidence degree theory, the existence of at least one solution for a class type of p-Laplacian in the frame of conformable in the sense of Caputo at the resonance is subjected to different boundary conditions. Another important contribution of our thesis is the study of a fourth point singular boundary value problem class of p-Laplacian with conformable derivative by the upper and lower solutions method associated with the fixed point theorem in partially ordered sets. Necessary and sufficient conditions for the existence of at least one positive solution are established. Our work investigate the dependence of a solution on the parameters and were able to generalize some earlier results in the literature.

Key Words : Local fractional derivative; nonlocal fractional derivative; conformable derivative; fractional integral; nonlinear boundary value problem; nonlocal multipoint boundary value problem; p-Laplacian operator; necessary and sufficient conditions; singular nonlinear boundary value problem; positive solution; existence; uniqueness; continuous dependence and continuation of solutions; Green function method; upper and lower solutions method; fixed point theorems; coincidence theorems; coincidence degree; fredholm operators; index theories; Mawhin's continuation theorem; resonance; cone.

2020 Mathematics Subject Classification : 26A33, 34A08, 34A12, 34B15, 34B10, 34B16, 34B18, 47H10, 54H25, 47A53.

مسائل p-لابلاس الغير خطية ذات قيم حدية باستخدام حساب الإشتقاق الكسري المطابق

ملخص

يعد الجريان المضطرب في وسط مسامي مسألة ميكانيكية أساسية. لدراسة هذا النوع من المسائل ، في عام 1983 ، قدم ليبنسن معادلة p-لابلاس. نظرا للتطوير الملحوظ للإشتقاق أو التفاضل الكسري ، فإن التعميم الطبيعي لمعادلة p-لابلاس هو إستبدال المشتق العادي بمشتق كسري لإنتاج معادلة p-لابلاس الكسرية ، والتي يمكن اعتبارها حالة خاصة لتعميم معادلة p-لابلاس. الهدف من هذه الأطروحة هو تطوير مسألة القيم الحدية لمعادلة p-لابلاس بواسطة حساب الإشتقاق الكسري. باستخدام نظرية درجة المصادفة ، أثبتنا وجود حل واحد على الأقل لصنف من معادلات p-لابلاس قابل للتوافق بمفهوم كيبتو للمعادلات التفاضلية غير الخطية عند الرنين بنقطتين أو ثلاث نقاط والتي يمكن أن تخضع لقيم حدية مختلفة. بالإضافة إلى ذلك ، قمنا بدراسة فئة أخرى من مسائل القيم الحدية الحرجة رباعية النقاط لمؤثر p-لابلاس مع مشتق مطابق بواسطة طريقة الحلول العلوية والسفلية المرتبطة بنظرية النقطة الثابتة في مجموعات مرتبة جزئيًا ، أين توصلنا إلى وضع الشروط اللازمة والكافية لوجود حل موجب واحد على الأقل. أيضًا ، تحرينا عن ارتباط الحل بدرجة المعادلة التفاضلية المطابقة. عَمَلُنا في هذه الأطروحة يعمم بعض النتائج السابقة.

Problèmes du p-Laplacien non linéaires à valeurs aux limites dans le cadre des dérivées fractionnaires conformables

Résumé

L'écoulement turbulent dans un milieu poreux est un problème mécanique fondamental. Pour étudier ce type de problème, en 1983, L. S. Leibenson a introduit l'équation p-Laplacienne. Comme conséquence du développement intensif du dérivé fractionnaire, une généralisation naturelle de l'équation différentielle p-Laplacienne est de remplacer le dérivé ordinaire par un dérivé fractionnaire pour donner l'équation p-laplacienne fractionnaire, qui peut être considérée comme un cas particulier de la généralisation du p -Équation différentielle Laplacienne. L'objectif de cette thèse est de développer un problème de valeurs aux limites p-laplaciennes par un calcul fractionnaire. En utilisant la théorie du degré de coïncidence, l'existence d'au moins une solution pour un type de p-Laplacien avec dérivée conformable au sens de Caputo, les équations différentielles non linéaires à résonance à deux ou trois points peuvent être soumises à différentes conditions aux limites. De plus, nous étudions une autre classe de problème singulier de quatre valeurs limites de l'opérateur p-Laplacien avec dérivée conformable par la méthode des solutions supérieure et inférieure associée au théorème du point fixe dans des ensembles partiellement ordonnés, une condition nécessaire et suffisante pour l'existence d'au moins une solution positive est établie. Aussi, nous étudions la dépendance de la solution par rapport à l'ordre de l'équation différentielle conformable. Notre travail généralise certains résultats antérieurs de la littérature.

LIST OF ACRONYMS

BVP	:	Boundary Value Problem;
CBVP	:	Conformable Boundary Value Problem;
CCFD	:	Conformable -Caputo Fractional Derivative;
CD	:	Conformable Derivative;
CFD	:	Conformable Fractional Derivative;
DE	:	Differential Equation;
FBVP	:	Fractional Boundary Value Problem;
FC	:	Fractional Caculus;
FDE	:	Fractional Differential Equation;
FPLE	:	Fractional p-Laplacian Equation;
FPT	:	Fixed Point Theorems;
IVP	:	Initial Value Problem;
LFD	:	Local Fractional Derivative;
NLFD	:	Nonlocal Fractional Derivative;
ODE	:	Ordinary Differential Equation;
PLE	:	p-Laplacian Equation.

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Chapter 1

GENERAL INTRODUCTION

Contents

1.1 Introduction

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type problem, Leibenson [1] introduced the following model

$$u_t = \frac{\partial}{\partial x}\left(\frac{\partial\left(u^m\right)}{\partial x}\left|\frac{\partial\left(u^m\right)}{\partial x}\right|^{p-1}\right), \tag{1.1}$$

where $m \geq 2, 1/2 \leq p \leq 1$. Generally, when $m > 1$, equation (1.1) is called porous medium equation [2]; when $0 < m < 1$, called diffusion equation; when $m = 1$, called heat equation, which often appears in non-Newtonian liquid [3]. For the study of equation (1.1), ones reduced equation (1.1) into the following p-Laplace equation

$$\left(\varphi_p\left(u'\left(t\right)\right)\right)' = f\left(t, u\left(t\right), u'\left(t\right)\right), \ t \in (0, 1), \tag{1.2}$$

where $\varphi_p\left(s\right) = \left|s\right|^{p-2} s, \ \infty > p > 1, \ s \in \mathbb{R}$. Obviously, when $p = 2$, equation (1.2) becomes to the general second order differential equation.

The equations with p-Laplacian operator (1.2) arise in the modeling of different physical and natural phenomena, e.g., non-Newtonian mechanics, nonlinear elasticity and glaciology, population biology, combustion theory, nonlinear flow laws, system of Monge-Kantorovich partial differential equations.

p-Laplacian equation have been studied extensively over many years. This study investigates on the existing PLE, their analysis in the field of FC[(1)] using fractional derivatives, and finally proposes an extension of conformable derivative with fractional order.

On the other hand, multi-point BVPs of ordinary differential equations arise in a variety of different areas of applied mathematics and physics. For example, the vibrations of a guy

[(1)]Math keywords and entries in the new Mathematics Subject Classification 2010, related to FC.

26A33 Fractional derivatives and integrals.

05C72 Fractional graph theory, fuzzy graph theory.

33E12 Mittag Leffler functions and generalizations.

34A08 Fractional ordinary differential equations.

34K37 Functional-differential equations with fractional derivatives.

35R11 Fractional partial differential equations.

60G22 Fractional stochastic processes, including fractional Brownian motion.

wire of a uniform cross-section composed of N parts of different densities can be set up as a multi-point BVP [4]. Many problems in the theory of elastic stability can also be handled by the method of multi-point problems. In recent years, some researchers used coincidence degree theorem to study the existence of at least one solution for some multi-point BVPs [5]. Singular differential BVP arise from many branches of applied mathematics and physics [6, 7]; for example, gas dynamics, Newtonian fluid mechanics, nuclear physics, engineering sciences and so on can all be described using the above problems. In particular, the study of positive solutions for multi-point BVPs has attracted much attention in recent years.

1.2 Problem statement

Fractional calculus is a generalization of classical calculus. Due to the fractional order modeling in different physical fields, scientists have shown their interests in the exploration of its different features. The demand of FC is increasing day by day due to accuracy in modeling of hereditary problems in physics and engineering in the last two decades. Some of the recently well considered aspects are including; existence of positive solutions, analytical solutions numerical solutions of FDEs involving integral boundary conditions, local boundary conditions, non local boundary conditions, periodic boundary conditions, anti boundary conditions and multi points boundary conditions of fractional order.

Fractional-order derivatives have been extensively studied for many years in various fields such as physics, engineering, applied mathematics including mathematical models and analysis of solutions of fractional differential equations characterizing the behaviors of dynamic systems. Considerable development in the field of FDEs can be found in the literature [8, 9, 10]. The PLE is extended in the field of fractional calculus by the equation defined by

$$\mathcal{D}^{\alpha}_{0+,t}\left[\varphi_p\left(\mathcal{D}^{\beta}_{0+,\tau}\left[u\right]\right)\right] = f\left(t, u\left(t\right), -\mathcal{D}^{\beta}_{0+,t}\left[u\right]\right),\ t \in \left(0, 1\right), \tag{1.3}$$

where $\mathcal{D}^{\alpha}_{0+}$ and $\mathcal{D}^{\beta}_{0+}$ are the fractional derivatives, with $\varphi_p^{-1} = \varphi_q$, $1/p + 1/q = 1$, with f continuous (but not necessarily locally Lipschitz continuous).

Another important property of FDs is that they are nonlocal : a function's fractional derivative at a particular point is not just influenced by the function's behavior near that point. This novelty arising in fractional but not classical calculus has led to many applications in fields such as control theory and dynamical systems.

The generalized derivative constitutive relation (equation 1.3) may be viewed as an extension of the standard model (equation 1.2) in the sense that the derivatives are no longer limited to being of integer order.

Fractional order DEs with φ_p operator extensively attract the attentions of researchers of various fields such as physics, mechanics, electrodynamics and dynamical systems. Three types of generalized differential operators were applied to the FPLE in the state of-the-art, producing the fractional conformable in the sense of Caputo and the conformable differential equations.

In the past few decades, many important results relative to (1.3) with certain boundary value conditions have been obtained. However, to the best of our knowledge, there are relatively few results on boundary value problems for FPLEs.

However, the newly introduced CFD has never been applied to FPLE despite its numerous applications in physics, natural sciences and engineering [11, 12, 13]. An analysis of PLE modeled with CFD will definitely enrich further the literature on $p-$Laplacian equations.

In this thesis, we analyze $p-$Laplacian equations modeled with the recently developed new CCFD, also referred to as the fractional calculus. This implies that the problem of well posedness related to the new CCFD model applied to PLE is addressed here.

The innovations of this thesis can be shown in two points: firstly, comparing with the literature, we consider more general nonlinear p-Laplacian boundary value problems involving the conformable derivative.

Secondly, by using the coincidence degree theory of Mawhin-Ge, lower and upper solutions method and Guo-Krasnosel'skii theorem, we study the p-Laplacian equation with three kinds of boundary value conditions and obtain new existence result of solutions. A study of existence results, our targets are to get existence of solutions for fractional differential equations involving nonlinear operator φ_p with two types of fractional derivatives that is the conformable derivative and conformable derivative in the sense of Caputo.

Noting that when $p = 2$, it will degenerate into a linear problem, so this thesis enriches the existing results.

1.3 Motivation

Originated from a complaint made on previous CFD whose mathematical expressions appeared cumbersome [14, 15, 16], the CFD was developed as a simplified expression with non singular kernel. The use of conformable removed difficulties previously experienced by the older fractional (non-integer) order derivatives in solving fractional related models. Another benefit of the CFD is its suitability for analysis tools such as Laplace and Fourier transforms, and also the effective description of behavior in applications such as viscoelastic media, thermal media, and electromagnetic systems. Contrary to the CFD, previous fractional-order derivatives including the old Caputo and other variant of fractional-order derivatives fitted more at modeling mechanical phenomena such as damage, plasticity, fatigue and fluid flow.

The well-posedness of a PLFE model with the new CD will be investigated. The existing literature in the field of fractional calculus indicates that there is still more to be done. New approaches in proving the existence of generalized PLE have been studied. Our main focus is then on the investigation and evaluation of the new CFD and its possible subsequent remnements on PLE. Consequently, two needs drive our interest in the proposed research. There are as follows:

– The need to demonstrate that the PLE modeled with the new CFD is a wellposed problem and emphasize on a potential application in the field of applied sciences. Thus various techniques used in the literature will be investigated to prove the well-posedness of the PLE modeled with the new CFD.

– The need to investigate the existence, uniqueness of solutions (primarily positive solutions) and the dependence of the solution proofs of the recently developed fractional derivative equations. Recent extensions of the original fractional derivative equations such Riemann-Liouville and Caputo fractional derivative equations to the newly developed CFD will also be investigated.

The intended demonstration of the well-posedness for the problem formulated with the new CFD in modelling PLE is a valuable contribution in the state-of-the-art of fractional calculus for the reason that, to the best our knowledge, such demonstration for PLE with CFD has not been done. Our studies is only directed to the FPLE though non-linear evolution equation can be object of future work.

1.4 Research aim and objectives

It is generally known that the p-Laplacian equations are derived from nonlinear elastic mechanics and non-Newtonian fluid theory. In view of their significance in theory and practice, more and more attention is being paid to the existence of solutions for some fractional p-Laplacian problems. Thus, many important results have been achieved in this regard.

As already indicated in Section 1.1, the purpose of this study is to prove the well-posedness for the newly developed CFD model applied to PLE by taking advantage of the fact that the integral in CFD. Once well-posedness established, advanced analysis of derived solutions can be effected.

A problem is well-posed if it satisfies the following three properties which are: (1) existence of a solution, (2) the existing solution is unique, and (3) the behavior's solution depends continuously on the data and parameters i.e., the dependence of the solution. These three properties summarizes our research objectives as each of them will be investigated and demonstrated with respect to the PLE with CFD.

The main objectives of this thesis are:

(i) to analyze the basis of the method. This covers the first and second chapters which deal with the fractional operators, the functions used in fractional calculus, p-Laplacian operator, upper and lower solutions and fixed point theorems.

(ii) to develop new nonlinear p -Laplacian two -point local boundary value problems at resonance with FCD, we use the Green function method to obtain a general representation of solutions, Guo-Krasnosel'skii theorem and lower and upper solutions.

(iii) to develop new nonlinear p -Laplacian three -point boundary value problems with FCD, by using the coincidence degree theory of Mawhin..

(iv) to develop new nonlinear singular p -Laplacian four- point nonlocal boundary value problems with conformable derivative, by using the coincidence degree theory of Mawhin-Ge.

Our main results are contained in Chapters 4 and 5. Our work generalizes some earlier results in the literature.

1.5 Research Methodology

Various related topics such as applied functional analysis, classical calculus, fractional calculus, differential equations with linear p-Laplacian equations in particular are investigated. A literature review on the FCs and on the fractional differentiation in particular is included in the thesis for good understanding of the history and development of this field of study. Strengths, weaknesses and limitations related to FDs and their applications are identified and elaborated. We have also reviewed the full literature on differential equations and in particular the non linear PLEs. The evaluation methods applied to these p-Laplacian equations are presented and related theorems, existence, uniqueness (positive solutions) and the dependence of solutions established. The concepts of lower and upper solution have been considered for this matter, especially that the upper and lower method provides the necessary and sufficient conditions to determine well-posedness. Hence we have made use of both methods and mathematical tools at our disposal to establish the well-posedness of the conformable fractional derivative model as applied to non linear p- Laplacian equations. Our investigation has used, as a departure point, the FPLE defined as

$$ {}^{c}\mathcal{D}^{\alpha,\rho}_{0+,t}\left[\varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[u\right]\right)\right] = g\left(t\right),\ t\in\left(0,1\right), \tag{1.4} $$

where $f\left(t, u\left(t\right), -{}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[u\right]\right)$ is replaced by $g\left(t\right)$. Then the concepts of fixed point theorems, upper and lower solutions have been used to our models and establish conditions for existence and uniqueness of positive solutions.

1.6 Dissertation outline

This doctoral dissertation is structured as follows:

The first chapter summarizes the goal of our work. The subject and aims of a research are introduced in Chapter 1. Motivation and problem statement for the concept of FBVPs and FPLEs with CFDs are considered. An overview of research methodology is given. The structure of the thesis is presented.

In Chapter **2**, we introduce the theory of fractional calculus, we give the definitions of terminologies used as well as state the basic tool employed in the proof of our results. This chapter consists of four Sections. In Section one, we present "A brief visit to the history of the fractional calculus", and in Section two, we gives some "Functions used in fractional calculus". In Section three, we present some "Non local fractional derivatives" Finally, in the last Section, we present some "Local fractional derivatives".

In Chapter **3**, we introduce notations and some preliminary notions.: some basic concepts, and useful famous theorems and results (notations, definitions, lemmas, p Laplacian, and fixed point theorems, ...) which are used throughout this thesis.

While Chapter **4** , here, some results for a class of boundary value problems for nonlinear fractional differential equations and for non local boundary value problem in Banach space with fractional conformable derivative in sense Caputo are discussed. An example is given to illustrate the applicability of our main results. By using the coincidence degree theory due to Mawhin and constructing the suitable operators, the existence of solutions for boundary value problems of fractional differential equations at resonance, we investigate the existence of solutions for the fractional differential equation at resonance

$$ {}^{c}\mathcal{D}_{0+,t}^{\alpha,\rho}\left[\varphi_p\left({}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right] = f\left(t, u(t), -{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right),\ t \in (0,1), \tag{E$_1$} $$

can be subjected to different boundary conditions:

- with the condition for (E_1) is

$$ u(0) = u(1),\ {}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u](0) = 0,\ 0 < \beta < \alpha \leq 1, \tag{C$_1$} $$

- we also consider the boundary condition of the type

$$ u(0) = u(\eta),\ {}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u](1) = 0, 0 < \beta \leq 1 < \alpha \leq 2, \tag{C$_2$} $$

where ${}^{c}\mathcal{D}_{0+}^{\alpha,\rho}$ and ${}^{c}\mathcal{D}_{0+}^{\beta,\rho}$ are the conformable derivatives in the sense of Caputo with $0 < \rho, .\varphi_p^{-1} = \varphi_q$, $1/p + 1/q = 1$, with f continuous (but not necessarily locally Lipschitz continuous).

By employing a fractional conformable integral operator and the coincidence degree theory due to Mawhin and constructing the suitable operators, the existence of at least one solution for a type of boundary value problem with p-Laplacian of fractional conformable in the sense of Caputo differential equations at resonance with Dirichlet condition is obtained. An examples are given to illustrate our results.

The existence results of nonlinear classical p-Laplacian equation follow as a special case of our results. We also aim at showing important connections of the results here with those including Riemann–Liouville fractional and classical integrals.

In Chapter **5**, is based on the published works“*Nonlinear singular p -Laplacian four- point nonlocal boundary value problems with conformable derivative* ” (M. Bouloudene et al.). This chapter studies a class of fourth point singular boundary value problem of p -Laplacian operator with conformable derivative nonlinear differential equations.

We investigate the following boundary value problems of conformable nonlinear differential equations with p-Laplacian operator and a nonlinear term dependent on the fractional derivative of the unknown function

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)(t)=f\left(t,u(t),-\mathbf{T}_{0+}^{\alpha}u(t)\right),\ t\in(0,1), \tag{E$_2$}$$

with the four-point boundary conditions

$$u(0)=0,\ u(1)=b_1u(\xi_1),\ \mathbf{T}_{0+}^{\alpha}u(0)=0,\ \mathbf{T}_{0+}^{\alpha}u(1)=b_2\mathbf{T}_{0+}^{\alpha}u(\xi_2), \tag{C$_3$}$$

where $\mathbf{T}_{0+}^{\beta}$ and $\mathbf{T}_{0+}^{\alpha}$ are the conformable derivatives with $1<\alpha,\beta\leq 2, 1<\alpha\leq\alpha+\beta-1,\ 0\leq b_1,b_2\leq 1,\ 0<\xi_1,\xi_2<1$.

By using the upper and lower solutions method and Krasnosel'skii's fixed point theorems on cones, necessary and sufficient conditions for the existence of $C^2([0,1])$ positive solutions are obtained. Our nonlinearity f may be singular at $t=0$ and/or $t=1$. Example is given to illustrate the main results. At the end of this chapter, we investigate the dependence of the solution on the order of the conformable differential equation and on the initial condition.

Lastly, conclusion and possible directions for future work with a bibliography at the end.

1.7 List of Publication and Manuscripts

The work is presented as a series of one published paper (see [17]), one submitted manuscript, and one manuscript in preparation.

– Nonlinear singular p-Laplacian boundary value problems in the frame of conformable derivative Discrete & Continuous Dynamical Systems - S doi: 10.3934/dcdss.2020442

– Nonlinear p -Laplacian two -point local boundary value problems with fractional conformable derivative in the sense of Caputo, with F. Jarad, Y. Adjabi and T. Abdeljawad, submitted on 2021.

– Nonlinear p -Laplacian three-point local boundary value problems with fractional conformable derivative in the sense of Caputo, with Y. Adjabi and F. Jarad, To be submitted.

Chapter 2

BACKGROUND TO FRACTIONAL CALCULUS

Contents

2.1 A brief overview of generalized derivatives

Fractional calculus is the field of mathematical analysis which deals with the investigation and application of integrals and derivatives of arbitrary order. It may be considered as an old but novel topic. It is an old topic since, starting from the speculations of Liebnitz (1695) and Euler (1738), it has been developed up to the current level.

Many known mathematicians contributed to this theory over the years. Thus, 30 September 1695 is the exact date of birth of the "fractional calculus". Therefore, the fractional calculus it its origin in the works by Leibnitz, L'Hopital (1695), Bernoulli (1697), Euler (1730), and Lagrange (1772). Some years later, Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Grunwald (1867), Letnikov (1868), Nekrasov (1888), Hadamard (1892), Heaviside (1892), Hardy (1915), Weyl (1917), Riesz (1922), P. Levy(1923), Davis (1924), Kober (1940), Zygmund (1945), Kuttner (1953), J. L. Lions (1959), and Liverman (1964)... have developed the basic concept of fractional calculus.

But it is a novel topic since only around 30 years ago it has become an object of specialized conferences and treatises. In June 1974, Ross has organized the "First Conference on Fractional Calculus and its Applications" at the University of New Haven, and edited its proceedings in 1974. Thereafter, Oldham and Spanier published the first monograph devoted to "Fractional Calculus" (1974). The integrals and derivatives of non-integer order, and the fractional integrodifferential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics. There exists the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbas and Marichev which was published in Russian in 1987 and in English in 1993 (for more details see [10, 11, 12, 13]) The works devoted substantially to fractional differential equations are : the book of Nishimoto (1991), Miller and Ross (1993), Kiryakova (1994), Rubin (1975), of Podlubny (1999), by Kilbas et al. (2006), by Diethelm (2010), by Ortigueira (2011), by Abbas et al. (2012), and by Baleanu et al. (2012).

Fractional calculus is a generalization of differentiation and integration to arbitrary order (non-integer) fundamental operator D_{a+} where ; $a \in \mathbb{R}$. Several approaches to fractional derivatives exist [(1)]: Riemann-Liouville, Hadamard, Grunwald-Letnikov, Weyl and Caputto

[(1)] A differential equation is fractional if it involves an operator that can be considered to be between a etc. The Caputo fractional derivative is well suitable to the physical interpretation of initial conditions and boundary conditions. We refer readers, for example, to the books such as [11] and references therein. Fractional derivatives of a function with respect to another function have been considered in the classical monograph by Samko et al. [10] and Kilbas et al. [11] as generalization of Riemann-Liouville. This fractional derivative is different from the other classical fractional derivative because the kernel is in terms of function. Recently they were reconsidered by Almeida in [18] where the Caputo-type regularization of the existing definition and some interesting properties are provided. Several properties of this operator could be found in [11, 10, 19, 20, 21]. For some special cases of ψ, we obtain the Caputo fractional derivative [11], the Caputo-Hadamard fractional derivative [22] and the Caputo–Erdélyi–Kober fractional derivative [23]. Nevertheless, the complexity and the lack of some basic properties satisfied by usual derivative have leaded the scientists to improve new local fractional derivatives and integrals. The authors in [14, 15] introduced the so-called conformable derivative. Very recently, a new variant of the fractional conformable integral operator was introduced by F. Jarad et al. in [16, 2017]. Later, many different definitions of fractional derivatives were derived and presented in [24].

The research on fractional differential equations is very important in both theory and applications [25]. By using nonlinear analysis tools, some scholars established the existence, uniqueness, multiplicity and qualitative properties of solutions, we refer the readers to [26, 27, 28] and the references therein for fractional differential equations, [29, 30, 31] for fractional differential systems, fractional two-point boundary value problems [32], fractional boundary value problems at resonance [33, 34], fractional multi-point problems with nonresonance [35, 36], fractional initial value problems [37], fractional impulsive problems [38], fractional inclusion problems [39, 40, 41], fractional integral boundary value problems [42, 43, 44, 45, 46, 47, 48, 49, 50], fractional problems with lower and upper solution [51, 52, 53], fractional control problems [25, 54, 55, 56] and fractional integro-differential equations [37, 40, 57, 58, 59, 60, 62].

The boundary value problems defined by fractional differential equations have been extensively studied over the last years. Particularly, the study of solutions of fractional differential

$(k-1)$th and kth order differential operator, for some positive integer k, and it is said to be of fractional-order if this operator is the highest order operator in the equation.

and integral equations is the key topic of applied mathematics research. Many interesting results have been reported regarding the existence, uniqueness, multiplicity and stability of solutions or positive solutions by means of some fixed point theorems, such as the Krasnosel'skii fixed point theorem, the Schaefer fixed point theorem and the Leggett-Williams fixed point theorem.

2.2 Functions used in fractional calculus

Mathematically, special functions are functions defined on $\mathbb{R}$ or $\mathbb{C}$ and they possess not only series representations, but also integral representations. So we need some various special functions that have appeared in this thesis.

2.2.1 Gamma function $\Gamma(z)$

A gamma function $\Gamma(z)$ can be defined in many ways. The Gamma function $\Gamma(z)$ is the most widely used of all the special functions: it is usually discussed first because it appears in almost every integral or series representation of other advanced mathematical functions. We take as its definition the integral formula

Let $z \in \mathbb{R}_+^*$, the representation of the Gamma function is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{2.1}$$

and

$$\Gamma(z+1) = z\Gamma(z). \tag{2.2}$$

The relation (2.2), yields the useful result

$$\Gamma(n+1) = n!, \;\; n = 0, 1, 2, ...$$

which shows that gamma function is the generalization of factorial function.

2.2.2 Beta function $\mathbf{B}(z, w)$

Let $z, w > 0$, the standard representation of the Beta function is

$$\mathbf{B}(z; w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt. \tag{2.3}$$

The Beta function is a complex function of two complex variables whose analyticity properties will be deduced later, as soon as the relation with the Gamma function has been established.

$$\mathbf{B}(w; z) = \mathbf{B}(z; w)$$

and

$$\forall z, w > 0, \ \mathbf{B}(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

This relation is of fundamental importance. Furthermore, it allows us to obtain the analytical continuation of the Beta function. The proof of (2.3) can easily be obtained by writing the product $\Gamma(z)\Gamma(w)$ as a double integral that is to be evaluated introducing polar coordinates.

2.2.3 Mittag–Leffler function $E_{\alpha,\beta}(z)$

Recently, Mittag-Leffler functions show its close relation to fractional calculus and especially to fractional problems which come from applications. This new era of research attract many scientists from different point of view (see, for example, [11, 12, 13]).

In 1903, the Swedish mathematician G. Mittag-Leffler introduced the one parametric Mittag-Leffler function $E_\alpha(z)$ defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha > 0, \ z \in \mathbb{C}. \tag{2.4}$$

A first generalization of this function was proposed by Wiman in 1905, and he defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + k\alpha)}, \alpha > 0, \beta \in \mathbb{R}, z \in \mathbb{C}.$$

When $\alpha, \beta > 0$ the series is convergent. Later, this function was rediscovered and intensively studied by R. P. Agarwal in 1953 and others, This generalization is referred to as two-parameter Mittag-Leffler function.

Another generalization of the Mittag-Leffler function (2.4) can be found in the contemporary monographs of R. Gorenflo et al. [63, 2014].

2.3 Non local fractional derivatives

Fractional-order operators are nonlocal, i.e. the value of a fractional derivative of a function at a point in the domain depends on values of the function throughout the domain.

2.3.1 Fractional derivatives with singular kernel

In the NLFD, there are several definitions for the operators of integration and differentiation of arbitrary order with singular kernel.

Grunwald-Letnikov fractional derivative

The Grunwald-Letnikov fractional derivative with fractional order α is defined as follows

$$\begin{aligned} {}^{GL}D^{\alpha}u\left(t\right) &= \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}\left(-1\right)^{k}\binom{\alpha}{k}u\left(t-kh\right) \\ &= \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}\frac{\Gamma\left(k-\alpha\right)}{\Gamma\left(k+1\right)\Gamma\left(-\alpha\right)}u\left(t-kh\right),\ \ \alpha>0, \end{aligned} \tag{2.5}$$

with

$$\left(-1\right)^{k}\binom{\alpha}{k}=\frac{-\alpha\left(1-\alpha\right)\left(2-\alpha\right)...\left(k-1-\alpha\right)}{k!}=\frac{\Gamma\left(k-\alpha\right)}{\Gamma\left(k+1\right)\Gamma\left(-\alpha\right)},\ \ 0\leq n-1<\alpha<n. \tag{2.6}$$

Consequently, if $u\in C^{n}\left[a,t\right]$ and by applying the integration by parts, we get

$${}^{GL}D^{\alpha}u\left(t\right)=\sum_{k=0}^{n-1}\frac{u^{(k)}\left(a\right)\left(t-a\right)^{k-\alpha}}{\Gamma\left(k-\alpha+1\right)}+\frac{1}{\Gamma\left(n-\alpha\right)}\int_{a}^{t}\left(t-\tau\right)^{n-\alpha-1}u^{(n)}\left(\tau\right)\mathrm{d}\tau, \tag{2.7}$$

where $n-1<\alpha<n\in\mathbb{Z}^{+}$.

Riemann-Liouville fractional operators

The Riemann-Liouville fractional integral is most frequently used definition of fractional calculus [64].

The left-sided Riemann-Liouville integral operator of order $\beta>0$, of a continuous function $u:[0,\infty)\longrightarrow\mathbb{R}$ is given by

$$J_{t}^{\beta}u\left(t\right)=\frac{1}{\Gamma\left(\beta\right)}\int_{a}^{t}\left(t-\tau\right)^{\beta-1}u\left(\tau\right)\mathrm{d}\tau, \tag{2.8}$$

provided that the right side is pointwise defined on $\mathbb{R}^{+}$.

The corresponding left-sided Riemann–Liouville NLFD of order $n-1 \leq \alpha < n$, of a continuous function $u : [0, \infty) \longrightarrow \mathbb{R}$ is given by

$$D_{a+}^{\alpha} u(t) \equiv \frac{d^{\alpha} u(t)}{d(t-a)^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{n} \int_{a}^{t} (t-\tau)^{n-\alpha-1} u(\tau)\, d\tau, \tag{2.9}$$

provided that the right side is pointwise defined on $\mathbb{R}^{+}$.

Caputo fractional derivative

In the development of the theory of both fractional integration and derivation, as well in the related applications in pure mathematics, the Riemann-Liouville NLFD as defined in (2.9) played a important role. However, the solutions of problems in physics have required a revision of Riemann-Liouville fractional derivative which is difficult to be interpreted physically. Hence the Caputo fractional derivative was proposed in [65]. The Caputo fractional derivative with fractional order of $u(t)$ is defined as follows

$${}^{c}D_{a+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-1-\alpha} u^{(n)}(\tau)\, d\tau. \tag{2.10}$$

This definition is more practical for analytic purpose than the Grunwald-Letnikov fractional derivative. Moreover, the initial conditions in the Caputo approach takes the same form as in the classical differential equations.

Remark 1 *The two definitions of Reimann-Liouville and Caputo are not equivalent and their relation is correlated by the following expression*

$$\begin{aligned} {}^{c}D_{a+}^{\alpha} u(t) &= D_{a+}^{\alpha} \left[u(\tau) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (\tau-a)^{k} \right](t) \qquad (2.11) \\ &= \left(I_{a+}^{n-\alpha} (D^{n} u) \right)(t), \; (a < t < b). \end{aligned}$$

Since there is no clear physical meaning for which the initial conditions are expressed in fractional derivatives, it is preferable to use the Caputo definition in a wide class of practical applications, depending on the nature of the material of system at hand [11].

Hadamard fractional operators

J. Hadamard [66], introduced a new definition of NLFDs and integrals in which he claims:

$$\left(\mathcal{D}_{a+}^{\alpha}u\right)(t)=\frac{1}{\Gamma\left(n-\alpha\right)}\delta^{n}\int_{a}^{t}\left(\ln\frac{t}{\tau}\right)^{n-\alpha-1}u\left(\tau\right)\frac{d\tau}{\tau},\ \alpha\in\left[n-1,n\right) \tag{2.12}$$

and

$$\left(\mathcal{J}_{a+}^{\alpha}u\right)(t)=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{t}\left(\ln\frac{t}{\tau}\right)^{\alpha-1}u(\tau)\frac{d\tau}{\tau},\ \left(0\leq a\right),\ 0<\alpha\leq 1, \tag{2.13}$$

where $\delta=t\dfrac{d}{dt}$ is the so-called δ-derivative.

Generalized fractional integral operator

The generalized fractional integral operator of order for $\alpha\in\left(0,1\right],\rho>0,a\geq 0$ and $t\in\left(a,\infty\right[$ given by [67]

$$\left(\mathcal{J}_{a+}^{\alpha,\rho}u\right)(t)=\frac{\rho^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_{a}^{t}\left(t^{\rho}-\tau^{\rho}\right)^{\alpha-1}u\left(\tau\right)\frac{d\tau}{\tau^{1-\rho}}, \tag{2.14}$$

and the generalized fractional derivative operator

$$\left(\mathcal{D}_{a+}^{\alpha,\rho}u\right)(t)=\frac{\rho^{\alpha}}{\Gamma\left(n-\alpha\right)}\gamma^{n}\int_{a}^{t}\left(t^{\rho}-\tau^{\rho}\right)^{n-\alpha-1}u\left(\tau\right)\frac{d\tau}{\tau^{1-\rho}},\ \alpha\in\left[n-1,n\right), \tag{2.15}$$

where $\gamma=\left(t^{1-\rho}\frac{d}{dt}\right)$ is the so-called γ-derivative.

The relation between these two fractional latter operators is as follows:

$$\left(\mathcal{D}_{a+}^{\alpha,\rho}u\right)(t)=\gamma^{n}\left(\mathcal{J}_{a+}^{n-\alpha,\rho}u\right)(t),\ \alpha\in\left[n-1,n\right). \tag{2.16}$$

The generalized operators (2.14)-(2.15) depend on extra parameter $\rho>0$, which by taking $\rho\to 0^{+}$ reduces to the Hadamard fractional operator and for parameter $\rho=1$ becomes the Riemann–Liouville fractional operator.

The left-sided Caputo type generalized fractional derivatives of u of order α defined by

$$\left({}^{c}\mathcal{D}_{a+}^{\alpha,\rho}u\right)(t)=\mathcal{J}_{a+}^{n-\alpha,\rho}\left(\gamma^{n}u\right)(t),\ \alpha\in\left[n-1,n\right). \tag{2.17}$$

Fractional conformable operators

The left-sided FC integral operator of order for $\alpha\in\left(0,1\right],\rho>0,a\geq 0$ and $t\in\left(a,\infty\right[$ given by

$$\mathfrak{J}_{a+,t}^{\alpha,\rho}\left[u\right]=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{t}\left(\frac{\left(t-a\right)^{\rho}-\left(\tau-a\right)^{\rho}}{\rho}\right)^{n-\alpha-1}u\left(\tau\right)\frac{d\tau}{\left(\tau-a\right)^{1-\rho}} \tag{2.18}$$

and the left-sided FCD operator in Riemann-Liouville setting, respectively, by

$$\mathcal{D}_{a+,t}^{\alpha,\rho}[u] = \frac{1}{\Gamma(n-\alpha)}\left(T_a^{\rho,n}\right)\int_a^t \left(\frac{(t-a)^\rho - (\tau-a)^\rho}{\rho}\right)^{n-\alpha-1} u(\tau)\frac{d\tau}{(\tau-a)^{1-\rho}},\ \alpha\in[n-1,n), \tag{2.19}$$

where $T^{\rho,n} = T^\rho \circ T^\rho \circ ... \circ T^\rho$ n times and $n = [\alpha]+1$ and $T^{\rho,n}$ is the left and right conformable differential operators presented in (2.34).

The left FCDs in the sense of Caputo is given by

$$\begin{aligned} {}^c\mathcal{D}_{a+,t}^{\alpha,\rho}[u] &= \mathfrak{J}_{a+}^{n-\alpha,\rho}\left(T_{a+}^{\rho,n}u\right)(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\frac{(t-a)^\rho - (\tau-a)^\rho}{\rho}\right)^{n-\alpha-1}\left(T_a^{\rho,n}\right)u(\tau)\frac{d\tau}{(\tau-a)^{1-\rho}}. \end{aligned} \tag{2.20}$$

Lemma 1 *[16]Let $n \geq \alpha > n-1,\ \alpha \notin \mathbb{N}$.*

$$\mathfrak{J}_{a+,t}^{\alpha,\rho}\left({}^c\mathcal{D}_{a+,\tau}^{\alpha,\rho}[u]\right) = u(t) - \sum_{k=0}^{n-1}\frac{\left(T_a^{\alpha,k}u\right)(a)}{\rho^k k!}(t-a)^{\rho k},\ \text{for } t\in(a,b]. \tag{2.21}$$

Lemma 2 *For $\beta > 0$, the general solution of the Caputo fractional differential equation*

$$\left({}^c\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right) = 0, \tag{2.22}$$

is given by

$$u(t) = c_0 + c_1 t + ... + c_{n-1}t^{n-1}, \tag{2.23}$$

where $c_i\,(i = 1,...,n-1)$ and $n = [\beta]+1$.

Lemma 3 *[16] when $u(t) = (t-a)^{\rho(\beta-1)}$ and $\beta > 0$, we have*

$$\begin{aligned} \left(\mathfrak{J}_{a+,t}^{\alpha,\rho}[u]\right) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \left((t-a)^\rho - (\tau-a)^\rho\right)^{\alpha-1}u(\tau)\frac{d\tau}{(\tau-a)^{1-\rho}} \\ &= \frac{\Gamma(\beta)}{\rho^\alpha\Gamma(\alpha+\beta)}(t-a)^{\rho(\alpha+\beta-1)}. \end{aligned} \tag{2.24}$$

$\psi-$Generalized fractional integral operator

The left-sided factional integrals and fractional derivatives of a function U with respect to another function ψ in the sense of Riemann-Liouville are defined as follows [11]

$$\left(J_{a+,t}^{\alpha,\psi}\right)[u] = \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(\tau)\left(\psi(t)-\psi(\tau)\right)^{\alpha-1}u(\tau)\,\mathrm{d}\tau \tag{2.25}$$

and

$$\left(D_{a+,t}^{\alpha,\psi}\right)[u]=\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\left(J_{a+,t}^{n-\alpha,\psi}\right)[u], \tag{2.26}$$

respectively, where $n=[\alpha]+1$, where $u,\psi\in\mathcal{C}^n[a,T]$ two functions such that ψ is increasing and $\psi'(t)\neq 0$, for all $t\in[a,T]$.

We propose the remarkable paper [21] in which some generalizations using ψ-fractional integrals and derivatives are described. In particular, we have

$$\begin{cases} \text{if} \quad \psi(t)\longrightarrow t, & \text{then} \quad J_{a+,t}^{\alpha,\psi}\longrightarrow J_{a+,t}^{\alpha}, \\ \text{if} \quad \psi(t)\longrightarrow \ln t, & \text{then} \quad J_{a+,t}^{\alpha,\psi}\longrightarrow {}^{H}J_{a+,t}^{\alpha}, \\ \text{if} \quad \psi(t)\longrightarrow t^{\rho}, & \text{then} \quad J_{a+,t}^{\alpha,\psi}\longrightarrow {}^{\rho}J_{a+,t}^{\alpha},\ \rho>0, \end{cases}$$

where $J_{a+,t}^{\alpha}, {}^{H}J_{a+,t}^{\alpha}, {}^{\rho}J_{a+,t}^{\alpha}$ are classical Riemann–Liouville, Hadamard and Katugampola fractional operators.

2.3.2 Fractional derivatives with non-singular kernel

Caputo-Fabrizio fractional derivative

Among existing NLFDs, the most commonly used are the Riemann-Liouville fractional derivative and the Caputo fractional derivative also known as the old Caputo derivative. The new Caputo-Fabrizio fractional derivative without singular kernel is simply an extension of the old Caputo fractional derivative where the kernel of the integral is reformulated. The definition of the new Caputo-Fabrizio fractional derivative presented in this section is extracted from [68].

For $n=1$ and a being an initial value other than 0 such that $a\in(-\infty,t]$, the equation (2.10) then becomes

$${}^{c}D_{a+}^{\alpha}u(t)=\frac{1}{\Gamma(1-\alpha)}\int_{a}^{t}(t-\tau)^{-\alpha}u'(\tau)\,d\tau, \tag{2.27}$$

with $u\in H^1(a,b)$.

By changing the kernel $(t-\tau)^{-\alpha}$ with the function $\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)$ and $1/\Gamma(1-\alpha)$ with $\mathbf{M}(\alpha)/(1-\alpha)$, the CFFD is defined by

$${}^{CF}\mathbf{D}_{t}^{\alpha}[u]=\frac{\mathbf{M}(\alpha)}{(1-\alpha)}\int_{a}^{t}\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right]\mathrm{D}_{\tau}[u]\,d\tau, \tag{2.28}$$

with $\alpha \in [0,1]$ and $u \in H^1(a,b)$ and $\mathbf{M}(\alpha)$ is a normalization function such that $\mathbf{M}(0) = \mathbf{M}(1) = 1$.

When u does not belong to $\mathbb{H}^1(a,b)$, the equation (2.28) is re-formulated for $u \in L^1(-\infty,b)$ and for $\alpha \in [0,1]$, and gives

$$ {}^{CF}\mathbf{D}_t^{\alpha}[u] = \frac{\alpha \mathbf{M}(\alpha)}{(1-\alpha)} \int_{-\infty}^{t} \exp\left[-\frac{1}{1-\alpha}(t-\tau)\right] (u(t) - u(\tau))\, d\tau. \tag{2.29} $$

Atangana-Baleanu fractional derivative

Atangana and Baleanu published an article [69] in which they proposed a new fractional derivative with a kernel that is non-local and non-singular. They introduced two versions, i.e:

$$ {}^{ABC}\mathbf{D}_t^{\alpha}[u] = \frac{\mathbf{B}(\alpha)}{(1-\alpha)} \int_{a}^{t} E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right] \mathrm{D}_{\tau}[u]\, d\tau, \tag{2.30} $$

where $u \in \mathbb{H}^1(0,1)$, $0 < \alpha < 1$ and $\mathbf{B}(\alpha)$ is normalization function which satisfy the properties $\mathbf{B}(0) = 1$, $\mathbf{B}(1) = 1$ and

$$ \mathbf{B}(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}. $$

The fractional integral associate to the ABC-fractional derivative with no-singular and non-local kernel is defined by

$$ \mathbf{I}_t^{\alpha}[u] = \frac{(1-\alpha)}{\mathbf{B}(\alpha)} u(t) + \frac{\alpha}{\mathbf{B}(\alpha)} J_t^{\alpha}[u],\ 0 < \alpha < 1, \tag{2.31} $$

where J_t^{α} is the left Riemann−Liouville fractional integral given in (2.8).

The Atangana−Baleanu in Riemann−Liouville fractional derivative of u of order $\alpha \in (0,1)$, defined by

$$ {}^{ABR}D_{a^+}^{\alpha} u(t) = \frac{\mathbf{B}(\alpha)}{1-\alpha} \frac{d}{dt} \int_{a}^{t} u(s) E_{\alpha}\left(-\gamma (t-s)^{\alpha}\right) ds, $$

where $u \in \mathbb{H}^1(a,b)$, $\gamma = \alpha/(1-\alpha)$ and $E_{\alpha}(.)$ is the Mittag−Leffler function.

2.4 Local fractional derivatives

Local fractional calculus (is also called Fractal calculus) was first introduced by Kolwankar and Gangal. It is explain the behavior of continuous but nowhere differentiable function.

They proposed particular notation that they had used in their publication for the LFD of a function defined on fractal sets [70, 71].

Unfortunately, as noted in section 2.3, fractional derivatives are not local in nature. On the other hand it is desirable and occasionally crucial to have local character in wide range of applications ranging from the structure of differentiable manifolds to various physical models. Secondly the fractional derivative of a constant is not zero, consequently the magnitude of the fractional derivative changes with the addition of a constant. The appropriate new notion of fractional differentiability must bypass the hindrance due to these two properties. These difficulties were remedied by introducing the notion LFD in [72] as follows.

Definition 1 *If, for a function* $u : [0, 1] \longrightarrow \mathbb{R}$*, the limit*

$$D^{\alpha}u(t) = \lim_{t \to \tau} \frac{d^{\alpha}\left(u(t) - u(\tau)\right)}{d\left(t - \tau\right)^{\alpha}}, \tag{2.32}$$

exists and is finite, then we say that the local fractional derivative (LFD) of order α $(0 < \alpha < 1)$*, at* $t = \tau$*, exists.*

Advantage of defining LFD in this manner lies in its local nature and hence allowing the study of pointwise behaviour of functions (also see [73] and the references therein).

2.4.1 Conformable operators

Among the inconsistencies of the existing fractional derivatives are:

(1) Most of the fractional derivatives except Caputo-type derivatives, do not satisfy ${}^{c}D_{a}^{\alpha}(1) = 0$, if α is not a natural number.

(2) All fractional derivatives do not obey the familiar Product Rule for two functions.

(3) All fractional derivatives do not obey the familiar Quotient Rule for two functions.

(4) All fractional derivatives do not obey the Chain Rule.

(5) Fractional derivatives do not have a corresponding Rolle's Theorem.

(6) Fractional derivatives do not have a corresponding Mean Value Theorem.

(7) All fractional derivatives do not obey: $D_{a}^{\alpha}D_{a}^{\beta}u = D_{a}^{\alpha+\beta}u$, in general.

(8) The Caputo definition assumes that the function u is differentiable.

To overcome some of these and other difficulties, Khalil et al. [14], came up with an interesting idea that extends the familiar limit definition of the derivative of a function given by the following.

Definition 2 *The left-sided conformable derivative of order $\alpha \in (0,1]$ is given by*

$$T_{0+}^{\alpha} u(t) = \lim_{\varepsilon \to 0} \frac{u(t+\varepsilon t^{1-\alpha}) - u(t)}{\varepsilon}, \quad T_{0+}^{\alpha} u(0) = \lim_{t \to 0^+} T_{0+}^{\alpha} u(t). \tag{2.33}$$

The properties of $\left(T_{0+}^{\alpha} u\right)$ can be found in [14, 15].

Definition 3 *Let $\alpha \in (0,1]$. A differential operator T_{0+}^{α} is conformable if and only if T^0 is the identity operator and T^1 is the classical differential operator.*

Definition 4 *Let $\alpha \in (n, n+1]$ and u be a n -differentiable function at $t > 0$, then the left sided conformable derivative of order α at $t > 0$ is given by*

$$\left(\mathbf{T}_{0+}^{\alpha} u\right)(t) = \left(T^{\alpha-n}\right) u^{(n)}(t) = \lim_{\delta \to 0} [u^{(n)}(t + \delta t^{n+1-\alpha}) - u^{(n)}(t)] / \left(\delta t^{n+1-\alpha}\right). \tag{2.34}$$

Lemma 4 *Let $t > 0$, $\alpha \in (n, n+1]$. The function u is $(n+1)$-differentiable if and only if u is α-differentiable, moreover, $\left(\mathbf{T}_{0+}^{\alpha} u\right)(t) = t^{n+1-\alpha} u^{(n+1)}(t)$*

Remark 2 *As a basic example, given $\alpha \in (n, n+1]$, we have, $\mathbf{T}_{0+}^{\alpha}\left(t^k\right) = 0$ where $k = 0, 1, ..., n$.*

Definition 5 *Let $\alpha \in (n, n+1]$. The left sided conformable integral of order α at $t > 0$ of a function $u \in C\left((0,+\infty), \mathbb{R}\right)$ is given by*

$$\mathbf{I}_{0+}^{\alpha} u(t) = J_{0+}^{n+1}\left(t^{\alpha-n-1} u(t)\right) = \frac{1}{n!} \int_0^t (t-s)^n s^{\alpha-n-1} u(s)\, \mathrm{d}s, \tag{2.35}$$

when $u^{(n)}(t)$ exists.

The following lemma play a fundamental role in obtaining an equivalent integral representation to the BVP (E_2-C_3).

Lemma 5 *Let $\alpha \in (n, n+1]$. If $u \in C(0,1]$ and $\mathbf{T}_{0+}^{\alpha} u \in L^1[0,1]$, then*

$$\mathbf{I}_{0+}^{\alpha} \mathbf{T}_{0+}^{\alpha} u = u(t) + \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k = u(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1}, \ \textit{for } t \in (0,1], \tag{2.36}$$

where $c_k = \frac{u^{(k)}(0)}{k!}$ and n is the smallest integer greater than or equal to α ($n = [\alpha] + 1$).

Lemma 6 *Let $t_2 > t_1 \geq 0$ and $u : [t_1, t_2] \to \mathbb{R}$ be a function with the properties that*

(i) u is continuous on $[t_1, t_2]$

(ii) u is α-differentiable on (t_1, t_2) for some $\alpha \in (0,1)$. Then there exists $\tau \in (t_1, t_2)$ such that

$$\left(T_{0+}^{\alpha} u\right)(\tau) = \frac{u(t_2) - u(t_1)}{\frac{1}{\alpha}\left(t_2^{\alpha} - t_1^{\alpha}\right)}. \tag{2.37}$$

Lemma 7 *Let $\alpha \in (0,1]$ and u, v be α -differentiable at a point $t > 0$. Then*

(i) $\left(T_{0+}^{\alpha}\right)(r_1 u + r_2 v) = r_1 \left(T_{0+}^{\alpha} u\right) + r_2 \left(T_{0+}^{\alpha} v\right), \ r_1, r_2 \in \mathbb{R}$.

(ii) $\left(T_{0+}^{\alpha}\right)(r_1) = 0$ for all constant functions $u(t) = r_1$.

(iii) $\left(T_{0+}^{\alpha}\right)(fg) = v\left(T_{0+}^{\alpha} u\right) + u\left(T_{0+}^{\alpha} v\right)$.

2.4.2 Local non conformable fractional derivative

Definition 6 *Given a function $u : [0, +\infty) \longrightarrow \mathbb{R}$. Then the N-derivative of u of order α is defined by [74, 75]*

$$N u_U^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon U(t, \alpha)) - u(t)}{\varepsilon}, \tag{2.38}$$

for all $t > 0, \alpha \in (0,1)$ being $U(t, \alpha)$ is some function. Here we will use some cases of U defined in function of $E_{\alpha,\beta}$ the classic definition of Mittag-Leffler function with $\alpha, \beta > 0$.

For example, if $U(t, \alpha) = t^{\alpha} E_{\alpha, \alpha+1}(at)$, then

$$\lim_{\alpha \to 1} N u_U^{(\alpha)}(t) = u'(t)\, t E_{1,2}(at), \tag{2.39}$$

i.e., a non-conformable derivative.

2.4.3 Other definitions for local fractional derivatives

For example, in [76], a concept similar to the Caputo fractional derivative is presented, but the first order derivative $u'(t)$ is replaced by another operator:

$$u^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon t) - u(t)}{\varepsilon^{\alpha}}. \tag{2.40}$$

In [77], the LFD is given by

$$u^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{u(t \exp(\varepsilon t^{-\alpha})) - u(t)}{\varepsilon} \tag{2.41}$$

and some fundamental properties like the algebraic rules or the mean value theorem are obtained.

In [78], the same concept of LFD is considered and an anti-derivative operator is defined, as well some applications to quantum mechanics; in [70], the LFD is defined by the expression

$$u^{(\alpha)}(t) = \lim_{\tau \to t} D^{\alpha} \left(u(\tau) - u(t) \right), \tag{2.42}$$

where D^{α} denotes the Riemann–Liouville fractional derivative.

In [79], the LFD is given by

$$u^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon k(t)^{1-\alpha}) - u(t)}{\varepsilon}. \tag{2.43}$$

In [80], Anderson et al. introduces conformable fractional as follows.

Definition 7 *Let $\alpha \in [0,1]$ and let the functions k_0 , $k_1 : [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous such that*

$$\lim_{\alpha \to 0^+} k_1(\alpha, t) = 1 , \qquad \lim_{\alpha \to 0^+} k_0(\alpha, t) = 0 \qquad \forall t \in \mathbb{R}$$

and

$$\lim_{\alpha \to 1^-} k_1(\alpha, t) = 0 , \qquad \lim_{\alpha \to 1^-} k_0(\alpha, t) = 1 \qquad \forall t \in \mathbb{R},$$

with

$$k_1(\alpha, t) \neq 0 , \ \alpha \in [0,1) , \ k_0(\alpha, t) \neq 0 , \ \alpha \in [0,1) , \ \forall t \in \mathbb{R}.$$

Then the following differential operator ${}^{c}T_{\alpha}$, defined via

$${}^{c}T_{\alpha} u(t) = k_1(\alpha, t) u(t) + k_0(\alpha, t) u'(t). \tag{2.44}$$

Chapter 3

EVALUATION METHODS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

Contents

3.1 Introduction

Various methods for evaluating FDEs including ODEs exist in the literature [81] and is explored in the thesis. To name few, we have [82]: the interactive method which is effective in solving only simple FDEs with real order,

(2) the Laplace transform method which is suitable for evaluating the FDE based IVPs.

(3) In spite of Adomian decomposition method, Homotopy analysis method, explicit numerical method and the Variational iterative method.

(4) In the Theory of DFs, the concept of fixed point and fixed point theorems are indispensable as they are deployed to show the existence of solutions to given problems. In common practice, given a BVP to establish the existence of solutions one transforms the given BVP as an equivalent fixed point problem to a integral operator

$$\mathbb{F}u\left(t\right)=\int_{0}^{1}G\left(t,s\right)f\left(s,u\left(s\right)\right)ds, \tag{3.1}$$

primarily in the function space $C\left([0,1]\right)$. So that the problem of existence of solutions reduces to seeking fixed points to the operator, $\mathbb{F}$ so defined.

(5) The theory of lower and upper solutions is known to be an easy, elementary method to deal with second order boundary value problems. The premises of the lower and upper solutions method can be traced back to Picard. In 1890 for partial differential equations and in 1893 for ordinary differential equations, he introduced monotone iterations from a lower solution. The method of upper and lower solutions is extensively developed for lower order equations; see Ako [83], Gaines [84], Jackson [85], Mawhin [86], and Nagumo [87]. More recently, Thompson [88] has continued the development of these methods with applications to fully nonlinear BVPs. Working applications, the use of lower and upper solutions method faces the difficulty to exhibit such functions. It replaces a difficult problem, "how to find a solution of a boundary value problem", by a no way easier problem, "how to find lower and upper solutions".

(6)Topological degree theory is a very powerful tool used in proving various existence results for nonlinear partial differential equations and it has received a lot of attention in the literature, see the references in [89]. Gaines and Mawhin introduced coincidence degree theory in 1970s in analyzing functional and differential equations [4, 90]. Mawhin has continued studies on this theory later on and has made so important contributions on this subject since then this theory is also known as Mahwin's coincidence degree theory. Coincidence theory is very powerful technique especially in existence of solutions problems in nonlinear equations. It has especially so broad applications in the existence of periodic solutions of nonlinear differential equations so that many researchers have used it for their investigations see [91] and references therein .

3.2 p-Laplacian operator

The classical p-Laplacian operator is a well known nonlinear operator mostly used in nonlinear analysis. The nonlinear p-Laplacian operator is defined as $\frac{1}{p}+\frac{1}{q}=1$ then $\varphi_p\left(s\right)=\left|s\right|^{p-2}s,\ \infty>p>1,\ s\in\mathbb{R}$ and $\varphi_q=\varphi_p^{-1}$. Let some properties for φ_p operator. Then

Lemma 8 *[36]For any $u,v\in\mathbb{R}$ and $1<p,q<\infty,\ 1/p+1/q=1,$ we have $\varphi_q=\varphi_p^{-1}$,*

(i) If $1<p\leq 2$, $uv>0$ and $\left|u\right|,\ \left|v\right|\geq l>0$ then

$$\left|\varphi_p\left(u\right)-\varphi_p\left(v\right)\right|\leq\left(p-1\right)l^{p-2}\left|u-v\right|. \tag{3.2}$$

(ii) If $p>2$, $\left|u\right|,\ \left|v\right|<L$ then

$$\left|\varphi_p\left(u\right)-\varphi_p\left(v\right)\right|\leq\left(p-1\right)L^{p-2}\left|u-v\right|. \tag{3.3}$$

Lemma 9 *[92]For any $u,v\geq 0$,*

(i) If $1<p<2$ then

$$\left|\varphi_p\left(u+v\right)\right|\leq\varphi_p\left(v\right)+\varphi_p\left(v\right). \tag{3.4}$$

(ii) If $p\geq 2$ then

$$\left|\varphi_p\left(u+v\right)\right|\leq 2^{p-2}\left(\varphi_p\left(v\right)+\varphi_p\left(v\right)\right). \tag{3.5}$$

Lemma 10 *[92]Let $t,\tau\in[t_1,t_2]$, we have*

(i) If $1 < q \leq 2$, then

$$\left|\varphi_q(t+\tau) - \varphi_q(\tau)\right| \leq 2^{2-q} |t|^{q-1}. \tag{3.6}$$

(ii) If $q > 2$, then

$$\left|\varphi_q(t+\tau) - \varphi_q(\tau)\right| \leq (q-1)\left(|t| + |\tau|\right)^{q-2} |t|. \tag{3.7}$$

From the definition of φ_p then for any $t, \tau \in \mathbb{R}$, we have

$$\varphi_q(|t| + |\tau|) \leq C_p\left(\varphi_q(|t|) + \varphi_q(|\tau|)\right),$$

where

$$C_p = \begin{cases} 1 & \text{if} \quad p > 2 \\ 2^{\frac{2-p}{p-1}} & \text{if} \quad 1 < p \leq 2. \end{cases}$$

We also need the following lemma to obtain our results.

Lemma 11 *If $a, b \geq 0, r > 0$, then*

$$(a+b)^r \leq \max\left\{2^{r-1}, 1\right\}(a^r + b^r)$$

or

$$(a+b)^r \leq \lambda^{r-1} a^r + \mu^{r-1} b^r, \;\; \lambda + \mu = 1. \tag{3.8}$$

Proof. Obviously, without loss of generality, we can assume that $a, b > 0, r \neq 1$. Let $h(t) = t^r, \; t \in [0, +\infty)$

(i) If $r > 1$, then $h(t)$ is convex on $(0, +\infty)$, and so

$$\left(\lambda \frac{a}{\lambda} + \mu \frac{b}{\mu}\right)^r \leq \lambda\left(\frac{a}{\lambda}\right)^r + \mu\left(\frac{b}{\mu}\right)^r, \;\; \frac{1}{\lambda} + \frac{1}{\mu} = 1.$$

(ii) If $0 < r < 1$, then $h(t)$ is concave on $[0, +\infty)$, and so

$$h(a) = h\left(\frac{b}{a+b}0 + \frac{a}{a+b}(a+b)\right) \geq \frac{b}{a+b}h(0) + \frac{a}{a+b}h(a+b) = \frac{a}{a+b}h(a+b),$$

$$h(b) = h\left(\frac{a}{a+b}0 + \frac{b}{a+b}(a+b)\right) \geq \frac{a}{a+b}h(0) + \frac{b}{a+b}h(a+b) = \frac{b}{a+b}h(a+b).$$

Thus, $h(a) + h(b) \geq h(a+b)$ namely,

$$(a+b)^r \leq a^r + b^r.$$

By (i), (ii) above, we know that the conclusion of Lemma 11 is true. ■

3.3 Solution and positive solution

By a solution of a BVP, we mean a function u satisfying the given DE together with the boundary conditions imposed. The solution is called positive if $u(t) > 0$ for all t, except at the endpoints.

3.4 Lower and upper solutions

The method of lower and upper solutions is an elementary but powerful tool in the existence theory of initial and periodic problems for semilinear DEs for which a maximum principle holds, even in cases where no special structure is assumed on the nonlinearity.

Now, we introduce the following definitions of a couple of the lower and upper solutions for fourth-order p-Laplacian conformable boundary value problem (E_2-C_3). Denote

$$E = \left\{u : u \in C^2\left([0,1]\right) \text{ and } \varphi_p\left(\mathbf{T}_{0+}^{\alpha} u\right) \in C^2\left([0,1]\right)\right\}.$$

Definition 8 *Let $\underline{u}\left(t\right), \overline{u}\left(t\right) \in E$, we say that $\underline{u}(t)$ is called a lower solution of operator $\mathbb{F}$ if $\underline{u}(t) \leq \mathbb{F}\underline{u}(t)$,and $\overline{u}\ (t)$ is called an upper solution of operator $\mathbb{F}$ if $\overline{u}(t) \geq \mathbb{F}\overline{u}(t)$.*

Definition 9 *A function $\underline{u}$ is called a lower solution of the conformable boundary value problem (E_2-C_3), if $\underline{u}\left(t\right) \in E$ and $\underline{u}\left(t\right)$ satisfies*

$$\begin{cases} \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\underline{u}\left(t\right)\right)\right)\left(t\right) - f\left(t, \underline{u}\left(t\right), \mathbf{T}_{0+}^{\gamma}\underline{u}\left(t\right)\right) \leq 0, \ t \in \left(0,1\right), \\ \underline{u}\left(0\right) \leq 0, \ \underline{u}\left(1\right) - b_1\underline{u}\left(\xi_1\right) \leq 0, \left(\mathbf{T}_{0+}^{\alpha}\right)\underline{u}\left(0\right) \geq 0, \ \left(\mathbf{T}_{0+}^{\alpha}\right)\underline{u}\left(1\right) - b_2\left(\mathbf{T}_{0+}^{\alpha}\right)\underline{u}\left(\xi_2\right) \geq 0. \end{cases} \tag{3.9}$$

This condition is motivated by the fact that an lower solution for an equation of the type $\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\left(\mathbf{T}_{0+}^{\alpha}\right)\underline{u}\left(t\right)\right)\right)\left(t\right) = f\left(t, \underline{u}\left(t\right), \mathbf{T}_{0+}^{\gamma}\underline{u}\left(t\right)\right)$ can be obtained if one has an lower solution for another equation of the form

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\left(\mathbf{T}_{0+}^{\alpha}\right)\underline{u}\left(t\right)\right)\right)\left(t\right) = g\left(t, \underline{u}\left(t\right), \mathbf{T}_{0+}^{\gamma}\underline{u}\left(t\right)\right), \ 0 < \alpha, \beta \leq 2 \text{ and } 0 < \gamma < \alpha + \beta,$$

with

$$g\left(s, \underline{u}\left(s\right), \mathbf{T}_{0+}^{\gamma}\underline{u}\left(s\right)\right) \leq f\left(s, \underline{u}\left(s\right), \mathbf{T}_{0+}^{\gamma}\underline{u}\left(s\right)\right) \text{ for all } s.$$

An upper solution $\overline{u}$ is defined by reversing inequalities in the previous definition.

The existence of a lower solution over the upper solution implies the existence of solutions lying between both functions.

3.5 Some basic fixed point theorems

In the existence results of fractional differential equations, scientists are utilizing various types of fixed point approaches. We concerned in this section to give some FPTs which will be used throughout in this thesis.

3.5.1 Schauder fixed point theorem

Theorem 1 (Schauder's fixed point theorem) *Let E be a Banach space and $D \subset E$, a convex, closed and bounded set. If $T : D \to D$ is a continuous operator such that $T(D) \subset E$, $T(D)$ is relatively compact, then T has at least one fixed point in D.*

Let Y be a closed bounded convex subset of a normed space X. Let $T : Y \longrightarrow X$ be a compact map such that $T(Y) \subset Y$. Then there is a point $u \in Y$.such that $Tx = u$.

3.5.2 Guo-Krasnosel'skii theorem

Lemma 12 (Guo-Krasnosel'skii theorem) *[93]Let P be a positive cone in a real Banach space E. Let Ω_1, Ω_2 be bounded open balls of E centered at the origin with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right) \to P$ is a completely continuous operator such that*

$$\|Au\| \leq \|u\| \quad \forall u \in P \cap \partial\Omega_1 \quad \textit{and} \quad \|Au\| \geq \|u\| \quad \forall u \in P \cap \partial\Omega_2,$$

or

$$\|Au\| \geq \|u\| \quad \textit{for } P \cap \partial\Omega_1 \quad \textit{and} \quad \|Au\| \leq \|u\| \quad \textit{for } P \cap \partial\Omega_2,$$

hold. Then A has a fixed point in $P \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$.

For $u \in C[0,1]$, the corresponding norm is $\|u\|_0 = \max\{|u(t)| : t \in [0,1]\}$. And for $u \in C^2[0,1]$, the corresponding norm is

$$\|u\| = \max\left\{\|u\|_0, \left\|T_{0+}^2 u\right\|_0\right\}. \tag{3.10}$$

3.5.3 Leray-Schauder degree theory

Let $\Omega \in \mathbb{R}^n$ be open and bounded and $u \in C^1(\bar{\Omega})$. If $p \notin u(\partial\Omega)$ then the Brouwer degree $\deg(u, \Omega, p)$ is a tool that describes the number of solutions for equation $u(t) = p$.

Definition 10 *[89]Let $u \in C^1(\bar{\Omega})$, $p \in \mathbb{R}^n$ be given with $p \notin u(\partial\Omega)$, and $p \notin u(S_u)$. The Brouwer degree of u at p with respect to Ω, deg(u, Ω, p), is defined by*

$$\deg(u, \Omega, p) = \sum_{t \in u^{-1}(p)} \operatorname{sgn} J_u(t), \tag{3.11}$$

where deg$(u, \Omega, p) = 0$ if $u^{-1}(p) = \phi$, $J_u(t)$ is the Jacobian of u at t and $S_u\left(\bar{\Omega}\right)$ is the set of all critical points of u in $\bar{\Omega}$

$$S_u\left(\bar{\Omega}\right) = \left\{t \in \Omega : J_u(t) = 0\right\}. \tag{3.12}$$

Theorem 2 *[89]The Leray-Schauder degree has the following properties:*

(i) (Normality). deg$(I, \Omega, 0) = 1$ if and only if $0 \in \Omega$

(ii).(Solvability). If deg$(I - M, \Omega, 0) \neq 0$, then $Mu = u$ has a solution in Ω .

(iii) (Homotopy). Let $\mathrm{H}(u, \lambda) : \bar{\Omega} \times [0, 1] \longrightarrow X$ be continuous compact and $\mathrm{H}(u, \lambda) \neq u$ for all $(u, \lambda) \in \partial\bar{\Omega} \times [0, 1]$. Then deg$(I - \mathrm{H}(., \lambda), \Omega, 0) \neq 0$ doesn't depend on $\lambda \in [0, 1]$.

Lemma 13 *[90]The Leray-Schauder degree of a linear isomorphism is equal to ± 1.*

3.5.4 Coincidence degree theory of Mawhin

Coincidence degree theory provides a method for proving the existence of solution of the equation $Lu = Nu$ where $L : \mathrm{dom}L \subset X \longrightarrow Y$ is a linear Fredholm mapping of index equal to zero and N is a (completely continuous, possibly nonlinear) mapping represents a wide variety of problems including nonlinear ordinary, partial and functional differential equations, which is defined on the closure of a bounded open subset of X and takes values from Y, X and Y being Banach spaces over the real. When L^{-1} exists, then this equation reduces to $u = L^{-1}Nu$ which is included in the class of Hammerstein operators and is under the scope of fixed point theory.

Some nonlinear problems arising in many areas of the applied sciences can be formulated under a mathematical point of view involving the study of solutions of equations of the form

$$\text{Find } u \in X \text{ such that } Lu = Nu, \tag{3.13}$$

where X is a nonempty set, Y is a Banach space, and $L, N : X \to Y$ are two mappings. The problem of finding a solution for equation (3.13) is known as a coincidence problem.

Equations of the form $Lu = Nu$, where $L : \mathrm{dom}L \subset X \longrightarrow Y$ is linear and $N : \mathrm{dom}N \subset X \longrightarrow Y$ not necessarily linear, with X and Y vector spaces, appear in the abstract formulation of many problems of analysis or applied mathematics and have been extensively studied.

Continuation theorem for linear operator

An operator $P : X \to X$ is said to be an algebraic projection if P is linear and idempotent, i.e. $P^2 = P$. Let $P : X \to X$ and $Q : Y \to Y$ be two algebraic projections.

Now, we briefly recall some notations and an abstract existence result, which can be found in [36].

Let X, Y be real Banach spaces, $\mathrm{L} : \mathrm{domL} \subset X \to Y$ be a Fredholm operator with index zero, where the index of a Fredholm operator L is defined by,

$$\text{Index } L := \dim \ker L - co\dim Im\ L$$

and $P : X \to X,\ Q : Y \to Y$ be projectors such that $\mathrm{Im}\, P = \ker \mathrm{L}, \mathrm{Im}\, \mathrm{L} = \ker Q$. then

$X = \ker \mathrm{L} \oplus \ker P, Y = \mathrm{Im}\, \mathrm{L} \oplus \mathrm{Im}\, Q$, and

$$\mathrm{L}|\mathrm{domL} \cap \ker P : \mathrm{domL} \cap \ker P \to \mathrm{Im}\, \mathrm{L},$$

is invertible. We denote the inverse by K_P .

If Ω is an open bounded subset of X such that $\mathrm{domL} \cap \bar{\Omega} \neq 0$, then the map $\mathrm{N} : X \to Y$ will be called L-compact on $\bar{\Omega}$ if $Q\mathrm{N}\left(\bar{\Omega}\right)$ is bounded and $K_P(\mathfrak{I} - Q)\mathrm{N} : \bar{\Omega} \to X$ is compact.

In this section, we take $Y = C[0,1]$, with the norm $\|y\|_\infty = \max\{|y(t)| : t \in [0,1]\}$, and $X = \left\{u : u, \left(\mathcal{D}_{0+}^{\alpha,\rho}\right) u \in Y\right\}$ with the norm $\|u\|_X = \max\left\{\|u\|_\infty, \left\|\left(\mathcal{D}_{0+}^{\alpha,\rho}\right) u\right\|_\infty\right\}$. By means of the linear functional analysis theory, we can prove that X is a Banach space.

Definition 11 *Let X and Y be normed spaces. A linear operator* $\mathrm{L} : \mathrm{dom}\ \mathrm{L} \subset X \longrightarrow Y$ *is said to be a Fredholm operator of index zero provided that*

(**i**) $\mathrm{Im}L$ is a closed subset of Y,

(**ii**) $\dim\ker L = co\dim\mathrm{Im}(L) < +\infty$.

Definition 12 *Let X be a normed space. A linear operator $P : X \longrightarrow X$ is said to be a projection if $P \circ P = P$. In this case, $I - P : X \longrightarrow X$ is also a projection and* $\ker(P) = \mathrm{Im}(I-P)$, $\mathrm{Im}(P) = \ker(I-P)$, *where I is the identity operator on X.*

It follows from Mawhin's equivalent theorem $\mathrm{L}u = \mathrm{N}u$ for $u \in \bar{\Omega}$ that the equation is equivalently converted into the fixed point equation $u = \phi(u)$ for $u \in \bar{\Omega}$ where $\phi = P + (JQ + K_{P,Q}N)$ is a completely continuous operator. This can be solvable thanks to the following theorem, called Mawhin's continuation theorem.

Theorem 3 (Mawhin's theory of coincidence) *[4]Let X, Y be real Banach spaces,* L *: dom $\subset X \longrightarrow Y$ be a Fredholm operator of index zero and* $\mathrm{N} : X \longrightarrow Y$ *be* L*-compact on* Ω*. Assume that the following conditions are satisfied*

(C_1) $\mathrm{L}u \neq \lambda \mathrm{N}u$ for every $(u, \lambda) \in [(\mathrm{Dom}\ \mathrm{L} \setminus \ker\mathrm{L}) \cap \partial\Omega] \times (0,1)$;

(C_2) $\mathrm{N}u \notin \mathrm{Im}L$, for every $u \in \ker L \cap \partial\Omega$;

(C_3) $\deg(JQ\mathrm{N}|_{\ker\mathrm{L}}, \Omega \cap \ker\mathrm{L}, 0) \neq 0$ where, $Q : Y \to Y$ is a projection such that $\mathrm{Im}L = \ker Q$, $J : \mathrm{Im}\, Q \longrightarrow \ker\mathrm{L}$ be a linear isomorphism with $J(\theta) = \theta$;

Then the equation $\mathrm{L}u = \mathrm{N}u$ has at least one solution in $\mathrm{dom}\ \mathrm{L} \cap \bar{\Omega}$.

Continuation theorem for quasi-linear operator

First, we recall Mawhin's continuation theorem which our study is based upon.

Next we introduce an extension of Mawhin's continuation theorem [36] which allows us to deal with the more general abstract operator equations, such as BVPs of p-Laplacian equations.

A new continuation theorem for the existence of solutions to p-Laplacian BVP at resonance

Now, we briefly recall some notations and an abstract existence result, which can be found in [36].

Definition 13 *Let X be a real Banach space and let $X_1 \subset X$.The operator $P : X \to X_1$ is said to be a projector provided that for $P^2 = P$ and $P(ax_1 + bx_2) = aP(u_1) + bP(u_2)$, for $u_1, u_2 \in X$ and $a, b \in \mathbb{R}$. The operator $Q : Y \to Y_1$ is said to be a semi-projector provided $Q^2 = Q$.*

Let X, Y be real Banach spaces, $\mathrm{M} : \mathrm{domM} \subset X \to Y$ be a map and $P : X \to X_1$ be a projector, $Q : Y \to Y_1$ be semi-projector, such that $\operatorname{Im} P = \ker \mathrm{M}$, $\operatorname{Im} \mathrm{M} = \ker Q$. then

$X = \ker \mathrm{M} \oplus \ker P, Y = \operatorname{Im} \mathrm{M} \oplus \operatorname{Im} Q$, and

$$\mathrm{M}|\mathrm{domM} \cap \ker P : \mathrm{domM} \cap \ker P \to \operatorname{Im} \mathrm{M},$$

is invertible. We denote the inverse by K_P .

If Ω is an open bounded subset of X with the origin $\theta \in \Omega$ such that $\mathrm{dom}M \cap \bar{\Omega} \neq 0$, then the map $\mathrm{N}_\lambda : X \to Y$ will be called M-compact on $\bar{\Omega}$ if $Q\mathrm{N}_\lambda\left(\bar{\Omega}\right)$ is bounded and $K_P(\mathfrak{I} - Q)\mathrm{N}_\lambda : \bar{\Omega} \to X$ is compact.

In this thesis, we take $Y = C[0,1]$, with the norm $\|y\|_\infty = \max\{|y(t)| : t \in [0,1]\}$, and $X = \left\{u : u, \left(\mathcal{D}^{\alpha,\rho}_{0+,t}\right) u \in Y\right\}$ with the norm $\|u\|_X = \max\left\{\|u\|_\infty, \left\|\left(\mathcal{D}^{\alpha,\rho}_{0+,t}\right) u\right\|_\infty\right\}$. By means of the linear functional analysis theory, we can prove that X is a Banach space.

Definition 14 *Let Y_1 be a subspace of Y. An operator $Q : Y \to Y_1$ is said to be a semi-projector provided that*

(i) $Q^2 y = Qy, \forall y \in Y,$

(ii) $Q(\lambda y) = \lambda Qy, \forall y \in Y, \lambda \in \mathbb{R}.$

Definition 15 *[94]Let X and Y be two Banach spaces with norms $\|.\|_X$ and $\cdot\ \|.\|_Y$, respectively. A continuous operator*

$$\mathrm{M} : X \cap \mathrm{domM} \to Y, \tag{3.14}$$

is said to be quasi-linear if

(i) $\operatorname{Im} M = M\,(X \cap \operatorname{dom} M)$ *is a closed subset of* Y*;*

(ii) $\ker M = \{u \in X \cap \operatorname{dom} M : Mx = 0\}$ *is linearly homeomorphic to* $\mathbb{R}^n$*;*$n < \infty$.

Definition 16 *[94]Let X and Y be to real Banach spaces and let $M : X \cap \operatorname{dom} M \to Y$ be a map. Assume that $X_1 = \ker M$ is a linear subspace of X and denote by X_2 its complement subspace, i. e.,$X = X_1 \oplus X_2$.Likewise, let Y_1 and Y_2 be two complementary linear subspaces of Y so that $Y = Y_1 \oplus Y_2$. Assume that $\dim X_1 = \dim Y_1$. Let $P : X \to X_1$ and $Q : Y \to Y_1$ be the corresponding orthogonal projectors. Denote by $J : Y_1 \to X_1$ a homeomorphism with $J(0) = 0$.The operator M is said to be quasi-linear if*

(i) $\dim \ker M = \dim M - 1(0) = n < \infty$; *where "*dim*" denotes dimension;*

(ii) $R(M) = \operatorname{Im} M = M(X \cap \operatorname{dom} M)$ *is a closed subset in* Y_2.

Definition 17 *[94]Let Ω be a bounded open subset of X,with $\theta \in \Omega$, and consider a parameter family of perturbation (generally nonlinear) $\mathrm{N}_\lambda : \bar{\Omega} \to Y$ with $N_1 = N$, $\lambda \in [0, 1]$. The continuous operator N_λ is said to be M-compact in $\bar{\Omega}$ if there exists subset Y_1 of Y satisfying $\dim Y_1 = \dim \ker M$ and an operator $R : \bar{\Omega} \times [0, 1] \to X_2$ being continuous and compact such that for $\lambda \in [0, 1]$,*

(i) $(I - Q)\mathrm{N}_\lambda(\bar{\Omega}) \subseteq \operatorname{Im} M \subset (I - Q)\, Y$;

(ii) $Q\mathrm{N}_\lambda u = \theta,\ \lambda \in (0, 1) \Longleftrightarrow QNx = \theta, \forall u \in \Omega$;

(iii) $R(0, u) = 0$ *and* $R(0, \lambda)|_{\sum_\lambda} = (I - P)|_{\sum_\lambda}$;

(iv) $M(P + R(., \lambda)) = (I - Q)\mathrm{N}_\lambda,\ \lambda \in [0, 1]$, *where X_2 is a the complement space of $\ker M$ in X, i.e., $X = \ker M \oplus X_2$; P, Q are two projectors satisfying $\operatorname{Im} P = \ker M$; $\operatorname{Im} Q = Z_1$; $N = N_1$;* $\sum_\lambda = \left\{u \in \bar{\Omega} : \mathrm{M}u = \mathrm{N}_\lambda u, u \in \bar{\Omega}, \lambda \in (0, 1]\right\}$.

Theorem 4 (Continuation theorem for quasi-linear operator) *[94]Let X, Y be real Banach spaces. Suppose M : dom(L) $\subset X \longrightarrow Y$ be a is a quasi-linear operator and N_λ : $\bar{\Omega} \longrightarrow Y$, $\lambda \in [0, 1]$ be M-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied*

($\mathbf{C}_1$) $\mathrm{M}u \neq \mathrm{N}_\lambda u\ \forall\, (u, y) \in [(\text{dom M}/ \ker \mathrm{M}) \cap \partial\Omega] \times (0, 1)$, where $\partial\Omega$ denotes the boundary of Ω;

($\mathbf{C}_2$) $\mathrm{N}_\lambda u \notin \mathrm{Im}M,\ \lambda \in (0,1)\,,\ \forall u \in \ker L \cap \partial\Omega$ or $Q\mathrm{N}u \neq 0,\ \forall u \in \ker M \cap \partial\Omega$;

($\mathbf{C}_3$) $\deg(JQ\mathrm{N}, \Omega \cap \ker \mathrm{M}, 0) \neq 0$ where $\mathrm{N} = \mathrm{N}_1$ and $Q : Y \to Y$ is a projection such that $\mathrm{Im}M = \ker Q$ and $J : \mathrm{Im}\, Q \longrightarrow \ker M$ is a homeomorphism.

Then the abstract equation $\mathrm{M}u = \mathrm{N}u$ has at least one solution in $\mathrm{dom}\ \mathrm{M} \cap \bar{\Omega}$.

Remark 3 *If* $\mathrm{M} = \mathrm{L}$ *is a linear operator, then the requirement that* L *is a Fredholm operator of index zero,* N *is L-compact on* $\bar{\Omega}$ *and* K *is inverse operator of* $\mathrm{L}_P = \mathrm{M}|_{\mathrm{dom M} \cap \mathrm{Im\, M}}$ *as is defined in Mawhin's theorem, implies* $\mathrm{N}_\lambda = \lambda \mathrm{N}$.

Remark 4 *We can write the BVPs with the form (4.2). If L is invertible, or* $\mathrm{Ker}L = \{0\}$, *(4.2) is called non-resonant problem. Otherwise, if* $\mathrm{Ker}L$ *is not a trivial space, then it is called resonant problem.*

Chapter 4

p -LAPLACIAN BOUNDARY VALUE PROBLEMS WITH CONFORMABLE DERIVATIVE IN THE SENSE OF CAPUTO

Contents

4.1 Introduction

Over the years, there have been a surge in the study of second order nonlinear ODEs of the form

$$u''(t) = f(t, u(t)), \; 0 < t < T \tag{4.1}$$

see ([66]) subject to different boundary conditions based on interest.

Hairong Lian et al. (see [95]) considered a three-point BVP of differential equation with a p-Laplacian given by

$$(\varphi_p(u'(t)))' = f(t, u(t), u'(t)), \; 0 < t < T \tag{4.2}$$

and

$$u(0) = u(\eta), u'(T) = 0, \; T \in (0, \infty].$$

The equation (4.2) subjected to different boundary condition has been studied by many authors.

A boundary value problem of differential equation is said to be at resonance if its corresponding homogeneous one has nontrivial solutions. For (4.2), it is easy to see that the following BVP

$$(\varphi_p(u'(t)))' = 0, \; 0 < t < T \tag{4.3}$$

and

$$u(0) = u(\eta), u'(T) = 0, \; T \in (0, \infty].$$

has solutions $\{u(t) = c, c \in \mathbb{R}\}$. When $c \neq 0$ they are nontrivial solutions. For multi-point BVP at resonance without p-Laplacian, there have been many existence results available in the references [5, 94]

This chapter considers the existence of solutions for two boundary value problems of FPLE at resonance. we investigate the existence of solutions for FPLE of the form

$${}^c\mathcal{D}^{\alpha,\rho}_{0+,t}\left[\varphi_p\left({}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right] = f\left(t, u(t), -{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right), \; t \in (0,1), \tag{4.4}$$

subject to either boundary value conditions

$$u(0) = u(1), {}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u](0) = 0, \; 0 < \beta < \alpha \leq 1, \tag{4.5}$$

or

$$u(0) = u(\eta), {}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u](1) = 0, 0 < \beta \leq 1 < \alpha \leq 2. \tag{4.6}$$

Bai and Lü [43] studied the existence and multiplicity of positive solutions of nonlinear fractional boundary value problem

$$\begin{cases} (D_t^{\alpha}) u(t) = f(t, u(t)) \text{ in } (0,1), 1 < \alpha \leq 2, \\ u(0) = 0 = u(1). \end{cases} \tag{4.7}$$

The Dirichlet boundary conditions are used in paper [96]. When the boundary values are not zero, Riemann-Liouville fractional derivative is not suitable.

In [97], Agarwal et al. have investigated the following the fractional boundary value problem

$$(D_t^{\alpha}) u(t) = -f\left(t, u(t), -\left(D_t^{\beta}\right) u(t)\right) \text{ in } (0,1) \tag{4.8}$$

and boundary conditions

$$u(0) = u(1) = 0, \tag{4.9}$$

where D_t^{β}, D_t^{α} are Riemann-Liouville fractional operators with $0 < \beta < \alpha \leq 1$, They established the existence results by the FPT in a cone. Related results for the case can be found in [97] and the references in these chapiter. As we shall see, a more specific result, which establishes the existence of solution continua of (4.8) which satisfy the above conditions, may be obtained also.

For fractional boundary value problems at resonance, Chen et al. (see [76]) considered a two-point BVP with p-Laplacian operator given by

$$\mathcal{D}_{0+,t}^{\alpha}\left[\varphi_p\left(\mathcal{D}_{0+,\tau}^{\beta}[u]\right)\right] = f\left(t, u(t), -\mathcal{D}_{0+,t}^{\beta}[u]\right), \ t \in (0,1), 0 < \beta, \alpha \leq 1, \tag{4.10}$$

with the condition

$$\mathcal{D}_{0+,t}^{\beta}[u](0) = 0 = \mathcal{D}_{0+,t}^{\beta}[u](1). \tag{4.11}$$

They obtained the existence results by using the coincidence degree theory.

However, there are few papers consider the boundary value problem for fractional order at resonance. Many works have been done to discuss the existence of solutions, positive solutions subject to Dirichlet, Sturm-Liouville, or nonlinear boundary value conditions.

The above result naturally prompts one to ponder if it is possible to establish similar existence results for BVP at resonance with a p-Laplacian under at most linearly increasing condition and other suitable conditions imposed on the nonlinear term.

Motivated by the work above, we consider the existence of solutions for two (or three) point boundary value problem for FPLE (E_1-C_1) (or (E_1-C_1)) at resonance. By using the coincidence degree theory, a new result on the existence of solutions for above FBVP (E_1-C_1) (or (E_1-C_1)) is obtained. These results extend the corresponding ones of ordinary differential equations of integer order.

In this section, we consider the following second order quasi-linear differential equation (E_1), subject to one of the following boundary conditions (C_1) and (C_1)

Our assumptions on the nonlinearity f will be the following:Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that

(A_1) there exist nonnegative functions $a, b, c \in Y$ such that

$$|f(t,u,v)| \leq a(t) + b(t)|u|^{p-1} + c(t)|v|^{p-1}, \forall t \in [0,1], (u,v) \in \mathbb{R}^2.$$

(A_2) There is $B > 0$ such that for all $|u| > B$, either

$$uf(t,u,0) < 0, \forall t \in [0,1]$$

or

$$uf(t,u,0) > 0, \forall t \in [0,1].$$

(A_3) There is $E > 0$ such that for all $|u| > E$ and $v \in \mathbb{R}$, either

$$f(t,u,v) < 0, \forall t \in [0,1]$$

or

$$f(t,u,v) > 0, \forall t \in [0,1].$$

(A_4) There exists constant $A > 0$ such that $\forall u \in \mathrm{domM} \cap \ker \mathrm{M}$; satisfying $|u(t)| > A$, for all $t \in [0,1]$, we have

$$\mathfrak{I}^{\beta,\rho}_{0+,\eta}\left[\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,\tau}\left[f\left(\tau, u(\tau), -{}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right] - \mathfrak{I}^{\alpha,\rho}_{0+,1}\left[f\left(\tau, u(\tau), -{}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right]\right)\right] \neq 0.$$

(A_5) There exists constant $D > 0$ such that for all $|c| > D$, either

$$c\mathfrak{I}_{0+,\eta}^{\beta,\rho}\left[\varphi_q\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f\left(\tau,c,0\right)\right]-\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[f\left(\tau,c,0\right)\right]\right)\right]<0$$

or

$$c\mathfrak{I}_{0+,\eta}^{\beta,\rho}\left[\varphi_q\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f\left(\tau,c,0\right)\right]-\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[f\left(\tau,c,0\right)\right]\right)\right]>0.$$

4.2 p -Laplacian two -point local boundary value problems with fractional conformable derivative in the sense of Caputo

4.2.1 Introduction

Motivated by the results of [1, 16, 97, 98, 99], in this section, by using the coincidence degree theory, we investigate the existence of solutions for the fractional differential equation at resonance

$$^{c}\mathcal{D}_{0+,t}^{\alpha,\rho}\left[\varphi_p\left(^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}\left[u\right]\right)\right]=f\left(t,u\left(t\right),-^{c}\mathcal{D}_{0+,t}^{\beta,\rho}\left[u\right]\right),\ t\in\left(0,1\right),0<\beta,\alpha\leq1, \tag{4.12}$$

with the condition for (4.12) is

$$u(0)=u(1),^{c}\mathcal{D}_{0+,t}^{\beta,\rho}\left[u\right](0)=0, \tag{4.13}$$

where $\left(^{c}\mathcal{D}_{0+,\tau}^{\alpha,\rho}\right)$ and $\left(^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}\right)$ are the conformable in the sense of Caputo derivative with $0<\beta\leq\beta+\alpha<1$, $0<\rho$,.with f continuous (but not necessarily locally Lipschitz continuous).

In this section, we establish sufficient conditions to guarantee the existence of at least one solution of nonlocal BVPs consisting of the fractional order differential equation with p-Laplacian (4.12) and one of following boundary conditions (4.13).

Note that, the nonlinear operator $^{c}\mathcal{D}_{0+}^{\alpha,\rho}\left[\varphi_p\left(^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}\right)\right]$ is reduced to the linear operator $\left(^{c}\mathcal{D}_{0+,\tau}^{\alpha+\beta,\rho}\right)$ when $p=2$ and the additive index law

$$^{c}\mathcal{D}_{0+,t}^{\alpha,\rho}\left[^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}\left[u\right]\right]=^{c}\mathcal{D}_{0+,t}^{\alpha+\beta,\rho}\left[u\right],$$

holds under some reasonable constraints on the function $u(t)$ (see [81]).

Furthermore, since ${}^{c}\mathcal{D}_{0+}^{\alpha,\rho}\left[\varphi_p\left({}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}\right)\right]$ is a nonlinear operator, the coincidence degree theory for linear differential operators with PBCs is invalid in the direct application to it.

In the special case $p = 2$ and $\beta = \alpha = 1$, the problem (4.12-4.13) becomes the two point BVPs of second order ordinary differential equation. When f is continuous, problem is nonsingular, the existence and uniqueness of positive solutions in this case have been studied by papers [42, 100]. The theorems we present include and extend some previous results.

When $\rho = 1$, Problem (4.12), (4.13) is reduced to second order two point BVP, which has been studied by many authors; see [11, 12].

In this section, we take

$$X = \left\{u : u \in C([0,1]) \text{ and } \varphi_p\left({}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right) \in C([0,1])\right\}. \tag{4.14}$$

For $u \in C[0,1]$, the corresponding norm is $\|u\|_0 = \max\{|u(t)| : t \in [0,1]\}$ and for $u \in X$, the corresponding norm is

$$\|u\| = \max\left\{\|u\|_0, \left\|{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right\|_0\right\}. \tag{4.15}$$

4.2.2 The solutions for the problem (E_1-C_1)

The problem (4.12-4.13) is equivalent to the following problem

$${}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u] = \varphi_q\left(\mathfrak{I}_{0+,t}^{\alpha,\rho}\left[f\left(\tau, u(\tau), -{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right] + \varphi_p\left({}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u](0)\right)\right), \tag{4.16}$$

with the initial condition for (4.16) is (4.13).

Define the operator $\mathrm{L} : \mathrm{Dom}\,\mathrm{L} \subset X \longrightarrow Y$ by

$$\mathrm{L}u = {}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u], \tag{4.17}$$

where

$$\mathrm{Dom}\mathrm{L} = \left\{u \in X : u(0) = u(1), {}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u](0) = 0\right\}. \tag{4.18}$$

Let $\mathrm{N} : X \longrightarrow Y$ be the Nemytski operator

$$\mathrm{N}u = \varphi_q\left(\mathfrak{I}_{0+,t}^{\alpha,\rho}\left[f\left(\tau, u(\tau), -{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]\right) \ \forall t \in [0,1]. \tag{4.19}$$

Then (4.16) is equivalent to the operator equation

$$\mathrm{L}u = \mathrm{N}u,\ \ u \in \mathrm{Dom L}. \tag{4.20}$$

The main goal in the coincidence degree theory is to search the existence of a solutions of the operator equation (4.16) in some bounded and open set Ω in some Banach space for L being a linear operator and N nonlinear operator using Leray-Schauder degree theory.

In this subsection, we need the following auxiliary lemmas to prove the existence of solutions to (4.16-4.13).

Lemma 14 *Let* L *be defined by (4.17), then*

$$\ker \mathrm{L} = \{c,\ \ c \in \mathbb{R}\} \tag{4.21}$$

and

$$\mathrm{Im}\,\mathrm{L} = \left\{ y \in Y : \int_0^1 (1-\tau^\rho)^{\beta-1} y(\tau) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} = 0 \right\}. \tag{4.22}$$

Proof. (i) By Lemma 2,

$$\mathrm{L}u = 0,\ \ {}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u] = 0, \tag{4.23}$$

has solution

$$u(t) = u(0) = c,\ \ \forall t \in [0,1]. \tag{4.24}$$

Combining with the boundary value conditions of (4.13), one has (4.21) hold.

(ii) If $y \in \mathrm{Im}L$, then there exists a function $u \in$ Dom L such that $\mathrm{L}u = y$

$$u(t) - u(0) = \mathfrak{I}^{\alpha,\rho}_{0+,t}[y], \tag{4.25}$$

$$u(t) = \int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\beta-1} y(\tau) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} + u(0). \tag{4.26}$$

From condition

$$u(1) = \int_0^1 \left(\frac{1-\tau^\rho}{\rho}\right)^{\beta-1} y(\tau) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} = u(0). \tag{4.27}$$

Thus, we get (4.22).

On the other hand, suppose $y \in Y$ and satisfies (4.22).

Let $u(t) = \mathfrak{I}_{0+,t}^{\beta,\rho}[y]$ then $u \in \text{Dom L}$ and

$$\mathrm{L}u(t) = \left({}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}\right)\left[\mathfrak{I}_{0+,\tau}^{\beta,\rho}[y]\right] = y(t) \in \text{DomL}.$$

■

Lemma 15 *Let* L *be defined by (4.17); then* L *is a Fredholm operator of index zero, and the linear continuous projector operators* $P : X \to X$ *and* $Q : Y \to Y$ *can be defined as*

$$(Pu)(t) = u(0), \ \forall t \in [0,1] \tag{4.28}$$

and

$$(Qy)(t) = w\left(\mathfrak{I}_{0+,1}^{\alpha,\rho}\right)y(t) = \rho\beta\int_0^1 (1-\tau^{\rho})^{\rho-1}\, y(\tau)\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}}, \tag{4.29}$$

where

$$w = \rho\beta.$$

Proof. (i) For any $u \in X$, we have

$$(Pu)(t) = u(0) \text{ and } \ker P = \{u \in X : u(0) = 0\}. \tag{4.30}$$

Obviously, $\operatorname{Im} P = \ker L$ and it is clear that $(P^2u)(t) = (Px)(t)$. $\forall u \in X$, it follows from $u = (u - Pu) + Pu$, that $X = \ker \mathrm{L} + \ker P$.

Since, for $u \in \ker L \cap \ker P \Longrightarrow u = 0$, which implies $\ker \mathrm{L} \cap \ker P = \{0\}$. Then we get

$$X = \ker \mathrm{L} \oplus \ker P. \tag{4.31}$$

(ii) For any $y \in Y$, we have

$$\begin{aligned} Q^2y(t) = Q(Qy(t)) = \rho\beta\int_0^1 (1-\tau^{\rho})^{\beta-1}\, Qy(\tau)\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ = Qy(t)\,\rho\beta\int_0^1 (1-\tau^{\rho})^{\beta-1}\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}} \\ = Qy(t). \end{aligned} \tag{4.32}$$

The next step is to prove $\ker Q = \operatorname{Im} \mathrm{L}$, It is clear that $\operatorname{Im} \mathrm{L} \subset \ker Q$ If $y \in \ker Q \subset Y$ then

$$Qy = 0 \Longrightarrow \rho\beta\int_0^1 (1-\tau^{\rho})^{\beta-1}\, y(\tau)\,\frac{\mathrm{d}\tau}{\tau^{1-\rho}} = 0.$$

Thus we get

$$y \in \operatorname{Im} \mathrm{L} \text{ and } \ker Q = \operatorname{Im} \mathrm{L}. \tag{4.33}$$

Let $y \in Y$, $y = (y - Qy) + Qy$ where $(y - Qy) \in \ker Q = \operatorname{Im} \mathrm{L}$,$Qy \in \operatorname{Im} Q$ It follows from $y = \ker Q + \operatorname{Im} \mathrm{L} = \operatorname{Im} \mathrm{L} + \ker Q$.

If $y \in \operatorname{Im} L \cap \operatorname{Im} Q$ then

$$\int_0^1 (1 - \tau^\rho)^{\beta - 1} y(\tau) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} = 0,$$

which implies that

$$y(t) \int_0^1 (1 - \tau^\rho)^{\beta - 1} \frac{\mathrm{d}\tau}{\tau^{1-\rho}} = \frac{y(t)}{\rho\beta} = 0, \ \forall t \in [0, 1].$$

Thus, we have

$$y = 0.$$

We can get that $\operatorname{Im} \mathrm{L} \cap \operatorname{Im} Q = \{0\}$. Then, we have

$$Y = \operatorname{Im} \mathrm{L} \oplus \operatorname{Im} Q. \tag{4.34}$$

Thus

$$\dim \ker \mathrm{L} = \dim \operatorname{Im} \mathrm{Q} = co \dim \operatorname{Im} \mathrm{L} = 1.$$

This means that L is a Fredholm operator of index zero.

$$\operatorname{Ind} \mathrm{L} = \dim \ker \mathrm{L} - co \dim \mathrm{L} = 1 - 1 = 0.$$

Furthermore, the operator $K_P : \operatorname{Im} L \rightarrow \operatorname{Dom} \mathrm{L} \cap \ker P$ can be written by

$$K_P y = \left(\mathfrak{I}^{\beta, \rho}_{0+, \tau} \right) y(t) \tag{4.35}$$

and

$$\mathrm{L}_P : \mathrm{L} \backslash \operatorname{dom}(\mathrm{L}) \cap \ker P \longrightarrow \operatorname{Im} \mathrm{L},$$

$$\mathrm{L}_P u = Lx. \tag{4.36}$$

Now, we will prove that K_P is the inverse of $(L|_{\operatorname{Dom} \mathrm{L} \cap \ker P})^{-1}$ From the definitions of P, K_P , it is easy to see that the generalized inverse of L is K_P.

In fact, for $y \in \operatorname{Im} L$, we have

$$\mathrm{L}_P K_P y =^c \mathcal{D}^{\beta, \rho}_{0+, \tau} \left[\mathfrak{I}^{\beta, \rho}_{0+, \tau} [y] \right] = y. \tag{4.37}$$

Moreover, for $u \in \mathrm{Dom\ L} \cap \ker P$, we have

$$u \in \mathrm{dom}\,(\mathrm{L}) \cap \ker P \Longrightarrow u\,(0) = 0 \text{ and } Px = 0.$$

By Lemma 1, we obtain that

$$K_P \mathrm{L}_P u = \mathfrak{I}^{\beta,\rho}_{0+,t} \left[{}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\,[u] \right] = u\,(t) - u\,(0)\,, \tag{4.38}$$

which together with $u(0) = 0$ yields that

$$\mathfrak{I}^{\beta,\rho}_{0+,t}\,[\mathrm{L}_P u] = u\,(t)\,. \tag{4.39}$$

Combining (4.37) with (4.39), we know that $K_P = \mathrm{L}_P^{-1}$. ■

Lemma 16 *Assume $\Omega \subset X$ is an open bounded subset such that* $\mathrm{Dom\ L} \cap \bar{\Omega} \neq \emptyset$*; then* N *is* L*-compact on* $\bar{\Omega}$*.*

Proof. Let

$$K_{P,Q} = K_P(I - Q)\mathrm{N}, \tag{4.40}$$

where I is identity operator. By the continuity of f and the definition of the operator N, there exists a constant $M > 0$ such that

$$\left| f\left(t, u\,(t)\,, -{}^{c}\mathcal{D}^{\beta,\rho}_{0+,t}\,[u]\right)\right| \leq M, \tag{4.41}$$

$$\left| \mathfrak{I}^{\alpha,\rho}_{0+,t} \left[f\left(\tau, u\,(\tau)\,, -{}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\,[u]\right)\right]\right| \leq M \tag{4.42}$$

and

$$\|(\mathrm{N}u)\,(t)\| \leq M, \forall u \in \bar{\Omega}, t \in [0,1]. \tag{4.43}$$

$$|(Q\mathrm{N}u)\,(t)| \leq w \int_0^1 (1 - \tau^{\rho})^{\beta - 1} \left|(\mathrm{N}u)\,(\tau)\right| \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \leq |\mathrm{N}u\,(t)| \leq M. \tag{4.44}$$

So, we get that $Q\mathrm{N}\left(\bar{\Omega}\right)$ is bounded.

$$|K_P(I - Q)\mathrm{N}u| \leq M, \forall u \in \bar{\Omega}, t \in [0,1]. \tag{4.45}$$

So, we get that $K_{P,Q}\left(\bar{\Omega}\right)$ is bounded.

Thus, in view of the Arzelà–Ascoli theorem, we need only prove that $K_{P,Q}\left(\bar{\Omega}\right) = K_P(I - Q)\mathrm{N}(\bar{\Omega}) \subset X$ is equicontinuous.

For $0 \leq t_1 < t_2 \leq 1, u \in \bar{\Omega}$, we have

$$|K_{P,Q}u(t_2) - K_{P,Q}u(t_1)| \leq |K_P\mathrm{N}u(t_2) - K_P\mathrm{N}u(t_1)| + |K_PQ\mathrm{N}u(t_2) - K_PQ\mathrm{N}u(t_1)|, \tag{4.46}$$

$$\begin{aligned} |K_P\mathrm{N}u(t_2) - K_P\mathrm{N}u(t_1)| &\leq \left| \tfrac{1}{\Gamma(\beta)} \int_0^{t_2} \left(\frac{t_2^\rho - \tau^\rho}{\rho}\right)^{\beta-1} \mathrm{N}u(\tau) \tfrac{\mathrm{d}\tau}{\tau^{1-\rho}} - \int_0^{t_1} \left(\frac{t_1^\rho - \tau^\rho}{\rho}\right)^{\beta-1} \mathrm{N}u(\tau) \tfrac{\mathrm{d}\tau}{\tau^{1-\rho}} \right| \\ &\leq \frac{1}{\Gamma(\beta)} \left| \int_0^{t_1} \left[\left(\frac{t_2^\rho - \tau^\rho}{\rho}\right)^{\beta-1} - \left(\frac{t_1^\rho - \tau^\rho}{\rho}\right)^{\beta-1} \right] \mathrm{N}u(\tau) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right| \\ &+ \left| \int_{t_1}^{t_2} \left(\frac{t_2^\rho - \tau^\rho}{\rho}\right)^{\beta-1} \mathrm{N}u(\tau) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right| \end{aligned} \tag{4.47}$$

$$\begin{aligned} |K_P\mathrm{N}u(t_2) - K_P\mathrm{N}u(t_1)| &\leq \frac{M}{\Gamma(\beta)} \left| \int_0^{t_1} \left[\left(\frac{t_2^\rho - \tau^\rho}{\rho}\right)^{\beta-1} - \left(\frac{t_1^\rho - \tau^\rho}{\rho}\right)^{\beta-1} \right] \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right| \\ &+ \left| \int_{t_1}^{t_2} \left(\frac{t_2^\rho - \tau^\rho}{\rho}\right)^{\beta-1} \frac{\mathrm{d}\tau}{\tau^{1-\rho}} \right| \\ &\leq \frac{M}{\rho^\beta \Gamma(\beta+1)} \left| \left[t_2^{\beta\rho} - (t_2^\rho - t_1^\rho)^\beta - t_1^{\beta\rho} \right] + (t_2^\rho - t_1^\rho)^\beta \right| \\ &\leq \frac{M}{\rho^\beta \Gamma(\beta+1)} \left| \left| t_2^{\beta\rho} - t_1^{\beta\rho} \right| + 2 \left| t_2^\rho - t_1^\rho \right|^\beta \right| \\ &\leq \frac{M}{\rho\beta^\rho \Gamma(\beta)} \left| t_2^{\beta\rho} - t_1^{\beta\rho} \right|. \end{aligned}$$

$$\begin{aligned} |K_{P,Q}u(t_2) - K_{P,Q}u(t_1)| &= |K_PQ\mathrm{N}u(t_2) - K_PQ\mathrm{N}u(t_1)| \\ &\leq \left| \tfrac{1}{\Gamma(\beta)} \int_0^{t_2} \left(\tfrac{t_2^\rho - \tau^\rho}{\rho}\right)^{\beta-1} Q\mathrm{N}u(\tau) \tfrac{\mathrm{d}\tau}{\tau^{1-\rho}} - \int_0^{t_1} \left(\tfrac{t_1^\rho - \tau^\rho}{\rho}\right)^{\beta-1} Q\mathrm{N}u(\tau) \tfrac{\mathrm{d}\tau}{\tau^{1-\rho}} \right| \\ &\leq \frac{M}{\rho^\beta \Gamma(\beta+1)} \left| \left| t_2^{\beta\rho} - t_1^{\beta\rho} \right| + 2 \left| t_2^\rho - t_1^\rho \right|^\beta \right|. \end{aligned} \tag{4.48}$$

Since $t^{\beta\rho}$ is uniformly continuous on $[0,1]$, we can obtain that $K_{P,Q}(\bar{\Omega}) \subset C[0,1]$ is equicontinuous.

Similar proof can show that ${}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[K_P(\mathfrak{I}-Q)\mathrm{N}(\bar{\Omega})\right] \subset C[0,1]$is equicontinuous. This, together with the uniformly continuity of $\varphi_q(s)$ on $[-M, M]$, yields that ${}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[K_{P,Q}(\bar{\Omega})\right] = \varphi_q({}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[K_P(\mathfrak{I}-Q)\mathrm{N})(\bar{\Omega})\right)] \subset C[0,1]$ is also equicontinuous.

We divide the prove into the following two cases.

Case 1. $1 < p \leq 2$, by Lemma 8 and (4.40-4.45), we have

$$\left| {}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[K_{P,Q}u\right](t_2) - {}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[K_{P,Q}u\right](t_1) \right| \leq |\mathrm{N}u(t_2) - \mathrm{N}u(t_1)| + |Q\mathrm{N}u(t_2) - Q\mathrm{N}u(t_1)|. \tag{4.49}$$

$$\begin{aligned}
|\mathrm{N}u(t_2)-\mathrm{N}u(t_1)| &\le \left|\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,\tau}[f_u]+\varphi_p\left({}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)(0)\right)-\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,\tau}[f_u]+\varphi_p\left({}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)(0)\right)\right|\\
&\le \tfrac{(q-1)M^{q-2}}{\Gamma(\alpha)}\left|\int_0^{t_2}\left(\tfrac{t_2^\rho-\tau^\rho}{\rho}\right)^{\rho\alpha-1}f_u(\tau)\tfrac{\mathrm{d}\tau}{\tau^{1-\rho}}-\int_0^{t_1}\left(\tfrac{t_1^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}f_u(\tau)\tfrac{\mathrm{d}\tau}{\tau^{1-\rho}}\right|\\
&\le \frac{(q-1)M^{q-1}}{\rho^\alpha\Gamma(\alpha+1)}\left[2\left|t_2^\rho-t_1^\rho\right|^\alpha+\left|t_2^{\alpha\rho}-t_1^{\alpha\rho}\right|\right]. \qquad (4.50)
\end{aligned}$$

Case 2. $p>2$, by (4.40-4.45), we have

(i) Suppose that

$$\left(\mathfrak{I}^{\alpha,\rho}_{0+,t_1}\right)[f_u]=0,$$

then $\exists\delta_1>0$, for $t_2\in[0,1]$ such that $0<t_2-t_1<\delta_1$ and $u\in\bar{\Omega}$, we have

$$\left(\mathfrak{I}^{\alpha,\rho}_{0+,t_2}\right)[f_u]>0$$

and

$$\begin{aligned}
\left|{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[K_{P,Q}u](t_2)-{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[K_{P,Q}u](t_1)\right| &\le |\mathrm{N}u(t_2)-\mathrm{N}u(t_1)|+|Q\mathrm{N}u(t_2)-Q\mathrm{N}u(t_1)|\\
&\le |\mathrm{N}u(t_2)-\mathrm{N}u(t_1)|. \qquad (4.51)
\end{aligned}$$

$$\begin{aligned}
|\mathrm{N}u(t_2)-\mathrm{N}u(t_1)| &\le |\mathrm{N}u(t_2)| \qquad (4.52)\\
&\le \left|\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,t_2}[f_u]\right)\right|\\
&\le \left|\left(\mathfrak{I}^{\alpha,\rho}_{0+,t_2}[f_u]\right)\right|^{q-1}\\
&\le \left|\mathfrak{I}^{\alpha,\rho}_{0+,t_2}[f_u]-\mathfrak{I}^{\alpha,\rho}_{0+,t_1}[f_u]\right|^{q-1}.
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{(\Gamma(\alpha))^{q-1}}\left|\int_0^{t_2}\left(\frac{t_2^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}f_u(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}-\int_0^{t_1}\left(\frac{t_1^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}f_u(\tau)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right| \qquad (4.53)\\
&\le \frac{M^{q-1}}{\rho^{(q-1)(\alpha-1)}(\Gamma(\alpha))^{q-1}}\left|\int_0^{t_1}\left[(t_2^\rho-\tau^\rho)^{\alpha-1}-(t_1^\rho-\tau^\rho)^{\alpha-1}\right]\frac{\mathrm{d}\tau}{\tau^{1-\rho}}+\int_{t_1}^{t_2}(t_2^\rho-\tau^\rho)^{\alpha-1}\frac{\mathrm{d}\tau}{\tau^{1-\rho}}\right|^{q-1}\\
&\le \frac{M^{q-1}}{\rho^{(q-1)\alpha}(\Gamma(\alpha+1))^{q-1}}\left|t_2^{\alpha\rho}-t_1^{\alpha\rho}\right|^{q-1}.
\end{aligned}$$

(ii) If

$$\mathfrak{I}^{\alpha,\rho}_{0+,t_1}[f_u]\neq 0,$$

then, there exists two positives constants δ_2 and $l>0$ such that

$$\left(\mathfrak{I}^{\alpha,\rho}_{0+,t_2}\right)[f_u]\ge l>0,\ \forall t_2\in\,]t_1-\delta_2,t_1+\delta_2[\,. \qquad (4.54)$$

By Lemma 8, we have

$$\left|\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\right) K_{P,Q}u\left(t_{2}\right)-\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\right) K_{P,Q}u\left(t_{1}\right)\right| \leq \left|\mathrm{N}u\left(t_{2}\right)-\mathrm{N}u\left(t_{1}\right)\right|+\left|Q\mathrm{N}u\left(t_{2}\right)-Q\mathrm{N}u\left(t_{1}\right)\right| \leq \left|\mathrm{N}u\left(t_{2}\right)-\mathrm{N}u\left(t_{1}\right)\right|. \quad (4.55)$$

$$\begin{aligned}\left|\mathrm{N}u\left(t_{2}\right)-\mathrm{N}u\left(t_{1}\right)\right| &\leq \left|\varphi_{q}\left(\mathfrak{I}^{\alpha,\rho}_{0+,t_{2}}\left[f_{u}\right]\right)-\varphi_{q}\left(\mathfrak{I}^{\alpha,\rho}_{0+,t_{1}}\left[f_{u}\right]\right)\right| \quad (4.56)\\ &\leq (q-1)\, l^{q-2}\left|\mathfrak{I}^{\alpha,\rho}_{0+,t_{2}}\left[f_{u}\right]-\mathfrak{I}^{\alpha,\rho}_{0+,t_{1}}\left[f_{u}\right]\right| \\ &\leq (q-1)\, l^{q-2}\frac{M}{\rho^{\alpha+1}\left(\Gamma\left(\alpha\right)\right)}\left|t_{2}^{\alpha\rho}-t_{1}^{\alpha\rho}\right|,\ \forall t_{2}\in\left]t_{1},t_{1}+\delta_{2}\right[,\end{aligned}$$

where $\delta=\max\left\{\delta_{1},\delta_{2}\right\}$ this inequality hold for $t_{2}\in\left]t_{1}-\delta,t_{1}+\delta\right[$.

(iii) If

$$\mathfrak{I}^{\alpha,\rho}_{0+,t_{1}}\left[f_{u}\right]<0,$$

we have similar proof.

From 4.53) and (4.56) we see that $K_{P,Q}:\bar{\Omega}\rightarrow X$ is equicontinuous. Thus, we get that $K_{P}(I-Q)\mathrm{N}:\bar{\Omega}\rightarrow X$ is compact. ■

Lemma 17 *Suppose (A_1), (A_2) and (A_3) hold; then the set*

$$\Omega_{1}=\left\{u\in \mathrm{DomL}\backslash\ker \mathrm{L}:\mathrm{L}u=\lambda \mathrm{N}u \textit{ for some } \lambda\in(0,1)\right\}, \quad (4.57)$$

is bounded.

Proof. Take $u\in\Omega_{1}$, then $\mathrm{L}u=\lambda\mathrm{N}u$, and $\mathrm{N}u\in \mathrm{Im}L=\ker Q$.

By (4.22), we have

$$\mathfrak{I}^{\beta,\rho}_{0+,1}\left[f_{u}\right]=0,$$

then by the integral mean value theorem, there exists a constant $\xi\in(0,1)$ such that

$$f\left(\xi,u\left(\xi\right),-{}^{c}\mathcal{D}^{\beta,\rho}_{0+,\xi}\left[u\right]\right)=0$$

So from (A_3), we get

$$\left|u\left(\xi\right)\right|\leq E.$$

From $u\in \mathrm{domL}$, we have

$$\mathfrak{I}^{\beta,\rho}_{0+,t}\left[{}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[u\right]\right]-\mathfrak{I}^{\beta,\rho}_{0+,\xi}\left[{}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[u\right]\right]=u\left(t\right)-u\left(\xi\right) \quad (4.58)$$

and

$$\left|\mathfrak{I}_{0+,t}^{\beta,\rho}\left[{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right]\right| = \frac{1}{\Gamma(\beta)}\left|\int_0^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\beta-1c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\frac{d\tau}{\tau^{1-\rho}}\right|$$
$$\leq \frac{t^{\rho\beta}}{\rho^\beta\Gamma(\beta+1)}\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_\infty.$$

Thus, we have

$$\begin{aligned}|u(t)| &= \left|u(\xi)+\mathfrak{I}_{0+,t}^{\beta,\rho}\left[{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right]-\mathfrak{I}_{0+,\xi}^{\beta,\rho}\left[{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right]\right| \\ &\leq E+\frac{2}{\rho^\beta\Gamma(1+\beta)}\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_\infty,\end{aligned} \tag{4.59}$$

then

$$\|u\|_\infty \leq E+\frac{2}{\rho^\beta\Gamma(1+\beta)}\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_\infty. \tag{4.60}$$

By $\mathrm{L}u=\lambda\mathrm{N}u$, we get

$${}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]=\lambda\varphi_q\left({}^{\rho}\mathfrak{I}_{0+,t}^{\alpha}[f_u]\right). \tag{4.61}$$

Applying the operator φ_p to the two sides of (4.61), one has

$$\begin{aligned}\varphi_p\left({}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right) &= \varphi_p\left(\lambda\varphi_q\left(\mathfrak{I}_{0+,t}^{\alpha,\rho}[f_u]\right)\right) \\ &= \varphi_p(\lambda)\left(\mathfrak{I}_{0+,t}^{\alpha,\rho}[f_u]\right) \\ &= \lambda^{p-1}\mathfrak{I}_{0+,t}^{\alpha,\rho}[f_u].\end{aligned} \tag{4.62}$$

From (A_1) and (4.62), we get

$$\begin{aligned}\left|\varphi_p\left({}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right)\right| &= \lambda^{p-1}\left|\mathfrak{I}_{0+,t}^{\alpha,\rho}[f_u]\right| \\ &\leq \frac{\lambda^{p-1}}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}[f_u](\tau)\frac{d\tau}{\tau^{1-\rho}} \\ &\leq \tfrac{\lambda^{p-1}}{\Gamma(\alpha)}\left[\|a\|_\infty+\|b\|_\infty\|u\|_\infty^{p-1}+\|c\|_\infty\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_\infty^{p-1}\right]\int_0^t\left(\tfrac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\tfrac{d\tau}{\tau^{1-\rho}} \\ &\leq \tfrac{1}{\rho^\alpha\Gamma(\alpha+1)}\left[\begin{array}{c}\|a\|_\infty+\|b\|_\infty\left(E+\frac{2\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_\infty}{\rho^\beta\Gamma(\beta+1)}\right)^{p-1} \\ +\|c\|_\infty\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_\infty^{p-1}\end{array}\right].\end{aligned} \tag{4.63}$$

If $1<p<2$, from Lemma 9, we have

$$\begin{aligned}\left|\varphi_p\left({}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right)\right| \leq{}& \tfrac{\left(\|a\|_\infty+\|b\|_\infty E^{p-1}\right)}{\rho^\alpha\Gamma(\alpha+1)} \\ &+\tfrac{1}{\rho^\alpha\Gamma(\alpha+1)}\left[\left(\|b\|_\infty\left(\tfrac{2}{\rho^\beta\Gamma(\beta+1)}\right)^{p-1}+\|c\|_\infty\right)\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_\infty^{p-1}\right]\end{aligned}$$

Moreover $\left|\varphi_p\left({}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right)\right| = \left|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right|^{p-1}$ then

$$\begin{aligned}\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|^{p-1}_{\infty} \leq & \frac{\|a\|_{\infty} + \|b\|_{\infty} E^{p-1}}{\rho^{\alpha}\Gamma(\alpha+1)} \\ & + \tfrac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\left[\left(\|b\|_{\infty}\left(\frac{2}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1} + \|c\|_{\infty}\right)\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|^{p-1}_{\infty}\right].\end{aligned} \tag{4.64}$$

So

$$\left(1 - \tfrac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\left(\|b\|_{\infty}\left(\tfrac{2}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1} + \|c\|_{\infty}\right)\right)\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|^{p-1}_{\infty} \leq \tfrac{\|a\|_{\infty}+\|b\|_{\infty}E^{p-1}}{\rho^{\alpha}\Gamma(\alpha+1)}.$$

If $0 < R_1 = \left(1 - \tfrac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\left(\|b\|_{\infty}\left(\tfrac{2}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1} + \|c\|_{\infty}\right)\right)$ then

$$\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|_{\infty} \leq L_1 = \left(\frac{\|a\|_{\infty} + \|b\|_{\infty} E^{p-1}}{\rho^{\alpha}\Gamma(\alpha+1) R_1}\right)^{1-p} \tag{4.65}$$

and ,

$$\|u\|_{\infty} \leq L_2 = E + \frac{2}{\rho^{\beta}\Gamma(\beta+1)} L_1. \tag{4.66}$$

If $p \geq 2$ then

$$\begin{aligned}\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|^{p-1}_{\infty} \leq & \frac{\|a\|_{\infty} + 2^{p-2}\|b\|_{\infty} E^{p-1}}{\rho^{\alpha}\Gamma(\alpha+1)} \\ & + \tfrac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\left[\left(\|b\|_{\infty}\left(\tfrac{2^{p-2}}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1} + \|c\|_{\infty}\right)\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|^{p-1}_{\infty}\right].\end{aligned} \tag{4.67}$$

So

$$\left(1 - \tfrac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\left(\|b\|_{\infty}\left(\tfrac{2}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1} + \|c\|_{\infty}\right)\right)\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|^{p-1}_{\infty} \leq \tfrac{\|a\|_{\infty}+2^{p-2}\|b\|_{\infty}E^{p-1}}{\rho^{\alpha}\Gamma(\alpha+1)}.$$

If $0 < R_2 = \left(1 - \tfrac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\left(\|b\|_{\infty}\left(\tfrac{2^{p-2}}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1} + \|c\|_{\infty}\right)\right)$ then

$$\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|_{\infty} \leq l_1 = \left(\frac{\|a\|_{\infty} + 2^{p-2}\|b\|_{\infty} E^{p-1}}{\rho^{\alpha}\Gamma(\alpha+1) R_2}\right)^{1-p} \tag{4.68}$$

and ,

$$\|u\|_{\infty} \leq l_2 = E + \frac{2}{\rho^{\beta}\Gamma(\beta+1)} l_1. \tag{4.69}$$

Using (4.65), (4.66), (4.68) and (4.69), we have

$$\|u\|_X \leq \max\left\{\|u\|_{\infty}, \left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right\|_{\infty}\right\} \leq \max\{L_2, L_1, l_2, l_1\} = L_3. \tag{4.70}$$

Therefore Ω_1 is bounded. ■

Lemma 18 *Suppose (A_2) holds, then the set*

$$\Omega_2 = \{u : u \in \ker \mathrm{L}, \mathrm{N}u \in \mathrm{Im}\,\mathrm{L}\}, \tag{4.71}$$

is bounded.

Proof. For $u \in \Omega_2$, we have $u(t) = c, c \in \mathbb{R}$ and $\mathrm{N}u \in \mathrm{Im}L= \ker \mathrm{Q}$.Then we get

$$Q\mathrm{N}\,(u) = \rho\beta \int_0^1 (1-\tau^\rho)^{\beta-1} |\mathrm{N}u| \frac{\mathrm{d}\tau}{\tau^{1-\rho}} = 0. \tag{4.72}$$

Then by the integral mean value theorem, there exists a constant $\xi \in (0,1)$ such that $\mathrm{N}u\,(\xi) = 0$ that implies

$$\int_0^\xi (\xi^\rho - \tau^\rho)^{\beta-1} f\,(\tau, c, 0) \frac{\mathrm{d}\tau}{\tau^{1-\rho}} = 0.$$

By same mean value theorem, we obtain $s \in (0,\xi)$ such that $f\,(s,c,0) = 0$ which together with (A_2) implies $|c| \le B$. Thus, we have

$$\|u\|_X \le \max\{B, 0\} = B. \tag{4.73}$$

Hence, Ω_2 is bounded. The proof is complete. ■

Lemma 19 *Suppose the first part of (A_2) holds; then*

$$\Omega_3^+ = \{u \in \ker \mathrm{L} : \lambda x + (1-\lambda)Q\mathrm{N}u = 0, \lambda \in [0,1]\}, \tag{4.74}$$

is bounded.

Proof. For $u \in \Omega_3^+$, we have $u(t) = c,\ c \in \mathbb{R}$ and which implies $\varphi_p \left({}^c\mathcal{D}^{\beta,\rho}_{0+,t}\,[u]\right)(0) = 0$ and

$$\mathrm{N}u = \mathrm{N}c = \varphi_q \left(\mathfrak{I}^{\alpha,\rho}_{0+,t}\,[f\,(\tau, c, 0)]\right). \tag{4.75}$$

If $\lambda = 1$, then $u = c = 0$.

If $\lambda = 0$, by same analysis of Lemma 17 we have Ω_3^+ is bounded, i.e., $|c| \le B$ because of the first part of (A_2).

If $\lambda \in (0,1)$, we have

$$\lambda c + (1-\lambda)Q\left(\varphi_q \left(\mathfrak{I}^{\alpha,\rho}_{0+,t}\,[f\,(\tau, c, 0)]\right)\right) = 0, \tag{4.76}$$

thus we obtain $Q\left(\varphi_q\left(\mathfrak{I}_{0+,t}^{\alpha,\rho}\left[f\left(\tau,c,0\right)\right]\right)\right)=0$. According the similar prof of Lemma 17 , we have Ω_3^+ is bounded, we can also obtain $|c|\leq B$. Otherwise,

if $|c|>B$, in view of the first part of (A_2), one has

$$\lambda c^2+(1-\lambda)\int_0^1\left(1-\tau^\rho\right)^{\alpha-1}cf\left(\tau,c,0\right)\frac{\mathrm{d}\tau}{\tau^{1-\rho}}>0, \tag{4.77}$$

which contradicts to (4.76). Therefore, Ω_3^+ is bounded. The proof is complete. ∎

Remark 5 *If the second part of (A_2) holds, then the set*

$$\Omega_3^-=\{u\in\ker\mathrm{L}:-\lambda Ix+(1-\lambda)JQ\mathrm{N}u=0,\lambda\in[0,1]\}, \tag{4.78}$$

is bounded.

Theorem 5 *Let $f:[0,1]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous. Assume that (A_1) and (A_2) hold, then Then BVP (4.12-4.13) has at least one solution, provided that*

$$\frac{1}{\rho^\alpha\Gamma\left(\alpha+1\right)}\left(\|b\|_\infty\left(\frac{2^{p-2}}{\rho^\beta\Gamma\left(\beta+1\right)}\right)^{p-1}+\|c\|_\infty\right)<1,\ \ \textit{if } p\geq2,$$

or

$$\frac{1}{\rho^\alpha\Gamma\left(\alpha+1\right)}\left(\|b\|_\infty\left(\frac{2}{\rho^\beta\Gamma\left(\beta+1\right)}\right)^{p-1}+\|c\|_\infty\right)<1,\ \ \textit{if } 1<p<2.$$

Proof of Theorem 5. Set

$$\Omega=\{u\in X:\|u\|_X<\kappa=\max\{L_3,B\}+1\}. \tag{4.79}$$

Obviously, $\Omega_1\cup\Omega_2\cup\Omega_3\subset\Omega$, or $\Omega_1\cup\Omega_2\cup\Omega_3^-\subset\Omega$. It follows from Lemmas 15 and 16 that L (defined by (4.17) is a Fredholm operator of index zero and N (defined by (4.19)) is L-compact on Ω. By Lemmas 17 and 18, we get that the following two conditions are satisfied

(i) $\mathrm{L}u\neq\lambda\mathrm{N}u\ \forall\,(u,y)\in[(\mathrm{DomL}/\ker\mathrm{L})\cap\partial\Omega]\times(0,1)$

(ii) $\mathrm{N}u\notin\mathrm{Im}L,\ \forall u\in\ker L\cap\partial\Omega$

It remains verifying the condition (C_3) of Theorem 3. In order to do that, let

$$\mathrm{H}(u,\lambda)=\pm\lambda x+(1-\lambda)Q\mathrm{N}u. \tag{4.80}$$

Basing on Lemma 19, we have

$$\mathrm{H}(u,\lambda) \neq 0, \forall u \in \partial\Omega \cap \ker \mathrm{L}. \tag{4.81}$$

Thus, by the homotopy property of degree, we have

$$\begin{aligned} \deg(Q\mathrm{N}|_{\ker \mathrm{L}}, \Omega \cap \ker \mathrm{L}, 0) &= \deg(\mathrm{H}(\cdot,0), \Omega \cap \ker \mathrm{L}, 0) \\ &= \deg(\mathrm{H}(\cdot,1), \Omega \cap \ker \mathrm{L}, 0) \\ &= \deg(\pm I, \Omega \cap \ker \mathrm{L}, 0) \neq 0. \end{aligned} \tag{4.82}$$

Consequently, by using Theorem 3, the operator equation $\mathrm{L}u = \mathrm{N}u$ has at least one solution in Dom $\mathrm{L} \cap \Omega$. Namely, BVP (4.12-4.13) has at least one solution in X. The proof is complete. ■

4.2.3 Example

In this subsection, we present one example to indicate how our theorem can be applied to concrete problems.

Example 1 *Let us consider the following fractional differential equation at resonance*

$${}^{c}\mathcal{D}_{0+}^{2/3,1/2}\left[\varphi_3\left({}^{c}\mathcal{D}_{0+,\tau}^{3/4,1/2}[u]\right)\right] = -\frac{1}{2}t + \frac{t}{2}u^2(t) + \frac{t}{4}\sin^2\left({}^{c}\mathcal{D}_{0+,t}^{3/4,1/2}[u]\right). \tag{4.83}$$

Corresponding to BVP (4.83-C_1), we have that $p = 3,\ \rho = 2,\ \alpha = 2/3,\ \beta =, 3/4$.

Choose $a(t) = -\frac{1}{2}t,\ b(t) = \frac{t}{2},\ c(t) = \frac{1}{4}t$ and $B = E = 1$. By simple calculation, we can get that $\|a\|_\infty = \|b\|_\infty = 1/2, \|c\|_\infty = 1/4$ and

$$0 < \frac{1}{\rho^\alpha \Gamma(\alpha+1)}\left(\|b\|_\infty \left(\frac{2^{p-2}}{\rho^\beta \Gamma(\beta+1)}\right)^{p-1} + \|c\|_\infty\right) = 0.76 < 1. \tag{4.84}$$

Obviously, the boundary value problem (4.83-C_1) satisfies all conditions of Theorem 5. Hence, it has at least one solution.

4.3 p -Laplacian three -point boundary value problems with fractional conformable derivative in the sense of Caputo

4.3.1 Introduction

Inspired and motivated by earlier works of authors like Hul et al. [101] and Z. Bai [102], In this section, the following problem for the fractional differential equation at resonance

$$ {}^{c}\mathcal{D}_{0+,t}^{\alpha,\rho}\left[\varphi_p\left({}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]=f\left(t,u(t),-{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right),\ t\in(0,1),0<\beta,\alpha\leq 1, \tag{4.85} $$

with the condition for (4.85) is

$$ u(0)=u(\eta),{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u](1)=0, \tag{4.86} $$

is considered. By using the coincidence degree theory, some existence results of solutions are established.

FBVP (4.85-4.86) happens to be at resonance in the sense that its associated linear homogeneous boundary value problem

$$ {}^{c}\mathcal{D}_{0+,t}^{\alpha,\rho}\left[\varphi_p\left({}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]=0, \tag{4.87} $$

with the condition (4.86) has a nontrivial solution $u(t)=c$, where $c\in\mathbb{R}$.

Due to the fact that the classical Mawhin's continuation theorem can't be directly used to discuss the BVP with nonlinear differential operator, in this section, we investigate the BVP (4.85-4.86) by applying an extension of Mawhin's continuation theorem due to Ge et al. [94].

We denote

$$ Y=C[0,1]\ \text{and}\ X=\left\{u\in Y:{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\in Y\right\}. \tag{4.88} $$

Define the operator $\mathrm{M}:\mathrm{Dom}\ \mathrm{M}\subset X\longrightarrow Y$ by

$$ (\mathrm{M}u)(t)=\left({}^{c}\mathcal{D}_{0+,t}^{\alpha,\rho}\right)\left[\varphi_p\left({}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right],\ t\in[0,1]. \tag{4.89} $$

where

$$ \mathrm{Dom}\mathrm{M}=\left\{u\in X:u(0)=u(\eta),{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u](1)=0\right\}. \tag{4.90} $$

Let

$$X_1 = \ker \mathrm{M} \ , \ X_2 = \{u \in X : u(0) = u(\eta) = 0\} \tag{4.91}$$

and

$$Y_1 = \mathbb{R} \ , \ Y_2 = \mathrm{Im}\, \mathrm{M}. \tag{4.92}$$

Let $\mathrm{N}_\lambda : Y \longrightarrow Y$ be the Nemytski operator

$$(\mathrm{N}_\lambda u) = \lambda f, \ \ \lambda \in [0,1], \ \forall t \in [0,1]. \tag{4.93}$$

Then BVP (4.85-4.86) is equivalent to the operator equation

$$\mathrm{M}u = \mathrm{N}u, \ \ u \in \mathrm{Dom}\mathrm{M}. \tag{4.94}$$

In this section, we need the following auxiliary lemmas to prove the existence of solutions to (4.94).

4.3.2 The solutions for the problem ($\mathbf{E}_1$-$\mathbf{C}_2$)

In order to obtain our main result, we firstly present and prove the following lemmas.

Lemma 20 *The operator* M *defined by (4.89) is quasi-linear and*

$$\ker \mathrm{M} = \{u \in \mathrm{Dom}\ \mathrm{M} \cap X : u(t) = c, c \in \mathbb{R}, \ t \in [0,1]\} \tag{4.95}$$

and

$$\mathrm{Im}\, \mathrm{M} = \left\{ y \in Y : \mathfrak{I}^{\beta,\rho}_{0+,\eta} \left[\varphi_q \left(\mathfrak{I}^{\alpha,\rho}_{0+,s}[y] - \mathfrak{I}^{\alpha,\rho}_{0+,1}[y]\right)\right] = 0 \right\}. \tag{4.96}$$

Proof. The proof will be given in the following two steps.

Step 1. kerM is linearly homeomorphic to $\mathbb{R}$. By Lemma 1, the homogeneous equation

$$(\mathrm{M}u)(t) = 0 \Longleftrightarrow {}^c\mathcal{D}^{\alpha,\rho}_{0+,t} \left[\varphi_p \left({}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right] = 0, \tag{4.97}$$

has solution

$$u(t) = \mathfrak{I}^{\beta,\rho}_{0+,t}[\varphi_q(c_0)] + c_1 = \frac{\varphi_q(c_0)}{\rho^\beta \Gamma(\beta+1)} t^{\rho\beta} + c_1. \tag{4.98}$$

Combining with the boundary value condition (4.86), one has that (4.95) holds.

It is easy to get (4.95). Thus $\dim\ker \mathrm{M} = 1$, i.e., ($\ker \mathrm{M} \simeq \mathbb{R}$).

Step 2. Im M is a closed subset of Y. If $y \in \mathrm{Im}M$, then there exists a function $u \in$ Dom M such that $(\mathrm{M}u)(t) = y(t)$, then

$$y(t) =^{c} \mathcal{D}^{\alpha,\rho}_{0+,t}\left[\varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right], \tag{4.99}$$

Basing on Lemma 1, we have

$$\mathfrak{I}^{\alpha,\rho}_{0+,t}[y] = \mathfrak{I}^{\alpha,\rho}_{0+,t}\left[{}^{c}\mathcal{D}^{\alpha,\rho}_{0+,s}\left[\varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right]\right],$$

then

$$\mathfrak{I}^{\alpha,\rho}_{0+,t}[y] = \varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)(t) - \varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)(0).$$

From condition

$$\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right)(1) = 0, \tag{4.100}$$

we obtain that (4.96).

On the other hand, suppose $y \in Y$ and satisfies (4.96) and let

$$u(t) = \mathfrak{I}^{\alpha,\rho}_{0+,t}\left[\varphi_q\left(\mathfrak{I}^{\beta,\rho}_{0+,\tau}[y]\right)\right], \tag{4.101}$$

then $u \in \mathrm{DomM}$ and $\mathrm{M}u =^{c} \mathcal{D}^{\alpha,\rho}_{0+,t}\left[\varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right] = y$, so $y \in \mathrm{Im}\,\mathrm{M}$, (4.96) is satisfied.

Then $\mathrm{Im}\,\mathrm{M} = \mathrm{M}(\mathrm{DomM})$ is closed subset of Y. Therefore, M is quasi-linear operator. ■

Lemma 21 *Let* M *be defined by (4.89); then* M *is quasi-linear operator and let the projector P and semi-projector Q by $P : X \to \ker M$ and $Q : Y \to \mathbb{R}$ can be defined as*

$$(Pu)(t) = u(0) \tag{4.102}$$

and

$$(Qy)(t) = w\varphi_p\left(\mathfrak{I}^{\beta,\rho}_{0+,\eta}\left[\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,s}[y] - \mathfrak{I}^{\alpha,\rho}_{0+,1}[y]\right)\right]\right), \tag{4.103}$$

where

$$w = \varphi_p\left(\mathfrak{I}^{\beta,\rho}_{0+,\eta}\left[\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,s}[1] - \mathfrak{I}^{\alpha,\rho}_{0+,1}[1]\right)\right]\right)^{-1}. \tag{4.104}$$

The operator $R : \overline{\Omega} \times [0,1] \to \mathrm{Dom}\,\mathrm{M} \cap \ker P$ can be written by $(\mathrm{N}_\lambda u = \lambda f)$

$$R(u,\lambda)(t) = \mathfrak{I}^{\beta,\rho}_{0+,t}\left[\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,s}[(I-Q)\mathrm{N}_\lambda u] - \mathfrak{I}^{\alpha,\rho}_{0+,1}[(I-Q)\mathrm{N}_\lambda u]\right)\right]. \tag{4.105}$$

Proof.

(i) For any $u \in X$, we have

$$(Pu)(t) = u(0).$$

It is clear that

$$P^2 u(t) = Pu(t), \ \forall t \in [0,1],$$

then P is a projector.

It follows from

$$\forall u \in X : u = (u - Pu) + Pu,$$

$$X = \ker \mathrm{M} + \ker P. \tag{4.106}$$

Since

$$\ker \mathrm{M} \cap \ker P = \{0\}, \tag{4.107}$$

we have

$$X = \ker \mathrm{M} \oplus \ker P. \tag{4.108}$$

(ii) For any $y \in Y$, we have

$$\forall y \in Y : Q : Y \longrightarrow Y,$$

$$Qy(t) = w\varphi_p \left\{ \left(\mathfrak{I}_{0+,\eta}^{\beta,\rho} \right) \left[\varphi_q \left(\left(\mathfrak{I}_{0+,s}^{\alpha,\rho} y \right) - \left(\mathfrak{I}_{0+,1}^{\alpha,\rho} y \right) \right) \right] \right\}. \tag{4.109}$$

Then we get

$$(Q(\lambda y)(t)) = (\lambda Q y(t))$$

and

$$Q^2 y(t) = Q(Qy(t)) = Qy(\tau),$$

that is, Q is semi-projector. Moreover, $X_1 = \operatorname{Im} P$ and $\operatorname{Im} \mathrm{M} = \ker Q$. ■

Lemma 22 *Let Ω be an open bounded set. Then $X_1 = \operatorname{Im} P$, $\operatorname{Im} \mathrm{M} = \ker Q$.and N_λ defined by (4.93) is* M *compact.*

Proof. Choose $X_2 = \ker P$ and $Y_1 = \operatorname{Im} Q$. Thus

$$\dim X_1 = \dim Y_1 = 1. \tag{4.110}$$

The remainder of the proof will be given in the following two steps.

Step 1. From $f \in C([0,1] \times \mathbb{R}^2 \times \mathbb{R}, \mathbb{R}), \forall \lambda \in [0,1]$, R is continuous and compact. By the definition of R, we obtain

$$ {}^{c}\mathcal{D}^{\alpha,\rho}_{0+,t}[R(u,\lambda)] = \varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,t}[(I-Q)\mathrm{N}_\lambda u] - \mathfrak{I}^{\alpha,\rho}_{0+,1}[(I-Q)\mathrm{N}_\lambda u]\right). \tag{4.111} $$

Clearly, the operators R, ${}^{c}\mathcal{D}^{\alpha,\rho}_{0+,t}[R]$ are compositions of the continuous operators. So R, ${}^{c}\mathcal{D}^{\alpha,\rho}_{0+,t}[R]$ are continuous in Y. Hence R is a continuous operator, and $R\left(\bar{\Omega}\right)$, ${}^{c}\mathcal{D}^{\alpha;\rho}_{0+}[R]\left(\bar{\Omega}\right)$ are bounded in Y.

$$ \left|f\left(\tau, u(\tau), -{}^{c}\mathcal{D}^{\beta;\rho}_{0+,\tau}[u]\right)\right| \le K, \text{ for } u \in \bar{\Omega},\ t \in [0,1]. \tag{4.112} $$

Furthermore, there exists a constant $K > 0$ such that

$$ \left|\mathfrak{I}^{\alpha,\rho}_{0+,t}[(I-Q)\mathrm{N}_\lambda u] - \mathfrak{I}^{\alpha,\rho}_{0+,1}[(I-Q)\mathrm{N}_\lambda u]\right| \le K,\ \forall u \in \bar{\Omega}, t \in [0,1]. \tag{4.113} $$

Thus, based on the Arzelà-Ascoli theorem, we need only to show $R\left(\bar{\Omega}\right) \subset X$ is equicontinuous.

For $0 \le t_1 \le t_2 \le 1$

$$ \begin{aligned} |R(u,\lambda)(t_2) - R(u,\lambda)(t_1)| &= \left|\mathfrak{I}^{\beta,\rho}_{0+,t_2}[1] - \mathfrak{I}^{\beta,\rho}_{0+,t_2}[1]\right| \\ &\quad\times \varphi_q\left[\mathfrak{I}^{\alpha,\rho}_{0+,\tau}[(I-Q)\mathrm{N}_\lambda u] - \mathfrak{I}^{\alpha,\rho}_{0+,1}[(I-Q)\mathrm{N}_\lambda u]\right] \\ &\le \frac{K^{q-1}}{\rho^\beta\Gamma(\beta+1)}\left[\left|t_2^{\rho\beta} - t_1^{\rho\beta}\right| + 2\left(t_2^\rho - t_1^\rho\right)^\beta\right]. \end{aligned} \tag{4.114} $$

As $t^{\rho\beta}$ is uniformly continuous in $[0,1]$, we obtain $R\left(\bar{\Omega}\right) \subset Y$ is equicontinuous. With similar proof we obtain

$$ \mathfrak{I}^{\alpha,\rho}_{0+t}[(I-Q)\mathrm{N}_\lambda]\left(\bar{\Omega}\right) - \mathfrak{I}^{\alpha,\rho}_{0+,1}[(I-Q)\mathrm{N}_\lambda]\left(\bar{\Omega}\right), \tag{4.115} $$

is equicontinuous.

φ_q is uniformly continuous on $[-K,K]$, then $\left({}^{c}\mathcal{D}^{\alpha,\rho}_{0+}\right)R(u,\lambda)$ is equicontinuous.Thus, R is compact.

Step 2. Equations (**i**) and (**ii**) of Definition 17 are satisfied.

Step 2.1 By $Q^2 = Q$, For $u \in \bar{\Omega}$, $Q(I-Q)\mathrm{N}_\lambda(u) = Q\mathrm{N}_\lambda(u) - Q^2\mathrm{N}_\lambda(u) = 0$ so $(I-Q)\mathrm{N}_\lambda(u) \in \ker Q = \operatorname{Im} M$. On the other hand, $\forall y \in \operatorname{Im} M$, clearly $Qy = 0$, so

$y = y - Qy = (I - Q)\, y$ then $y \in (I - Q)\, Y$. Namely

$$(I - Q)\, \mathrm{N}_\lambda \left(\bar{\Omega}\right) \subset \operatorname{Im} M \subset (I - Q)\, Y. \tag{4.116}$$

Hence (**i**) of Definition 17 holds.

Step 2.2 It is easy to verify that: $Q\mathrm{N}_\lambda u = 0,\ \ \lambda \in (0,1) \Longleftrightarrow Q\mathrm{N}u = 0, \forall u \in \Omega$. Since $Q\mathrm{N}_\lambda u = \lambda Q\mathrm{N}u$. Hence (**ii**) of Definition 17 holds too.

Step 2.3 $\forall u \in \sum_\lambda$, we have $\mathrm{M}u = \mathrm{N}_\lambda u \in \ker Q = \operatorname{Im} M$ So $Q\mathrm{N}_\lambda u = 0$

Thus we obtain

$$\begin{aligned} R\,(u,\lambda)\,(t) &= \mathfrak{I}^{\beta,\rho}_{0+,t}\left[\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,\tau}\left[(I-Q)\,\mathrm{N}_\lambda u\right] - \mathfrak{I}^{\alpha,\rho}_{0+,1}\left[(I-Q)\,\mathrm{N}_\lambda u\right]\right)\right] \\ &= \mathfrak{I}^{\beta,\rho}_{0+,t}\left[\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,\tau}\left[\mathrm{N}_\lambda u\right] - \mathfrak{I}^{\alpha,\rho}_{0+,1}\left[\mathrm{N}_\lambda u\right]\right)\right] \\ &= \mathfrak{I}^{\beta,\rho}_{0+,t}\left[\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,\tau}\left[\mathrm{M}u\right] - \mathfrak{I}^{\alpha,\rho}_{0+,1}\left[\mathrm{M}u\right]\right)\right] \\ &= \mathfrak{I}^{\beta,\rho}_{0+,t}\left[\varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[u\right]\right)\right] \\ &= u\,(t) - u\,(0) \\ &= \left[(I-P)\,u\right](t)\,. \end{aligned} \tag{4.117}$$

Then

$$R\,(u,\lambda)\,(t) = \left[(I-P)\,u\right](t)\,, \tag{4.118}$$

which implies that

$$R(.,\lambda)|_{\sum_\lambda} = (I-P)|_{\sum_\lambda}\,. \tag{4.119}$$

When $\lambda = 0$, we have $\mathrm{N}_\lambda u\,(t) = 0$, which yields

$$R(u,0)\,(t) = 0,\ \ \forall u \in \bar{\Omega}. \tag{4.120}$$

Hence (**iii**) of Definition 17 holds

Step 2.4. $\forall u \in \bar{\Omega}$, we have

$$\begin{aligned} \mathrm{M}(Px + R(u,\lambda))\,(t) &={}^{c} \mathcal{D}^{\alpha,\rho}_{0+,t}\left[\varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[Pu + R(u,\lambda)\right]\right)\right] \\ &={}^{c} \mathcal{D}^{\alpha,\rho}_{0+,t}\left[\varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}\left[(R(u,\lambda)\right]\right)\right] \\ &={}^{c} \mathcal{D}^{\alpha,\rho}_{0+,t}\left[\varphi_p\left(\varphi_q\left[\mathfrak{I}^{\alpha,\rho}_{0+,s}\left[(I-Q)\,\mathrm{N}_\lambda u\right] - \mathfrak{I}^{\alpha,\rho}_{0+,1}\left[(I-Q)\,\mathrm{N}_\lambda u\right]\right]\right)\right] \\ &= \left({}^{c}\mathcal{D}^{\alpha,\rho}_{0+,t}\right)\left[\mathfrak{I}^{\alpha,\rho}_{0+,s}\left[(I-Q)\,\mathrm{N}_\lambda u\right] - \mathfrak{I}^{\alpha,\rho}_{0+,1}\left[(I-Q)\,\mathrm{N}_\lambda u\right]\right] \\ &= (I-Q)\mathrm{N}_\lambda u\,(t)\,, \end{aligned} \tag{4.121}$$

which implies that (**iv**) of Definition 17 holds. Hence, N_λ is M-compact in $\bar{\Omega}$. ■

Remark 6 *By the definition of Q we can easily obtain that Q is not a projector but satisfies $Q(I-Q)y = Q(y-Qy) = 0, y \in Y$.*

Lemma 23 *Suppose (A_1), (A_4) hold; then the set*

$$\Omega_1 = \{u \in \operatorname{dom}(\mathrm{M}) \setminus \ker \mathrm{M} : \mathrm{M}u = \mathrm{N}_\lambda u, \lambda \in (0,1)\}, \tag{4.122}$$

is bounded.

Proof. Take $u \in \Omega_1$, then $\mathrm{M}u = \mathrm{N}_\lambda u$, and $\mathrm{N}_\lambda u \in \mathrm{Im}M$. then $Q\mathrm{N}_\lambda = 0$. By (4.96), we have

$$\mathfrak{I}^{\beta,\rho}_{0+,\eta}\left[\varphi_q\left(\mathfrak{I}^{\alpha,\rho}_{0+,\tau}\left[f\left(\tau,u,-{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right]-\mathfrak{I}^{\alpha,\rho}_{0+,1}\left[f\left(\tau,u,-{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right]\right)\right]=0. \tag{4.123}$$

From (A_4) $\exists A$ and $\exists \zeta \in [0,1]$ such that $u(\zeta) \leq A$. Moreover,

$$u(t)-u(\zeta)=\mathfrak{I}^{\beta,\rho}_{0+,1}\left[{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right]-\mathfrak{I}^{\beta,\rho}_{0+,\zeta}\left[{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right], \tag{4.124}$$

then

$$\begin{aligned}|u(t)| &\leq |u(\zeta)|+\left|\mathfrak{I}^{\beta,\rho}_{0+,1}\left[{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right]-\mathfrak{I}^{\beta,\rho}_{0+,\zeta}\left[{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right]\right|\\ &\leq A+\frac{2}{\rho^\beta\Gamma(\beta+1)}\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right\|_\infty.\end{aligned}$$

So, we have

$$\|u\|_\infty \leq A+\frac{2}{\rho^\beta\Gamma(\beta+1)}\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right\|_\infty, \tag{4.125}$$

since $\mathrm{M}u=\mathrm{N}_\lambda u$ and ${}^c\mathcal{D}^{\beta,\rho}_{0+,\tau}[u](1)=0$

$$\mathfrak{I}^{\alpha,\rho}_{0+,t}[\mathrm{M}u]-\mathfrak{I}^{\alpha,\rho}_{0+,1}[\mathrm{M}u]=\mathfrak{I}^{\alpha,\rho}_{0+,t}[\mathrm{N}_\lambda u]-\mathfrak{I}^{\alpha,\rho}_{0+,1}[\mathrm{N}_\lambda u]=\varphi_p\left({}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right), \tag{4.126}$$

then

$$\begin{aligned}\left|\varphi_p\left({}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right)\right| &\leq 2\lambda\|\mathrm{N}u\|\,\mathfrak{I}^{\alpha,\rho}_{0+,1}[1] \\ &\leq\left(\frac{2}{\rho^\alpha\Gamma(\alpha+1)}\right)\left[\|a\|_\infty+\|u\|_\infty^{p-1}\|b\|_\infty+\|c\|_\infty\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|_\infty^{p-1}\right].\end{aligned} \tag{4.127}$$

If $p<2$

$$\left(1-\frac{2}{\rho^\alpha\Gamma(\alpha+1)}\left[\left(\frac{2}{\rho^\beta\Gamma(\beta+1)}\right)^{p-1}\|b\|_\infty+\|c\|_\infty\right]\right)\left\|{}^c\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right\|_\infty^{p-1}\leq\left(\frac{2}{\rho^\alpha\Gamma(\alpha+1)}\right)\left[\|a\|_\infty+A^{p-1}\|b\|_\infty\right].$$

Thus from $\frac{2}{\rho^{\alpha}\Gamma(\alpha+1)}\left[\left(\frac{2}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1}\|b\|_{\infty}+\|c\|_{\infty}\right]<1$ we obtain

$$\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_{\infty}\leq T_{1} \tag{4.128}$$

and

$$\|u\|_{\infty}\leq A+\frac{2T_{1}}{\rho^{\beta}\Gamma(\beta+1)}. \tag{4.129}$$

If $p\geq 2$ we obtain

$$\left(1-\frac{2}{\rho^{\alpha}\Gamma(\alpha+1)}\left[\frac{1}{2}\left(\frac{4}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1}\|b\|_{\infty}+\|c\|_{\infty}\right]\right)\left\|{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right\|_{\infty}^{p-1}\leq\left(\frac{2}{\rho^{\alpha}\Gamma(\alpha+1)}\right)\left[\|a\|_{\infty}+2^{p-2}A^{p-1}\|b\|_{\infty}\right]$$

has from $\frac{2}{\rho^{\alpha}\Gamma(\alpha+1)}\left[\frac{1}{2}\left(\frac{4}{\rho^{\beta}\Gamma(\beta+1)}\right)^{p-1}\|b\|_{\infty}+\|c\|_{\infty}\right]<1$ we obtain

$$\left\|{}^{c}\mathcal{D}_{0+t\tau}^{\beta,\rho}[u]\right\|_{\infty}\leq T_{2} \tag{4.130}$$

and

$$\|u\|_{\infty}\leq A+\frac{2T_{2}}{\rho^{\beta}\Gamma(\beta+1)}. \tag{4.131}$$

Combining (4.130) with (4.131), we have

$$\|u\|_{X}\leq\max\left\{T_{1},T_{2},A+\frac{2T_{1}}{\rho^{\beta}\Gamma(\beta+1)},A+\frac{2T_{2}}{\rho^{\beta}\Gamma(\beta+1)}\right\}=T. \tag{4.132}$$

Therefore, Ω_1 is bounded. The proof is complete. ■

Lemma 24 *Suppose (A_5) holds, then the set*

$$\Omega_{2}=\{u\in\ker\mathrm{M}:\mathrm{N}_{\lambda}u\in\operatorname{Im}\mathrm{M}\}=\{u\in\ker\mathrm{M}:Q\mathrm{N}_{\lambda}u=0\}, \tag{4.133}$$

is bounded.

Proof. For $u\in\Omega_2$, we have $u(t)=c, c\in\mathbb{R}$ and $\mathrm{N}_{\lambda}u\in\mathrm{Im}M$. Then we get $Q\mathrm{N}_{\lambda}u=0$ and

$$\mathfrak{I}_{0+,\eta}^{\beta,\rho}\left[\varphi_{q}\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}[f(\tau,c,0)]-\mathfrak{I}_{0+,1}^{\alpha,\rho}[f(\tau,c,0)]\right)\right]=0, \tag{4.134}$$

with (A_5) implies $|c|\leq D$

$$\|u\|_{X}=D. \tag{4.135}$$

Hence, Ω_2 is bounded. The proof is complete. ■

Lemma 25 *Suppose the first part of (A_2) holds; then the set*

$$\Omega_3^+ = \{u \in \ker \mathrm{M} : \lambda x + (1-\lambda)Q\mathrm{N}u = 0, \lambda \in [0,1]\}, \tag{4.136}$$

is bounded.

Proof. For $u \in \Omega_3^+$, we have $u(t) = c, c \in \mathbb{R}$ and

$$\mathrm{N}u = \mathrm{N}c = \varphi_q\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f\left(\tau, c, 0\right)\right]\right). \tag{4.137}$$

If $\lambda = 1$, we have $c = 0$. If $\lambda = 0$ and by the same analysis of lemma 23, $|c| \leq B$, then Ω_3^+ is bounded.

If $\lambda \in [(0, 1)$ then

Case 1: $cf\left(\tau, c, 0\right) > 0$. In this case $\Omega_3^+ = \{u \in \ker \mathrm{M} : \lambda x + (1-\lambda)Q\mathrm{N}u = 0\}$ then

$$\lambda c + (1-\lambda)Q\mathrm{N}c = 0,\ \forall \lambda \in [(0,1) \Longrightarrow |c| \leq B. \tag{4.138}$$

Otherwise, if $|c| > B$ then $c\mathrm{N}c > 0$ and

$$\lambda c^2 + (1-\lambda)Qc\mathrm{N}c = c\left(\lambda c + (1-\lambda)Qc\mathrm{N}c\right) > 0, \tag{4.139}$$

which contradicts to (4.138). Therefore, Ω_3^+ is bounded. From the above Case 1 and Case 2, we can know that Ω_3^+ is bounded. The proof is complete. ■

Remark 7 *If the second part of (A_2) holds, then the set*

$$\Omega_3^- = \{u \in \ker \mathrm{M} : -\lambda x + (1-\lambda)Q\mathrm{N}u = 0, \lambda \in [0,1]\}, \tag{4.140}$$

is bounded.

Theorem 6 *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Assume that (A_1- A_5) hold, then Then BVP (4.85-4.86) has at least one solution, provided that*

$$\left(\frac{2}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}\right)\left[\frac{1}{2}\left(\frac{4}{\rho^{\beta}\Gamma\left(\beta+1\right)}\right)^{p-1}\|b\|_{\infty} + \|c\|_{\infty}\right] < 1,\ \ \textit{if}\ p \geq 2,$$

or

$$\left(\frac{2}{\rho^{\alpha}\Gamma\left(\alpha+1\right)}\right)\left[\left(\frac{2}{\rho^{\beta}\Gamma\left(\beta+1\right)}\right)^{p-1}\|b\|_{\infty} + \|c\|_{\infty}\right] < 1,\ \ \textit{if}\ 1 < p < 2.$$

Proof of Theorem 6. Set

$$\Omega = \{u \in X : \|u\|_X < \max\{K, B\} + 1\}. \tag{4.141}$$

Obviously, $\Omega_1 \cup \Omega_2 \cup \Omega_3^+ \cup \Omega_3^- \subset \Omega, . \forall (u, \lambda) \in \partial\Omega \times (0, 1)$ It follows from Lemmas 21 and 22 that M (defined by (4.89) is quasi-linear operator and N_λ (defined by (4.93) is M-compact on $\bar{\Omega}$.

By Lemmas 23 and 24, we get that the following two conditions are satisfied

(i) $\mathrm{M}u \neq \mathrm{N}_\lambda u \ \forall (u, y) \in [(\mathrm{DomM}/\ker \mathrm{M}) \cap \partial\Omega] \times (0, 1)$

(ii) $\mathrm{N}_\lambda u \notin \mathrm{Im}M, \forall u \in \ker L \cap \partial\Omega$

It remains verifying the condition ($\mathbf{C}_3$) of Theorem 4. In order to do that, Take the homotopy

$$\mathrm{H}(u, \lambda) = \lambda x + (1 - \lambda) JQ\mathrm{N}u, \ u \in \bar{\Omega} \cap \ker \mathrm{M}, \ \lambda \in [0, 1], \tag{4.142}$$

where $J : \mathrm{Im}\, Q \longrightarrow \ker \mathrm{M}$ is a homeomorphism with $J(c) = c, \ c \in \mathbb{R}, \ \forall u \in \partial\Omega \cap \ker \mathrm{M}$ we have $u = c_0, \ |c_0| = \bar{B} > B_1$, then

$$\mathrm{H}(u, \lambda) = \lambda c_0 + (1 - \lambda) Q(-f)(c_0), \ c_0 \in \bar{\Omega} \cap \ker \mathrm{M}, \ \lambda \in [0, 1]. \tag{4.143}$$

If $\lambda = 1$, then $\mathrm{H}(u, \lambda) = c_0 \neq 0$.

If $\lambda \neq 1$, suppose $\mathrm{H}(u, \lambda) = 0$, then $Q(-f) = \dfrac{-\lambda c_0}{1 - \lambda}$. From $(I - Q)(-f) \in \mathrm{Im}\, \mathrm{M}$,

By using assumption (A_2) $cf(t, u, c) < 0, \forall t \in [0, 1]$, whenever.$|c| \geq B_1$, it is a contradiction, so $\mathrm{H}(u, \lambda) \neq 0$, $\forall u \in \partial\Omega \cap \ker \mathrm{M}$.

Thus, by the homotopy property of degree, we have

$$\begin{aligned} \deg(Q\mathrm{N}|\ker \mathrm{M}, \Omega \cap \ker \mathrm{M}, 0) &= \deg(\mathrm{H}(\cdot, 0), \Omega \cap \ker \mathrm{M}, 0) \\ &= \deg(\mathrm{H}(\cdot, 1), \Omega \cap \ker \mathrm{M}, 0) \\ &= \deg(\pm\mathfrak{I}, \Omega \cap \ker \mathrm{M}, 0) \neq 0. \end{aligned} \tag{4.144}$$

So that, the condition ($\mathbf{C}_3$) of Theorem 4 is satisfied.

Consequently, by using Theorem 4 , the operator equation $\mathrm{M}u = \mathrm{N}u$ has at least one solution in $\mathrm{Dom}\ \mathrm{M} \cap \Omega$. Namely, BVP (4.85-4.86) has at least one solution in X. The proof is complete. ■

Remark 8 *If denote the homotopy*

$$\mathrm{H}(u,\lambda) = -\lambda x + (1-\lambda) JQ\mathrm{N}u, \;\; u \in \bar{\Omega} \cap \ker \mathrm{M}, \; \lambda \in [0,1]. \tag{4.145}$$

By using assumption (A_2) $cf(t,u,c) > 0, \forall t \in [0,1]$, whenever.$|c| \geq B_1$, we also have $\mathrm{H}(u,\lambda) \neq 0$, $\forall u \in \partial\Omega \cap \ker \mathrm{M}$.

4.3.3 Example

Example 2 *Consider the following BVP for conformable fractional p-Laplacian equation*

$${}^{c}\mathcal{D}^{\alpha,\rho}_{0+,t}\left[\varphi_p\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,\tau}[u]\right)\right] = \frac{t}{8} + \frac{1}{16}u(t)^{\frac{1}{3}} + \frac{1}{32}\sin^2\left({}^{c}\mathcal{D}^{\beta,\rho}_{0+,t}[u]\right), \; t \in (0,1). \tag{4.146}$$

we have

$$|f(t,u,y)| = \left|\frac{t}{8} + \frac{1}{16}u(t)^{\frac{1}{3}} + \frac{1}{32}\sin^2 y(t)\right| \leq \frac{5}{32} + \frac{1}{16}|u(t)|^{\frac{1}{3}}, \tag{4.147}$$

then

$$a(t) = \frac{5}{32}, \; b(t) = \frac{1}{16}, \; c(t) = 0.$$

Corresponding to BVP (4.146-4.86), we get that $p = \frac{4}{3}$, $\alpha = 1/2, \beta = 3/4, \rho = 1/2$.

Choose $a(t) = \frac{5}{32}$, $b(t) = \frac{1}{16}$, $c(t) = 0$. By a simple calculation, we can obtain that

$$\|a\|_\infty = \frac{5}{32}, \|b\|_\infty = \left\|\frac{1}{16}\right\|_\infty, \|c\|\infty = 0.$$

and

$$\left(\frac{2}{\rho^\alpha\Gamma(\alpha+1)}\right)\left[\left(\frac{2}{\rho^\beta\Gamma(\beta+1)}\right)^{p-1}\|b\|_\infty + \|c\|_\infty\right] = 0.307 < 1.$$

(**i**) Take $D = 9$ and $|c| > 9$

$$cf(t,c,0) > 0, \;\; \forall t \in [0,1], \;\; |c| > 9. \tag{4.148}$$

If $c > 9$ then

$$cf(t,c,0) = \frac{tc}{8} + \frac{1}{16}c^{\frac{4}{3}} > 0, \;\; \forall t \in [0,1], \;\; c > 9. \tag{4.149}$$

If $c < -9$ then

$$f(t,c,0) < \frac{1}{8} + \frac{1}{16}(-9)^{\frac{1}{3}} < 0 \Longrightarrow cf(t,c,0) > 0 \; \forall t \in [0,1], \; c < -9. \tag{4.150}$$

Thus

$$cf(t,c,0)>0,\ \forall t\in[0,1],\ |c|>9.$$

From (4.149) and (4.150), we have

$$\begin{aligned} &c\left(\mathfrak{I}_{0+,\eta}^{\beta,\rho}\left[\varphi_q\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f(\tau,c,0)\right]+\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[f(\tau,c,-0)\right]\right)\right]\right) \\ &=\frac{1}{c^2}\left(\mathfrak{I}_{0+,\eta}^{\beta,\rho}\left[\varphi_q\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[cf(\tau,c,0)\right]+\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[cf(\tau,c,-0)\right]\right)\right]\right)<0. \end{aligned} \tag{4.151}$$

So, the condition (A_5) holds.

(**ii**) Take $A=16$,

If $u(t)>A$ holds for any $t\in[0,1]$, then

$$f\left(t,u,{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right)\geq\frac{1}{16}u(t)^{\frac{1}{3}}>\frac{1}{16}A^{\frac{1}{3}}>0, \tag{4.152}$$

so

$$\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]-\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]<0, \tag{4.153}$$

we obtain

$$\mathfrak{I}_{0+,\eta}^{\beta,\rho}\left[\varphi_q\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]-\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]\right)\right]<0. \tag{4.154}$$

If $u(t)<-A$ holds for any $t\in[0,1]$, then

$$f\left(t,u,{}^{c}\mathcal{D}_{0+,t}^{\beta,\rho}[u]\right)\leq\frac{5}{32}+\frac{1}{16}u(t)^{\frac{1}{3}}<\frac{5}{32}-\frac{1}{16}A^{\frac{1}{3}}<0, \tag{4.155}$$

so

$$\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]-\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]>0, \tag{4.156}$$

then

$$\mathfrak{I}_{0+,\eta}^{\beta,\rho}\left[\varphi_q\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]-\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]\right)\right]>0. \tag{4.157}$$

From (4.154) and (4.157), if $|u(t)|>A$, we have

$$\mathfrak{I}_{0+,\eta}^{\beta,\rho}\left[\varphi_q\left(\mathfrak{I}_{0+,\tau}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]-\mathfrak{I}_{0+,1}^{\alpha,\rho}\left[f\left(\tau,u(\tau),-{}^{c}\mathcal{D}_{0+,\tau}^{\beta,\rho}[u]\right)\right]\right)\right]\neq 0. \tag{4.158}$$

Thus, the condition (A_4) holds. Obviously, BVP (4.146-4.86) satisfies all assumptions of Theorem 6. Hence, BVP (4.146-4.86) has at least one solution.

Remark 9 *The contents of this chapter is communicated in the form of two paper as mentioned below:*

- p -Laplacian two -point local boundary value problems with fractional conformable derivative in the sense of Caputo, communicated for publication.

- p -Laplacian three -point local boundary value problems with fractional conformable derivative in the sense of Caputo, communicated for publication.

Chapter 5

NONLINEAR SINGULAR p-LAPLACIAN FOUR- POINT BOUNDARY VALUE PROBLEMS WITH CONFORMABLE DERIVATIVE

Contents

5.1 Introduction

The theory of singular BVPs has become an important area of investigation in recent years (see [103] and references therein). The search for the existence of positive solutions and multiple positive solutions to nonlinear FBVPs with p-Laplacian operator by the use of techniques of nonlinear analysis, have been studied by several authors (see [43, 44]).

For example, in [33, 34], Zhang and Liu considered the following fourth-order four-point boundary value problem

$$(\varphi_p (u'' (t)))'' = f (t, u (t)), \ t \in (0, 1), \tag{5.1}$$

with the four-point boundary conditions

$$u (0) = 0, \ u (1) = b_1 u (\xi_1), \ u'' (0) = 0, \ u'' (1) = b_2 u'' (\xi_2). \tag{5.2}$$

Z. Bai and H. Lü [43] studied nonlocal FDEs with BVPs with p-Laplacian for existence and uniqueness of solutions and multiple solutions by using Avery-Peterson FPT for the problem

$$((\varphi_p (D_t^\alpha u)) (t))'' = -f \left(t, u (t), -\left(D_t^\beta\right) u (t)\right) \ \text{in} \ (0, 1), \tag{5.3}$$

where D_t^α, D_t^β represent Caputo derivative sense.

In recent years, some results have been obtained under different assumptions on f [43, 44, 50, 51, 52], as for FBVPs, in [104], J. Wang and H. Xiang have investigated the following the fractional boundary value problem

$$\left(D_t^\beta\right) (\varphi_p (D_t^\alpha u)) (t) = f (t, u (t)) \ \text{in} \ (0, 1), \tag{5.4}$$

and boundary conditions

$$u (0) = 0, \ u (1) = b_1 u (\xi_1), \ D_t^\alpha u (0) = 0, \ D_t^\alpha u (1) = b_2 D_t^\alpha u (\xi_2), \tag{5.5}$$

where D_t^α, D_t^β are Riemann-Liouville fractional operators with $1 < \alpha, \beta \leq 2$, (see also [105]).

Motivated by the above-mentioned works, we investigate the following BVPs of conformable nonlinear differential equations with p-Laplacian operator and a nonlinear term dependent on the fractional derivative of the unknown function

$$\mathbf{T}_{0+}^\beta \left(\varphi_p \left(\mathbf{T}_{0+}^\alpha u\right)\right) (t) = f \left(t, u (t), -\mathbf{T}_{0+}^\alpha u (t)\right), \ t \in (0, 1), \tag{5.6}$$

with the four-point boundary conditions

$$u(0) = 0,\ u(1) = b_1 u(\xi_1),\ \mathbf{T}^{\alpha}_{0+} u(0) = 0,\ \mathbf{T}^{\alpha}_{0+} u(1) = b_2 \mathbf{T}^{\alpha}_{0+} u(\xi_2), \tag{5.7}$$

where $\mathbf{T}^{\beta}_{0+}$ and $\mathbf{T}^{\alpha}_{0+}$ are the conformable derivatives with $1 < \alpha, \beta \leq 2, 1 < \alpha \leq \alpha+\beta-1,\ 0 \leq b_1, b_2 \leq 1,\ 0 < \xi_1, \xi_2 < 1$.

In the special case $p = \alpha = \beta = 2$ and $b_1 = b_2 = 0$, the problem (E_2-C_3) becomes the two point BVPs of fourth order ODE. When f is continuous, problem is nonsingular, the existence and uniqueness of positive solutions in this case have been studied by papers [42, 100]. The theorems we present include and extend some previous results.

The remainder of the thesis is organized as follows: Firstly, we present some necessary definitions and Lemmas that are needed in the subsequent sections. In Section 3, we construct the Green functions for the homogeneous CBVP corresponding to (E_2-C_3) and estimate the bounds for the Green functions. One of the difficulties here is that the corresponding Green's function is singular at $s = 0$. By applying the upper and lower solutions method associated with the Krasnosel'skii's fixed point theorem in a cone, the existence of at least one positive solution are established is dealt with in section 4. Furthermore, example is presented to illustrate the main results. In the final subsection of the this section, we look at the question as to how the solution u varies when we change the order of the conformable differential operator or the initial values and the dependence on parameters of nonlinear term f is also established.

It is well known that a powerful tool for proving existence results for nonlinear problems is the upper and lower solution method [51, 52, 106].

Our assumptions on the nonlinearity f will be the following:

A. The function f fulfill a Lipschitz condition with respect to the second and third variables,

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|, \tag{5.8}$$

such that $L_1, L_2 > 0$, (f is locally Lipschitz in $(0, 1) \times (0, +\infty) \times (-\infty, 0]$).

For convenience, we suppose that the following hypotheses are satisfied:

H. $f : ((0,1) \times (0,+\infty) \times \mathbb{R} \longrightarrow \mathbb{R}^+)$ is continuous. There exist constants $\lambda_i, \mu_i,\ i = 1, 2,$ $0 < \lambda_1 \leq \mu_1,\ 0 \leq \lambda_2 \leq \mu_2 < p-1, \lambda_1 + \lambda_2 > p-1$, such that

$$\begin{aligned} \sigma^{\mu_1} f(t,u,v) \leq f(t,\sigma u,v) \leq \sigma^{\lambda_1} f(t,u,v) \quad \text{if} \quad 0 < \sigma \leq 1, \\ \sigma^{\mu_2} f(t,u,v) \leq f(t,u,\sigma y) \leq \sigma^{\lambda_2} f(t,u,v) \quad \text{if} \quad 0 < \sigma \leq 1. \end{aligned} \tag{5.9}$$

It can be easily seen that $f(t,u,v)$ is non-decreasing with respect to u, v, and (5.9) are equivalent to

$$\begin{aligned} \sigma^{\lambda_1} f(t,u,v) \leq f(t,\sigma u,v) \leq \sigma^{\mu_1} f(t,u,v) \quad \text{if} \quad \sigma \geq 1, \\ \sigma^{\lambda_2} f(t,u,v) \leq f(t,u,\sigma y) \leq \sigma^{\mu_2} f(t,u,v) \quad \text{if} \quad \sigma \geq 1. \end{aligned} \tag{5.10}$$

for $(t,u,v) \in (0,1) \times [0,+\infty) \times [0,+\infty)$. From (5.9-5.10), we have

$$f(t,u_1,v_1) \leq f(t,u_2,v_2) \ \text{ for } u_1 \leq u_2,\ t \in (0,1)\,. \tag{5.11}$$

We say $u_1 \leq u_2$ if $u_1(t) \leq u_2(t)$ for $t \in [0,1]$.

Now we present the Green's function for boundary value problem of fractional differential equation.

5.2 Construction of Green's function of problem (E_2-C_3)

In this section, we obtain Green's function corresponding to the FDEs (E_2) with $1 < \alpha, \beta \leq 2$ subject to four-point boundary conditions (C_3) and estimate bounds for Green's function that will be used to prove our main theorems.

To study the nonlinear problem (E_2-C_3), we first consider the associated linear problem and obtain its solution.

Lemma 26 *Suppose $v(t) \geq 0, \alpha \in (1,2]$. The corresponding Green's function for the problem*

$$\begin{cases} \mathbf{T}^{\alpha}_{0+} u(t) + v(t) = 0, \\ u(0) = 0,\ u(1) = b_1 u(\xi_1)\,, \end{cases} \quad t \in (0,1)\,, \tag{5.12}$$

is given by

$$G_\alpha(t,s) = \mathcal{G}_\alpha(t,s) + \frac{b_1 t}{1-b_1\xi_1}\mathcal{G}_\alpha(\xi_1,s), \tag{5.13}$$

where

$$\mathcal{G}_\alpha(t,s) = \begin{cases} (1-t)s^{\alpha-1} & \text{if } \ 0 \le s \le t \le 1, \\ t(1-s)s^{\alpha-2} & \text{if } \ 0 \le t \le s \le 1, \end{cases} \tag{5.14}$$

where we assume the parameters satisfy $0 \le b_1 \le 1$, $0 < \xi_1 < 1$.

Moreover, if v *is not identically* 0 *on* $(0,1)$*, then* u *is concave, decreasing,* $u(t) > 0$ *and* $u(t) \ge \psi_1(t)\|u\|_0$ *for* $t \in [0,1]$*, where* $\psi_1(t) = t\left(\frac{b_1(1-\xi_1)}{1-b_1\xi_1}t + 1\right)$.

Proof. We will show that

$$u(t) = \int_0^1 G_\alpha(t,s)v(s)\mathrm{d}s, \tag{5.15}$$

for G_α given by (5.13), is a unique solution to the linear BVP (E_2-C_3).

By applying Lemma 5, we may reduce (E_2) to an equivalent integral equation

$$u(t) = -\mathbf{I}^\alpha v(t) + c_0 + c_1 t, \ \ c_0, c_1 \in \mathbb{R}. \tag{5.16}$$

From $u(0) = 0$ and (5.16), we have $c_0 = 0$. Consequently the general solution of (E_2) is

$$u(t) = -\mathbf{I}^\alpha_{0+} v(t) + c_1 t = -\int_0^t (t-s)\, s^{\alpha-2} v(s)\,\mathrm{d}s + c_1 t. \tag{5.17}$$

By (5.17), one has

$$u(1) = -\int_0^1 (1-s)\, s^{\alpha-2} v(s)\,\mathrm{d}s + c_1, \ \ u(\xi_1) = -\int_0^{\xi_1} (\xi_1 - s)\, s^{\alpha-2} v(s)\,\mathrm{d}s + c_1\xi_1.$$

And from $u(1) = b_1 u(\xi_1)$, then we have

$$c_1 = \frac{1}{1-b_1\xi_1}\left[\int_0^1 (1-s)\, s^{\alpha-2} v(s)\,\mathrm{d}s - b_1\int_0^{\xi_1} (\xi_1 - s)\, s^{\alpha-2} v(s)\,\mathrm{d}s\right].$$

So, the unique solution of problem (E_2-C_3) is

$$\begin{aligned}
u(t) &= -\int_0^t (t-s)\, v(s) \frac{ds}{s^{2-\alpha}} + \frac{t}{1-b_1\xi_1}\left[\int_0^1 (1-s)\, v(s)\frac{ds}{s^{2-\alpha}} - b_1\int_0^{\xi_1} (\xi_1 - s)\, v(s)\frac{ds}{s^{2-\alpha}}\right] \\
&= -\int_0^t (t-s)\, v(s) \frac{ds}{s^{2-\alpha}} + \frac{1}{1-b_1\xi_1}\int_0^1 t(1-s)\, v(s)\frac{ds}{s^{2-\alpha}} - b_1\int_0^{\xi_1} (\xi_1 - s)\, v(s)\frac{ds}{s^{2-\alpha}} \\
&= -\int_0^t (t-s)\, v(s) \frac{ds}{s^{2-\alpha}} + \frac{1}{1-b_1\xi_1}\left(\int_0^t t(1-s)\, v(s)\frac{ds}{s^{2-\alpha}} + \int_t^1 t(1-s)\, v(s)\frac{ds}{s^{2-\alpha}}\right) \\
&+ \frac{b_1 t}{1-b_1\xi_1}\left[\int_0^{\xi_1} (\xi_1(1-s) - (\xi_1 - s))\, v(s)\frac{ds}{s^{2-\alpha}} + \int_{\xi_1}^1 \xi_1 (1-s)\, v(s)\frac{ds}{s^{2-\alpha}}\right] \\
&= \int_0^t \frac{(1-t)\, v(s)\, ds}{s^{1-\alpha}} + \int_t^1 \frac{t(1-s)\, v(s)\, ds}{s^{2-\alpha}} \\
&+ \frac{b_1 t}{1-b_1\xi_1}\left[\int_0^{\xi_1} \frac{(1-\xi_1)\, v(s)\, ds}{s^{1-\alpha}} + \int_{\xi_1}^1 \frac{\xi_1(1-s)\, v(s)\, ds}{s^{2-\alpha}}\right] \\
&= \int_0^1 \left[\mathcal{G}_\alpha(t,s) + \frac{b_1 t}{1-b_1\xi_1}\mathcal{G}_\alpha(\xi_1,s)\right] v(s)\, ds \\
&= \int_0^1 G_\alpha(t,s) v(s)\, ds,
\end{aligned}$$

where $G_\alpha(t,s)$ is defined in (5.13).

For $t \in [0,1]$ we have $\mathcal{G}_\alpha(t,s) \geq 0$ and $G_\alpha(t,s) \geq 0$ for $s \in (0,1)$, hence, when v is not identically 0 on $[0,1]$, it follows that $u(t) > 0$.

Furthermore,

$$G_\alpha(t,s) = \begin{cases} bt(1-s)\, s^{\alpha-2} & \text{if } \ 0 \leq t \leq s \leq 1, \\ (ct+1)\, s^{\alpha-1} & \text{if } \ 0 \leq s \leq t \leq 1, \end{cases} \tag{5.18}$$

where

$$a = \frac{b_1}{1-b_1\xi_1}, \ b = 1 + a\xi_1, \ c = a(1-\xi_1) - 1, \tag{5.19}$$

when $0 \leq s \leq t \leq 1$ then

$$G_\alpha(t,s) = (ct+1)\, s^{\alpha-1} \geq (ct+1)(cs+1)\, s^{\alpha-1} = (ct+1)\, G_\alpha(s) \ \text{ with } 0 < (ct+1) < 1, \tag{5.20}$$

and when $0 \leq t \leq s \leq 1$ then

$$G_\alpha(t,s) = bt(1-s)\, s^{\alpha-2} \geq bts(1-s)\, s^{\alpha-2} = tG_\alpha(s), \ \ s \in (0,1). \tag{5.21}$$

From (5.20) and (5.21), we get

$$G_\alpha(t,s) \geq \psi_1(t)\, G_\alpha(s), \text{with } \psi_1(t) = t(ct+1), \tag{5.22}$$

and

$$u(t) = \int_0^1 G_\alpha(t,s)v(s)\mathrm{d}s \geq \psi_1(t) \int_0^1 G_\alpha(s)\, v(s)\mathrm{d}s \geq \psi_1(t) \|u\|_0\,, \tag{5.23}$$

where

$$\|u\|_0 = \max|u(t) : t \in [0,1]| \leq \int_0^1 G_\alpha(s)\, v(s)\mathrm{d}s, \tag{5.24}$$

with $G_\alpha(.) \equiv G_\alpha(.,.)$ as in (5.13). The proof is completed. ■

Lemma 27 *Suppose $U(t) \geq 0,\ 1 < \alpha, \beta \leq 2, 0 \leq b_1, b_2 \leq 1$ and $0 < \xi_1, \xi_2 < 1$. Then the unique solution of the following CBVP with the p-Laplacian operator*

$$\begin{cases} \mathbf{T}^\beta_{0+}(\varphi_p\left(\mathbf{T}^\alpha_{0+}u\right)(t)) + U(t) = 0, & t \in (0,1)\,, \\ \mathbf{T}^\alpha_{0+}u(0) = 0, \mathbf{T}^\alpha_{0+}u(1) = b_2\mathbf{T}^\alpha_{0+}u(\xi_2) \end{cases} \tag{5.25}$$

is given by

$$\varphi_p\left(\mathbf{T}^\alpha_{0+}u\right)(t) = \int_0^1 G_\beta(t,s)\, U(s)\, \mathrm{d}s,$$

where

$$G_\beta(t,s) = \mathcal{G}_\beta(t,s) + \frac{b_0 t}{1 - b_0\xi_2}\mathcal{G}_\beta(\xi_2, s) \tag{5.26}$$

and $b_0 = b_2^{p-1}$ and $\mathcal{G}_\beta(t,s)$ is defined in (5.14) with α replaced by β.

Moreover, if U is not identically 0 on $(0,1)$, then u is concave, decreasing, $u(t) > 0$ and $u(t) \geq \psi_2(t)\|u\|_0$ for $t \in [0,1]$, where $\psi_2(t) = t\left(\frac{b_0(1-\xi_2)}{1-b_0\xi_2}t + 1\right)$.

Proof. By a similar argument in the proof of Lemma 27, we can get Lemma 26. ■

Lemma 28 *Suppose $u(t) \geq 0,\ 1 < \alpha, \beta \leq 2, 0 \leq b_1, b_2 \leq 1$ and $0 < \xi_1, \xi_2 < 1$. Then the unique solution of the following CBVP with the p-Laplacian operator (E_2-C_3) given by*

$$u(t) = \int_0^1 G_\alpha(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, u(\tau), -\mathbf{T}^\alpha_{0+}u(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \tag{5.27}$$

where $G_\alpha(t,s)$ and $G_\beta(t,s)$ defined by (5.13) and (5.26) respectively.

Remark 10 *From the expression of (5.15) and (5.27), we can see that if all the conditions in Lemmas 26, 27 and 28 are satisfied, the solution is a $C^2[0,1]$ solution of the CBVP (E_2-C_3). Furthermore, if we denote $-\varphi_p\left(\mathbf{T}^\alpha_{0+}u\right)(t) = U(t)$, there holds $U \geq \psi_1(t)\|U\|_0$ for $t \in [0,1]$, where $\psi_1(t)$ is defined in Lemma 26.*

The properties of $\mathcal{G}_\alpha$ and G_α are collected in the following lemma.

Lemma 29 *Let $1 < \alpha \leq 2, 0 \leq b_1 \leq 1$ and $0 < \xi_1 < 1$. Then*

(i) *Let $\mathcal{G}_\alpha$ be as in (5.14). Then $\mathcal{G}_\alpha(0,s) = 0 = \mathcal{G}_\alpha(1,s)$ for $s \in [0,1]$.*

$$\mathcal{G}_\alpha(t,s) \leq \mathcal{G}_\alpha(s) \ \text{ or } \ \mathcal{G}_\alpha(t) \ \text{ for all } t,s \in [0,1] \tag{5.28}$$

and if $\delta \in \left(0, \frac{1}{2}\right)$ then

$$\min\left\{\mathcal{G}_\alpha(t,s) : \delta \leq t \leq 1-\delta\right\} \geq \mathcal{G}_\alpha(1-\delta)\,\mathcal{G}_\alpha(s), \tag{5.29}$$

with $\mathcal{G}_\alpha(.) \equiv \mathcal{G}_\alpha(.,.)$ is defined in (5.14).

(ii) *Function G_α defined by (5.13) is continuous on $[0,1] \times [0,1]$ satisfying*

(a) *For all $t,s \in [0,1]$, $G_\alpha(t,s) \geq 0$ and*

$$\phi_1(s)\,t \leq G_\alpha(t,s) \leq \phi_2(s)\,t, \tag{5.30}$$

where

$$\phi_1(s) = \frac{b_1 \mathcal{G}_\alpha(\xi_1,s)}{1-b_1\xi_1} \ \text{ and } \ \phi_2(s) = s^{\alpha-2} + \phi_1(s). \tag{5.31}$$

(b) *For all $t,s \in [0,1]$,*

$$G_\alpha(t)\,G_\alpha(s) \leq G_\alpha(t,s) \leq G_\alpha(s) \ \text{ or } \ G_\alpha(t) \tag{5.32}$$

and

$$G_\alpha(t,s) \leq \left(1 + \frac{b_1 t}{1-b_1\xi_1}\right)\mathcal{G}_\alpha(s) \leq \left(\frac{1-b_1(1-\xi_1)}{1-b_1\xi_1}\right)\mathcal{G}_\alpha(s). \tag{5.33}$$

(c) *If $\delta \in \left(0, \frac{1}{2}\right)$ then*

$$\min\left\{G_\alpha(t,s) : \delta \leq t \leq 1-\delta\right\} \geq G_\alpha(1-\delta)\,G_\alpha(s). \tag{5.34}$$

Proof. (i) Observing the expression of $\mathcal{G}_\alpha(t,s)$, it is clear that $\mathcal{G}_\alpha(t,s) > 0$ for $t,s \in (0,1)$, with

$$\mathcal{G}_\alpha(t,s) = \begin{cases} \mathcal{G}^1_\alpha(t,s) & \text{if } \ 0 \leq s \leq t \leq 1, \\ \mathcal{G}^2_\alpha(t,s) & \text{if } \ 0 \leq t \leq s \leq 1. \end{cases} \tag{5.35}$$

Next, for given $s \in (0,1)$ we consider the partial derivative of $\mathcal{G}_\alpha(t,s)$ with respect to t,

$$\partial_t \mathcal{G}_\alpha(t,s) = \begin{cases} -s^{\alpha-1} & \text{if } \ 0 \le s \le t \le 1, \\ (1-s)s^{\alpha-2} & \text{if } \ 0 \le t \le s \le 1. \end{cases} \tag{5.36}$$

This shows that $\mathcal{G}_\alpha(t,s)$ is decreasing with respect to t for $s \le t$, and increasing for $t \le s$. So,

$$\mathcal{G}_\alpha(t,s) \le \mathcal{G}_\alpha(s) \ \text{ or } \mathcal{G}_\alpha(t) \ \text{ for all } t,s \in (0,1)$$

and if $\delta \in \left(0, \frac{1}{2}\right)$ then it is easily to see that

$$\min\{\mathcal{G}_\alpha(t,s) : \delta \le t \le 1-\delta\} = \begin{cases} \mathcal{G}^1_\alpha(1-\delta,s), & \text{if } \ s \in [0,\delta], \\ \min\{\mathcal{G}^1_\alpha(1-\delta,s), \mathcal{G}^2_\alpha(\delta,s)\}, & \text{if } \ s \in [\delta, 1-\delta], \\ \mathcal{G}^2_\alpha(\delta,s), & \text{if } \ s \in [1-\delta, 1], \end{cases} \tag{5.37}$$

or

$$\min_{t\in[\delta,1-\delta]} \left\{\mathcal{G}_\alpha(t,s) : \delta \in \left(0,\tfrac{1}{2}\right)\right\} = \begin{cases} \mathcal{G}^1_\alpha(1-\delta,s), & \text{if } \ s \in [0,\theta], \\ \mathcal{G}^2_\alpha(\delta,s), & \text{if } \ s \in [\theta,1], \end{cases} \tag{5.38}$$

with $\theta \in [\delta, 1-\delta]$ is a solution of the equation $\mathcal{G}_\alpha(1-\delta,\theta) = 0$.

Consequently,

$$\min\{\mathcal{G}_\alpha(t,s) : \delta \le t \le 1-\delta\} \ge \mathcal{G}_\alpha(1-\delta)\,\mathcal{G}_\alpha(s).$$

(**ii**) It is easy to verify properties (**ii-a**).

(**ii-b$_1$**) If $\mathcal{G}_\alpha(t,s) = t(1-s)s^{\alpha-2}$ for all $t \in [0,s]$, then from (5.18) and (5.19), we have

$$G^1_\alpha(t,s) = bt\,(1-s)\,s^{\alpha-2}, \ \ s \in (0,1). \tag{5.39}$$

Continuity of G^1_α clearly follows from the definition of G^1_α. We start by differentiation $G^1_\alpha(t,s)$ with respect to $s \in [t,1]$ for every fixed $t \in (0,1)$, we can get

$$\partial_s G^1_\alpha(t,s) = (1 + a\xi_1)\,ts^{\alpha-3}\left((1-\alpha)\,s + (\alpha-2)\right).$$

This together with the fact that $G^1_\alpha(t,1) = 0$ implies that $G^1_\alpha(t,s) < 0$.

By fixing an arbitrary $s \in (0,1)$. Differentiating $G^1_\alpha(t,s)$ with respect to t, we get

$$\partial_t G^1_\alpha(t,s) = (1 + a\xi_1)\,ts^{\alpha-3}\left((1-\alpha)\,s + (\alpha-2)\right) > 0, \ G^1_\alpha(0,s) = 0.$$

Hence $G^1_\alpha(t,s)$ has maximum at point $t=s$,.we get

$$G^1_\alpha(t,s) \leq G^1_\alpha(t,t) \text{ or } G^1_\alpha(t,s) \leq G^1_\alpha(s,s), \tag{5.40}$$

also we observe that

$$G^1_\alpha(t,s) = bt\,(1-s)\,s^{\alpha-2} \geq bts\,(1-s)\,s^{\alpha-2} = tG(s,s), \text{ for all } t \leq s, \ s \in [0,1]. \tag{5.41}$$

To prove **(ii-c$_2$)** if $\mathcal{G}_\alpha(t,s) = (1-t)s^{\alpha-1}$ for all $s \in [0,t]$, then we have

$$G^2_\alpha(t,s) = (ct+1)\,s^{\alpha-1}, \ 0 < (ct+1) < 1, \ s \leq t. \tag{5.42}$$

In an entirely similar manner to **(ii-c$_1$)**, we get

$$\partial_s G^2_\alpha(t,s) = (\alpha-1)\,(ct+1)\,s^{\alpha-2} > 0, \ G^2_\alpha(t,0) = 0.$$

The other cases can be deal similarly. Now,

$$\partial_t G^2_\alpha(t,s) = (\alpha-1)\,(ct+1)\,s^{\alpha-2} < 0, \ G^2_\alpha(1,s) > 0,$$

we deduce that

$$G^2_\alpha(t,s) \leq G^2_\alpha(t,t) \text{ or } G^2_\alpha(t,s) \leq G^2_\alpha(s,s), \tag{5.43}$$

also we observe that

$$G^2_\alpha(t,s) = (ct+1)\,s^{\alpha-1} \geq (ct+1)\,(cs+1)\,s^{\alpha-1} = (ct+1)\,G_\alpha(s), \ 0 < (ct+1) < 1. \tag{5.44}$$

So, from **(ii-b$_1$)** and **(ii-b$_2$)** we conclude that

$$\psi_1(t)\,G_\alpha(s) \leq G_\alpha(t,s) \leq G_\alpha(s) \text{ or } G_\alpha(t) \text{ where } \psi_1(t) = t\,(ct+1) \text{ for all } t,s \in [0,1]. \tag{5.45}$$

To prove **(c),** if $\delta \in \left(0, \frac{1}{2}\right)$ then from (**i**), it is easily to see that

$$\min_{t\in[\delta,1-\delta]} \left\{G_\alpha(t,s) : \delta \in \left(0,\tfrac{1}{2}\right)\right\} = \begin{cases} G^1_\alpha(1-\delta,s), & \text{if } s \in [0,\theta], \\ G^2_\alpha(\delta,s), & \text{if } s \in [\theta,1], \end{cases} \text{ with } \theta \in [\delta, 1-\delta].$$

Consequently,

$$\min\left\{G_\alpha(t,s) : \delta \leq t \leq 1-\delta \text{ when } \delta \in \left(0,\tfrac{1}{2}\right)\right\} \geq G_\alpha(1-\delta)\,G_\alpha(s). \tag{5.46}$$

■

Lemma 30 *Suppose that (H) holds. Let $u(t)$ be a $C^2([0,1])$ positive solution of (E_2-C_3). Then there are constants a_1 and a_2, $0 < a_1 < 1 < a_2$ such that*

$$a_1 G_\alpha(t) \leq u(t) \leq a_2 G_\alpha(t) \quad or \quad a_1 t \leq u(t) \leq a_2 t \quad for \quad t \in [0,1]. \tag{5.47}$$

Proof. Assume that $u(t)$ is a $C^2([0,1])$ positive solution of (E_2-C_3) By Lemma 26, $u(t)$ given by

$$u(t) = \int_0^1 G_\alpha(t,s) \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s, \tag{5.48}$$

or

$$u(t) = \int_0^1 \mathcal{G}_\alpha(t,s) \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s + \frac{b_1 t}{1 - b_1 \xi_1} \int_0^1 \mathcal{G}_\alpha(\xi_1, s) \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s. \tag{5.49}$$

From (5.49) and (C_3), we have

$$u(0) = \int_0^1 \mathcal{G}_\alpha(0,s) \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s = 0, \tag{5.50}$$

$$u(1) = \int_0^1 \mathcal{G}_\alpha(1,s) \left(-\mathbf{T}_{0+}^\alpha u\right) \mathrm{d}s + \tfrac{b_1}{1-b_1\xi_1} \int_0^1 \mathcal{G}_\alpha(\xi_1,s) \left(-\mathbf{T}_{0+}^\alpha u\right) \mathrm{d}s = b_1 u(\xi_1). \tag{5.51}$$

Thus, it follows from (5.50) and (5.51) that

$$\frac{b_1}{1 - b_1 \xi_1} \int_0^1 \mathcal{G}_\alpha(\xi_1, s) \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s = b_1 u(\xi_1), \; \mathcal{G}_\alpha(0,s) = \mathcal{G}_\alpha(1,s). \tag{5.52}$$

Noticing $\left(\mathbf{T}_{0+}^\alpha u(s)\right) \leq 0, \; t \in [0,1]$ and (5.52), we have

$$u(t) = \int_0^1 \mathcal{G}_\alpha(t,s) \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s + t b_1 u(\xi_1) \geq t b_1 u(\xi_1).$$

On the other hand, from (5.32) and (5.48), we have

$$u(t) = \int_0^1 G_\alpha(t,s) \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s \leq G_\alpha(t) \int_0^1 \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s.$$

Now we choose

$$a_1 < \min\{1, b_1 u(\xi_1)\} \text{ and } a_2 > \max\left\{1, \int_0^1 \left(-\mathbf{T}_{0+}^\alpha u(s)\right) \mathrm{d}s \geq 1\right\}. \tag{5.53}$$

Therefore, we get (5.47). This completes the proof. ■

From Lemmas 26 and 27, it is easy to obtain the following lemma.

Lemma 31 *Suppose that $u \in C^2[0,1]$ is a function with $\mathbf{T}_{0+}^\alpha u(t) \geq 0, \; 1 < \alpha \leq 2, u(0) \geq 0, \; \mathbf{T}_{0+}^\alpha u(0) \geq 0, \; u(1) \leq b_1 u(\xi_1)$ and $\mathbf{T}_{0+}^\alpha u(1) \leq b_2 \mathbf{T}_{0+}^\alpha u(\xi_2)$. Then $u(t) \geq 0$ and $\mathbf{T}_{0+}^\alpha u(t) \leq 0$ for any $t \in [0,1]$.*

5.3 A necessary and sufficient condition for the existence of positive solution

In this section, by using the upper and lower solutions technique, Arzela-Ascoli theorem and Krasnosel'skii FPT, we establish the existence of positive solution to CBVP (E_2-C_3).

Theorem 7 *Suppose that (H) holds,* $f\left(s, G_{\alpha}(s), s^{2-\alpha}\right)$ *does not vanish identically on* $(0,1)$ *. Then a necessary and sufficient condition for problem* (E_2-C_3) *to have* $C^2[0,1]$ *positive solutions is that the following integral condition holds*

$$0<\int_0^1 \mathcal{G}_\beta(s) f\left(s, G_\alpha(s), s^{2-\alpha}\right) \mathrm{d} s<+\infty, \tag{5.54}$$

where $G_\alpha(.) \equiv G_\alpha(.,.)$ *and* $\mathcal{G}_\beta(.) \equiv \mathcal{G}_\beta(.,.)$ *defined by (5.13) and (5.14) respectively.*

The proof is divided into two parts, necessity and sufficiency.

Necessary. First we prove $\int_0^1 \mathcal{G}_\beta(s) f\left(s, G_\alpha(s), s^{2-\alpha}\right) ds<\infty$. Assume that u is a $C^2[0,1]$ positive solution of (E_2-C_3). By Lemma 30, there exist constants a_1 and a_2, $0<a_1<1<a_2$ such that (5.47) holds. Choose $\sigma>0$ such that

$$0<\sigma \leq 1, \quad M=\sup_{s \in[0,1]}\left|T_{0+}^2 u(s)\right| \quad \text{and} \quad \sigma M \leq 1. \tag{5.55}$$

Then, from (H) and (5.55) , we have

$$\begin{aligned} f\left(s, u(s),-\mathbf{T}_{0+}^\alpha u(s)\right) & \geq f\left(s, a_1 G_\alpha(s),-s^{2-\alpha} T_{0+}^2 u(s)\right) \\ & =f\left(s, a_1 G_\alpha(s),-\sigma^{-1} \sigma s^{2-\alpha} T_{0+}^2 u(s)\right) \\ & \geq a_1^{\mu_1} \sigma^{-\lambda_2}\left(-\sigma T_{0+}^2 u(s)\right)^{\mu_2} f\left(s, G_\alpha(s), s^{2-\alpha}\right) \\ & \geq a_1^{\mu_1} \sigma^{-\lambda_2}(\sigma M)^{\mu_2} f\left(s, G_\alpha(s), s^{2-\alpha}\right), \end{aligned}$$

which implies for $s \in(0,1)$

$$f\left(s, u(s),-\mathbf{T}_{0+}^\alpha u(s)\right) \geq \omega_1 f\left(s, G_\alpha(s), s^{2-\alpha}\right), \text{ where } \omega_1=a_1^{\mu_1} \sigma^{\mu_2-\lambda_2} M^{\mu_2}>0, \tag{5.56}$$

we also have

$$\begin{aligned} f\left(s,u\left(s\right),-\mathbf{T}_{0+}^{\alpha}u\left(s\right)\right) &\leq f\left(s,a_{2}G_{\alpha}\left(s\right),-s^{2-\alpha}T_{0+}^{2}u\left(s\right)\right) \\ &\leq a_{2}^{\mu_{1}}f\left(s,G_{\alpha}\left(s\right),-\sigma^{-1}\sigma s^{2-\alpha}T_{0+}^{2}u\left(s\right)\right) \\ &\leq a_{2}^{\mu_{1}}\sigma^{\lambda_{2}-\mu_{2}}\left(-T_{0+}^{2}u\left(s\right)\right)^{\lambda_{2}}f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right) \\ &\leq a_{2}^{\mu_{1}}\sigma^{\lambda_{2}-\mu_{2}}\left(M\right)^{\lambda_{2}}f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right), \end{aligned}$$

which implies for $s\in(0,1)$

$$f\left(s,u\left(s\right),-\mathbf{T}_{0+}^{\alpha}u\left(s\right)\right)\leq\omega_{2}f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right),\text{ where }\omega_{2}=a_{2}^{\mu_{1}}\sigma^{\lambda_{2}-\mu_{2}}M^{\lambda_{2}}>0. \tag{5.57}$$

According to (E_2), we have

$$\omega_{1}f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right)\leq f\left(s,u\left(s\right),-\mathbf{T}_{0+}^{\alpha}u\left(s\right)\right)=\mathbf{T}_{0+}^{\beta}\left(\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)\left(s\right). \tag{5.58}$$

By applying Lemmas 5, 7, we have

$$\omega_{1}\mathbf{I}_{0+}^{\beta}\left(f\left(s,G_{\alpha}\left(s\right),s^{2-\alpha}\right)\right)(t)\leq\left(\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)+\left(\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)\Big|_{t=0^{+}}+t\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\left(s\right)\right)\right)\Big|_{t=0^{+}}.$$

Moreover

$$\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\left(t\right)\right)=-\int_{0}^{1}G_{\beta}(t,\tau)f(\tau,u\left(\tau\right),-\mathbf{T}_{0+}^{\alpha}u\left(\tau\right))\mathrm{d}\tau\text{ for }t\in\left(0,1\right), \tag{5.59}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\left(t\right)\right)\right)=-\int_{0}^{t}\frac{\partial}{\partial t}G_{\beta}(t,\tau)f(\tau,u,-\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau-\int_{t}^{1}\frac{\partial}{\partial t}G_{\beta}(t,\tau)f(\tau,u,-\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau. \tag{5.60}$$

By (E_2), we have

$$\partial_{t}G_{\beta}(t,s)\leq1+\frac{b_{0}}{1-b_{0}\xi_{2}},$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi\left({}_{p}\left(\mathbf{T}_{0+}^{\alpha}u\left(t\right)\right)\right)\leq\left(\tfrac{1+b_{0}(1-\xi_{2})}{1-b_{0}\xi_{2}}\right)\int_{0}^{1}\mathcal{G}_{\beta}(\tau)f(\tau,u\left(\tau\right),-\mathbf{T}_{0+}^{\alpha}u\left(\tau\right))\mathrm{d}\tau.$$

Thus, from (E_2) and (5.59), we have $\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\left(s\right)\right)<0$ and $\mathbf{T}_{0+}^{\beta}\left(\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)(s)\geq0$ for $t\in(0,1)$, combining this with $\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\left(s\right)\right)(t)\in C^{1}\left[0,1\right]$, we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\left(s\right)\right)\right)(t)\Big|_{t=0^{+}}<0\quad\text{and}\quad\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_{p}\left(\mathbf{T}_{0+}^{\alpha}u\left(s\right)\right)\right)(t)\Big|_{t=1^{-}}>0,$$

with

$$\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)(t)\big|_{t=0^+}=0 \Longleftrightarrow \mathbf{T}_{0+}^{\alpha}u\,(0)=0, \varphi_p^{-1}\,(0)=0,$$

we deduce that

$$\begin{aligned} \mathbf{I}_{0+}^{\beta}\left(f\left(s,G_\alpha\,(s)\,,s^{2-\alpha}\right)\right)(t) &= \int_0^t (t-s)\,s^{\beta-2}f\left(s,G_\alpha\,(s)\,,s^{2-\alpha}\right)\mathrm{d}s \\ &\le \frac{1}{\omega_1}\left[\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\right)(t)+t\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\,(s)\right)\right)(t)\Big|_{t=0^+}\right]. \end{aligned} \tag{5.61}$$

Letting $t\longrightarrow 1$ in (5.61) we have

$$\int_0^1 (1-s)\,s^{\beta-2}f\left(s,G_\alpha\,(s)\,,s^{2-\alpha}\right)\mathrm{d}s \le \frac{1}{\omega_1}\left[\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\right)\big|_{t=1^-}+t\frac{\mathrm{d}}{\mathrm{d}t}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\,(s)\right)\right)\Big|_{t=0^+}\right]<\infty.$$

Second, we prove $\int_0^1 (1-s)\,s^{\beta-2}f\left(s,G_\alpha\,(s)\,,s^{2-\alpha}\right)ds>0$. The function $f\left(s,G_\alpha\,(s)\,,s^{2-\alpha}\right)\neq 0$ for all $s\in(0,1)$ yield

$$\int_0^1 \mathcal{G}_\beta\,(s)\,f\left(s,G_\alpha\,(s)\,,s^{2-\alpha}\right)\mathrm{d}s>0.$$

Therefore, we immediately get (5.54). The proof is complete. ■

Sufficiency. Suppose that (5.54) holds, we will divide our proof into tow steps.

Step 1. Auxiliary problem of (E_2-C_3)

$\forall u\,(t)\in C^2\,[0,1]\cap C^4\,[0,1]=\mathcal{F},\ t\in[0,1]$ we define an auxiliary function

$$F\,(u)\,(t)\equiv F\left(t,u,-\,\mathbf{T}_{0+}^{\alpha}u\right)=\begin{cases} f\left(t,\underline{u}\,(t)\,,-\mathbf{T}_{0+}^{\alpha}\underline{u}\,(t)\right) & \text{if} \quad u\,(t)<\underline{u}\,(t)\,,\\ f\left(t,u\,(t)\,,-\mathbf{T}_{0+}^{\alpha}u\,(t)\right) & \text{if} \quad u\,(t)\in\left[\underline{u}\,(t)\,,\overline{u}\,(t)\right],\\ f\left(t,\overline{u}\,(t)\,,-\mathbf{T}_{0+}^{\alpha}\overline{u}\,(t)\right) & \text{if} \quad u\,(t)>\overline{u}\,(t)\,. \end{cases} \tag{5.62}$$

The function $F\left(t,u\,(t)\,,-\mathbf{T}_{0+}^{\alpha}u\,(t)\right)$ is called a modification of $f\left(t,u\,(t)\,,-\mathbf{T}_{0+}^{\alpha}u\,(t)\right)$ associated with the coupled of lower and upper solutions $\underline{u}\,(t)$ and $\overline{u}\,(t)$. By the hypothesis (H) we have $F:\mathcal{F}\longrightarrow[0,+\infty)$ is continuous.(i.e., $F:(0,1)\times(0,+\infty)\times(-\infty,0)\longrightarrow\mathbb{R}^+$ is continuous.). Consider the auxiliary problem of (E_2-C_3)

$$\begin{cases} \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)(t)=F\left(t,u\,(t)\,,-\mathbf{T}_{0+}^{\alpha}u\right),\ t\in(0,1)\,,\\ u\,(0)=0,\ u\,(1)=b_1u\,(\xi_1)\,,\mathbf{T}_{0+}^{\alpha}u\,(0)=0,\ \mathbf{T}_{0+}^{\alpha}u\,(1)=b_2\mathbf{T}_{0+}^{\alpha}u\,(\xi_2)\,. \end{cases} \tag{5.63}$$

For convenience, we define linear operators as follows [6], [7]

$$A_2u\,(t)=\int_0^1 G_\beta\,(t,s)\,u\,(s)\,\mathrm{d}s \text{ and } A_1u(t)=\int_0^1 G_\alpha(t,s)u(s)\mathrm{d}s. \tag{5.64}$$

Obviously, by the proof of Lemma 28, the problem (5.63) is equivalent to the integral equation

$$u(t) = (A_1\varphi_q(A_2F))\,u(t),\ t \in (0,1). \tag{5.65}$$

By the definition (5.62) of F, we can get that $A_1\varphi_q(A_2f) : \mathcal{F} \to \mathcal{F}$ and $F(\mathcal{F})$ is bounded. By the continuity of $G_\alpha(t,s)$, we can show that $A_1\varphi_q(A_2)$ is a compact operator. So, $(A_1\varphi_q(A_2F))(\mathcal{F})$ is a relatively compact set. So $A_1\varphi_q(A_2) : \mathcal{F} \longrightarrow \mathcal{F}$ is a compact operator. Moreover, $u \in \mathcal{F}$ is a solution of (5.63) if and only if $(A_1\varphi_q(A_2F))\,u = u$. Using the Schauder's FPT, we assert that $A_1\varphi_q(A_2F)$ has at least one fixed point $u \in C^2[0,1]$, by $u(t) = (A_1\varphi_q(A_2F))\,u(t)$, we can get $u \in C^4[0,1]$.

1.1 Consider the problem

$$\begin{cases} \mathbf{T}^{\beta}_{0+}\left(\varphi_p\left(\mathbf{T}^{\alpha}_{0+}v\right)\right)(t) = f(t, G_\alpha(t), t^{2-\alpha}),\ t \in [0,1], \\ v(0) = 0,\ v(1) = b_1 v(\xi_1), \mathbf{T}^{\alpha}_{0+}v(0) = 0,\ \mathbf{T}^{\alpha}_{0+}v(1) = b_2\mathbf{T}^{\alpha}_{0+}v(\xi_2). \end{cases} \tag{5.66}$$

Let

$$v(t) = \int_0^1 G_\alpha(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)f(\tau, G_\alpha(\tau), \tau^{2-\alpha})\mathrm{d}\tau\right)\mathrm{d}s,\ t \in [0,1].$$

From the Lemma 30, (5.54) implies that

$$\exists 0 < a_3 < 1 < a_4 : 0 < v(t) < \infty \text{ and } a_3G_\alpha(t) \le v(t) \le a_4G_\alpha(t). \tag{5.67}$$

We will prove that the functions

$$\underline{u}(t) = k_1 v(t),\ \overline{u}(t) = k_2 v(t),\ t \in [0,1], \tag{5.68}$$

are lower and upper solutions of (E_2-C_3), respectively, here

$$k_1 \le \min\left\{1, \frac{1}{a_3}, \frac{1}{a_4}, \left(a_3^{\mu_1}\sigma^{\mu_2-\lambda_2}M^{\mu_2}\right)^{\frac{1}{1-\mu_1+\mu_2}}\right\}, \tag{5.69}$$

$$k_2 \ge \max\left\{1, \frac{1}{a_3}, \frac{1}{a_4}, \left(a_4^{\mu_2}\sigma^{\lambda_2-\mu_2}M^{\lambda_2}\right)^{\frac{1}{1-\lambda_1+\lambda_2}}\right\}. \tag{5.70}$$

This, by virtue of the assumption of the Lemma 29, (5.67) and (5.68), shows that

$$k_1a_3G_\alpha(t) \le \underline{u}(t) \le k_1a_4G_\alpha(t) \tag{5.71}$$

and

$$k_1a_3 \le \frac{\underline{u}(t)}{G_\alpha(t)} \le k_1a_4 \le 1,\ \frac{1}{k_2a_4} \le \frac{G_\alpha(t)}{\overline{u}(t)} \le \frac{1}{k_2a_3}.$$

By Lemma 7, shows that

$$-k_1 a_3 \mathbf{T}_{0+}^{\alpha} G_\alpha(t) \leq -\mathbf{T}_{0+}^{\alpha} \underline{u}(t) \leq -k_1 a_4 \mathbf{T}_{0+}^{\alpha} G_\alpha(t) \tag{5.72}$$

and

$$-k_2 a_4 \mathbf{T}_{0+}^{\alpha} G_\alpha(t) \leq -\mathbf{T}_{0+}^{\alpha} \overline{u}(t) \leq -k_2 a_3 \mathbf{T}_{0+}^{\alpha} G_\alpha(t). \tag{5.73}$$

Choose $0 < k_1 < 1$ small enough, and from (5.55), (5.71), (5.72) and (H) yield that

$$\begin{aligned} f\left(t, \underline{u}(t), -\mathbf{T}_{0+}^{\alpha} \underline{u}(t)\right) &= f(t, \left(\frac{\underline{u}(t)}{G_\alpha(t)}\right) G_\alpha(t), -\underline{u}''(t) t^{2-\alpha}) \\ &\geq f(t, k_1 a_3 G_\alpha(t), -\sigma^{-1}\sigma k_1 y t^{2-\alpha}) \\ &\geq k_1^{\mu_1+\mu_2} a_3^{\mu_1} \sigma^{\mu_2-\lambda_2} M^{\mu_2} f(t, G_\alpha(t), t^{2-\alpha}) \\ &\geq k_1 f(t, G_\alpha(t), t^{2-\alpha}),\ t \in (0,1). \end{aligned} \tag{5.74}$$

Similarly, choose $k_2 > 1$ large enough, we have

$$\begin{aligned} f\left(t, \overline{u}(t), -\mathbf{T}_{0+}^{\alpha} \overline{u}(t)\right) &\leq f(t, \frac{u(t)}{G_\alpha(t)} G_\alpha(t), -u''(t) t^{2-\alpha}) \\ &\leq f(t, k_1 a_4 G_\alpha(t), -\sigma^{-1}\sigma k_1 y t^{2-\alpha}) \\ &\leq k_1^{\lambda_1+\lambda_2} a_4^{\mu_2} \sigma^{\lambda_2-\mu_2} M^{\lambda_2} f(t, G_\alpha(t), t^{2-\alpha}) \\ &\leq k_2 f(t, G_\alpha(t), t^{2-\alpha}),\ t \in (0,1). \end{aligned} \tag{5.75}$$

Consequently, by Lemma 7, for $t \in (0,1)$

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} \underline{u}\right)\right)(t) = k_1 \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} v\right)\right) = k_1 f(t, G_\alpha(t), t^{2-\alpha}) \leq f\left(t, \underline{u}, -\mathbf{T}_{0+}^{\alpha} \underline{u}\right) \tag{5.76}$$

and

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} u\right)\right)(t) = k_2 \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha} v\right)\right) = k_2 f(t, G_\alpha(t), t^{2-\alpha}) \geq f\left(t, \overline{u}, -\mathbf{T}_{0+}^{\alpha} \overline{u}\right). \tag{5.77}$$

From (5.76) and (5.77), we obtain that for such choice of k_1 and k_2, $\underline{u}(t)$ and $\overline{u}(t)$ are, respectively, lower and upper solutions of (E_2-C_3) satisfying $0 < \underline{u}(t) \leq \overline{u}(t)$ for $t \in (0,1)$.

1.2. Let X be the Banach space $C^2[0,1]$ and the cone P in X be

$$P = \left\{ \begin{array}{l} u : u \in X, \varphi_p\left(\mathbf{T}_{0+}^{\alpha} u\right) \in X, -\varphi_p\left(\mathbf{T}_{0+}^{\alpha} u\right) \text{ is concave on } t \in (0,1), \\ u(t) \geq 0, -\mathbf{T}_{0+}^{\alpha} u(t) \geq 0 \text{ for } t \in [0,1],\ u \text{ satisfies } (C_3) \end{array} \right\}$$

If $u \in P$, then, it follows from Lemmas 26, 28 and (H) that

$$\begin{aligned} u(t) &= \int_0^1 G_\alpha (t,s) \left(-\mathbf{T}^\alpha_{0+} u(s)\right) \mathrm{d}s \\ &\le \int_0^1 G_\alpha (s) \left(s^{2-\alpha} \|u''\|_0\right) \mathrm{d}s \le \|u''\|_0 \int_0^1 G_\alpha (s) \left(s^{2-\alpha}\right) \mathrm{d}s. \end{aligned} \tag{5.78}$$

Thus, it is clear that

$$\|u\| = \|u''\|_0 \ \ \forall u \in P. \tag{5.79}$$

From Lemma 28 we have

$$\begin{aligned} u(t) &= \int_0^1 G_\alpha (t,s) \left(-\mathbf{T}^\alpha_{0+} u(s)\right) \mathrm{d}s \\ &\le \int_0^1 G_\alpha (t) \left(s^{2-\alpha} \|u''\|_0\right) \mathrm{d}s \le \frac{1}{3-\alpha} G_\alpha (t) \|u''\|_0 \ \text{ for } t \in [0,1]. \end{aligned} \tag{5.80}$$

Moreover, Remark 10 implies that

$$-\varphi_p \left(\mathbf{T}^\alpha_{0+} u\right)(t) \ge \psi(t)\, t^{2-\alpha} \varphi_p \left(\|u''\|_0\right) \ \text{ for } t \in [0,1]. \tag{5.81}$$

From (5.54), there exists an interval $[\delta, 1-\delta] \subset (0,1)$ such that

$$0 < \int_\delta^{1-\delta} \mathcal{G}_\beta (s) f\left(s, G_\alpha (s), s^{2-\alpha}\right) \mathrm{d}s < +\infty, \ \text{ where } \delta \in \left(0, \tfrac{1}{2}\right). \tag{5.82}$$

Note that from (5.13) and (5.14) we have

$$G_\alpha (t,s) \ge \mathcal{G}_\alpha (t,s) \ge \delta \text{ for } t,s \in [\delta, 1-\delta]. \tag{5.83}$$

Noting the continuity of G_β and F, we can choose $[\delta_1, \delta_2] \subset (0,1)$ such that

$$\int_0^1 G_\beta(s,\tau) F(\tau, u, -\mathbf{T}^\alpha_{0+} u) \ \mathrm{d}\tau \ge \int_{\delta_1}^{\delta_2} G_\beta(s,\tau) F(\tau, u, -\mathbf{T}^\alpha_{0+} u) \mathrm{d}\tau > 0.$$

Then, from (5.81) and (5.83), we obtain

$$u(t) = \int_0^1 G_\alpha (t,s) \left(-\mathbf{T}^\alpha_{0+} u(s)\right) \mathrm{d}s \ge \delta \|u''\|_0 \int_\delta^{1-\delta} \varphi_q \left(s^{2-\alpha}\right) \mathrm{d}s \ge m_1 \|u''\|_0 \ \text{ for } t \in [\delta, 1-\delta]. \tag{5.84}$$

where

$$m_1 = \delta \int_\delta^{1-\delta} \varphi_q \left(s^{2-\alpha}\right) \mathrm{d}s \in (0,1). \tag{5.85}$$

For any fixed $u \in P$, choose a positive number $\theta = \frac{1}{\|u\|+1} < 1$ Then, (5.83) yields

$$\frac{\theta u(t)}{G_\alpha (t)} \le \theta \|u''\|_0 = \theta \|u\| \le 1 \text{ for } t \in (0,1). \tag{5.86}$$

Thus, (5.55) and (H), imply

$$f(t,u(t),-\mathbf{T}_{0+}^{\alpha}u(t))=f(t,\frac{\theta u(t)}{\theta G_{\alpha}(t)}G_{\alpha}(t),\theta^{-1}\theta(-u''(t))t^{2-\alpha})$$
$$\leq \theta^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}f(t,G_{\alpha}(t),t^{2-\alpha}) \text{ for } t\in(0,1).$$

Therefore, from (5.32), we have

$$\int_0^1 G_{\beta}(t,s)f(s,u(s),-\mathbf{T}_{0+}^{\alpha}u(s))\mathrm{d}s \leq \theta^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}\left(\tfrac{1+b_0(1-\xi_2)}{1-b_0\xi_2}\right)\int_0^1 \mathcal{G}_{\beta}(s)f(s,G_{\alpha}(s),s^{2-\alpha})<\infty. \tag{5.87}$$

Now we prove that problem (5.63) has a positive solution $u^*\in X$ with $0<\underline{u}(t)\leq u^*\leq u(t)$. We consider the operator $A:X\longrightarrow X$ defined as follows

$$(Au)(t)=\int_0^1 G_{\alpha}(t,s)\varphi_q\left(\int_0^1 G_{\beta}(s,\tau)F(\tau,u(\tau),-\mathbf{T}_{0+}^{\alpha}u(\tau))\mathrm{d}\tau\right)\mathrm{d}s. \tag{5.88}$$

It is well known that a fixed point of the operator A is a solution of the problem (5.63). The following fixed point result of cone compression type due to Krasnosel'skii is fundamental for the solvability of problem (5.63).

From (5.88), Lemmas 26, 26 and 28 it is easy to see that $u\in P$ is a $C^2[0,1]$ nonnegative solution of the problem (5.63) if and only if u is a fixed point of A. Moreover

$$-\mathbf{T}_{0+}^{\alpha}(Au)(t)=\varphi_q\left(\int_0^1 G_{\beta}(t,\tau)F(\tau,u(\tau),-\mathbf{T}_{0+}^{\alpha}u(\tau))\mathrm{d}\tau\right) \text{ for } t\in[0,1]. \tag{5.89}$$

In the following, we divide the proof of the existence of fixed point of $A:X\longrightarrow X$ into three steps.

$(\mathbf{S_1})$ The operator $A:P\longrightarrow P$ is completely continuous.

$(\mathbf{S_{11}})$ $A:P\longrightarrow P$

If $u\in P$, it is clear that $Au\in X$,

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)\right)(t)\geq 0, \text{for } t\in(0,1) \text{ and } \left(\mathbf{T}_{0+}^{\alpha}(Au)\right)(t)\leq 0,\ (Au)(t)\geq 0 \text{ for } t\in[0,1],$$
$$(Au)(0)=0,\ (Au)(1)=b_1(Au)(\xi_1),\mathbf{T}_{0+}^{\alpha}(Au)(0)=0,\ \mathbf{T}_{0+}^{\alpha}(Au)(1)=b_2\mathbf{T}_{0+}^{\alpha}(Au)(\xi_2). \tag{5.90}$$

$(\mathbf{S_{12}})$ A is pre-compact in P.

Let Ω be a bounded set on u. Then there is $\rho>0$ such that $\|u\|\leq\rho$ for all $u\in\Omega$. We show that $(A\Omega)$ is a pre-compact set in P. Denote $\theta_\rho = \frac{1}{1+\rho}$. For all $u \in \Omega$

$$
\begin{aligned}
|(Au)(t)| &= \left| \int_0^1 G_\alpha(t,s)\,\varphi_q\left(\int_0^1 G_\beta(s,\tau) F(\tau, u(\tau), -\mathbf{T}^\alpha_{0+} u) \mathrm{d}\tau \right) \mathrm{d}s \right| \qquad (5.91)\\
&\le \left| \int_0^1 G_\alpha(s)\,\varphi_q\left(\int_0^1 G_\beta(\tau) F(\tau, u(\tau), \left(-\mathbf{T}^\alpha_{0+} u\right)(\tau)) \mathrm{d}\tau \right) \mathrm{d}s \right| \\
&\le \left(\int_0^1 G_\alpha(s)\,\mathrm{d}s \right) \left(\varphi_q\left(\int_0^1 G_\beta(\tau) f(\tau, u(\tau), \left(-\mathbf{T}^\alpha_{0+} u\right)(\tau)) \mathrm{d}\tau \right) \right) \\
&\le \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2} M^{\lambda_2} k_2 \right)^{\frac{1}{p-1}} \left(\int_0^1 G_\alpha(s)\,\mathrm{d}s \right) \varphi_q\left(\int_0^1 G_\beta(\tau) f(\tau, G_\alpha(\tau), \tau^{2-\alpha}) \mathrm{d}\tau \right) \\
&\le \frac{\left(\theta_\rho^{\lambda_2-\mu_1-\mu_2} M^{\lambda_2} k_2 \right)^{\frac{1}{p-1}}}{\left(\frac{1+b_0(1-\xi_2)}{1-b_0\xi_2} \right)^{\frac{1}{1-p}}} \left(\int_0^1 G_\alpha(s)\,\mathrm{d}s \right) \varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau) f(\tau, G_\alpha(\tau), \tau^{2-\alpha}) \mathrm{d}\tau \right) \\
&\le \mathcal{L}_\rho \left(\int_0^1 G_\alpha(s)\,\mathrm{d}s \right) < \infty,
\end{aligned}
$$

where

$$
\mathcal{L}_\rho = \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2} M^{\lambda_2} k_2 \right)^{\frac{1}{p-1}} \left(\frac{1+b_0(1-\xi_2)}{1-b_0\xi_2} \right)^{\frac{1}{p-1}} \varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau) f(\tau, G_\alpha(\tau), \tau^{2-\alpha}) \mathrm{d}\tau \right) \qquad (5.92)
$$

and

$$
\begin{aligned}
\left| \mathbf{T}^\alpha_{0+}(Au)(t) \right| &= \left| \varphi_q\left(\int_0^1 G_\beta(t,\tau) F(\tau, u(\tau), -\mathbf{T}^\alpha_{0+} u(\tau)) \mathrm{d}\tau \right) \right| \qquad (5.93)\\
&\le \varphi_q\left(\int_0^1 G_\beta(\tau) f(\tau, u(\tau), \left(-\mathbf{T}^\alpha_{0+} u\right)(\tau)) \mathrm{d}\tau \right) \\
&\le \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2} M^{\lambda_2} k_2 \right)^{\frac{1}{p-1}} \varphi_q\left(\int_0^1 G_\beta(\tau) f(\tau, G_\alpha(\tau), \tau^{2-\alpha}) \mathrm{d}\tau \right) \\
&\le \mathcal{L}_\rho < \infty.
\end{aligned}
$$

Then (Au) is uniformly bounded in X.

For all $u \in \Omega,\ t_1, t_2 \in [0,1]$,

$$
\begin{aligned}
|(Au)(t_2) - (Au)(t_1)| &= \int_0^1 |G_\alpha(t_2,s) - G_\alpha(t_1,s)|\,\varphi_q\left(\int_0^1 G_\beta(s,\tau) F(\tau, u, -\mathbf{T}^\alpha_{0+} u) \mathrm{d}\tau \right) \mathrm{d}s \qquad (5.94)\\
&\le \int_0^1 |G_\alpha(t_2,s) - G_\alpha(t_1,s)|\,\varphi_q\left(\int_0^1 G_\beta(s,\tau) f(\tau, u, \left(-\mathbf{T}^\alpha_{0+} u\right) \mathrm{d}\tau \right) \mathrm{d}s \\
&\le \mathcal{L}_\rho \left(\int_0^1 |G_\alpha(t_2,s) - G_\alpha(t_1,s)|\,\mathrm{d}s \right) \\
&\le \mathcal{L}_\rho\,|t_2 - t_1| < \infty.
\end{aligned}
$$

For arbitrary $\epsilon > 0$. Let $\delta_1 = \frac{\epsilon}{\mathcal{L}_\rho}$, then

$$|(Au)(t_2) - (Au)(t_1)| < \epsilon \quad \text{if} \quad |t_2 - t_1| < \delta_1. \tag{5.95}$$

At the same time, from

$$\begin{aligned} \left|\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)(t)\right| &= \left|\int_0^1 G_\beta(s,\tau)F(\tau, u(\tau), -\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau\right| \\ &\leq \left|\int_0^1 G_\beta(s,\tau) f\left(\tau, u(\tau), \left(-\mathbf{T}_{0+}^{\alpha}u\right)(\tau)\right)\mathrm{d}\tau\right| \\ &\leq \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2} M^{\lambda_2} k_2\right)\left(\int_0^1 G_\beta(\tau) f(\tau, G_\alpha(\tau), \tau^{2-\alpha})\mathrm{d}\tau\right) < \infty, \end{aligned} \tag{5.96}$$

it follows that for $\epsilon > 0$, there is $\delta_2 > 0$ such that $\forall u \in \Omega, t \in [0,1]$

$$\int_{1-\delta_2}^1 G_\beta(s,\tau)F(\tau, u, -\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau < \int_0^1 G_\beta(s,\tau) f\left(\tau, u(\tau), \left(-\mathbf{T}_{0+}^{\alpha}u\right)(\tau)\right)\mathrm{d}\tau < \frac{\epsilon}{3}. \tag{5.97}$$

On the other hand, (5.54) imply that for $u \in \Omega, t \in [0, 1-\delta_2]$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)(t) &= \int_0^t \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)(\tau)\right)\mathrm{d}\tau \\ &= \int_0^t \tau^{\beta-2}\tau^{2-\beta}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)(\tau)\right)\mathrm{d}\tau \\ &= \int_0^t \tau^{\beta-2}\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)(\tau)\right)\mathrm{d}\tau \\ &= \int_0^t \tau^{\beta-2}F(\tau, (Au)(\tau), -\mathbf{T}_{0+}^{\alpha}(Au))\mathrm{d}\tau \\ &\leq \int_0^t \tau^{\beta-2}f(\tau, (Au)(\tau), -\mathbf{T}_{0+}^{\alpha}(Au))\mathrm{d}\tau \\ &\leq \left(\theta_\rho^{\lambda_2-\mu_1-\mu_2} M^{\lambda_2} k_2\right)\left(\int_0^{1-\delta_2} \tau^{\beta-2} f(\tau, G_\alpha(\tau), \tau^{2-\alpha})\mathrm{d}\tau\right) = m_2. \end{aligned} \tag{5.98}$$

where k_2 is defined in (5.70).

Let $\delta = \min\left\{\delta_1, \delta_2, \frac{\epsilon}{3m_2}\right\}$. Then for $u \in \Omega, t_1, t_2 \in [0,1], 0 \leq t_2 - t_1 \leq \delta$.

Moreover, by mean value theorem, Lemma 6 ensures that, for any t_2, t_1 in $[0,1]$ with $t_1 < t_2$, there exists a point t in (t_1, t_2) such that

$$\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)(t_2) - \varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)(t_1) = \frac{1}{\beta-1}\left(t_2^{\beta-1} - t_1^{\beta-1}\right)\left[T_{0+}^{\beta-1}\varphi_p\left(\mathbf{T}_{0+}^{\alpha}(Au)\right)\right](t).$$

This, together with (5.98), yields

$$\begin{aligned}
\mathfrak{W} &= \left|\varphi_p\left(\mathbf{T}^{\alpha}_{0+}(Au)\right)(t_2) - \varphi_p\left(\mathbf{T}^{\alpha}_{0+}(Au)\right)(t_1)\right| \\
&= \left|\frac{1}{\beta-1}\left(t_2^{\beta-1} - t_1^{\beta-1}\right)\left[T^{\beta-1}_{0+}\varphi_p\left(\mathbf{T}^{\alpha}_{0+}(Au)\right)\right](t)\right| \\
&= \left|\frac{1}{\beta-1}\left(t_2^{\beta-1} - t_1^{\beta-1}\right)\left[t^{\beta-2}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_p\left(\mathbf{T}^{\alpha}_{0+}(Au)\right)(t)\right]\right| \\
&\leq \left|\frac{1}{\beta-1}\left(t_2^{\beta-1} - t_1^{\beta-1}\right)\left[t^{\beta-2}\int_0^t \tau^{\beta-2} f(\tau,(Au)(\tau), -\mathbf{T}^{\alpha}_{0+}(Au)(\tau))\mathrm{d}\tau\right]\right| \\
&\leq M_1 k_2\left|t_2^{\beta-1} - t_1^{\beta-1}\right| \leq \frac{\epsilon}{3} < \epsilon,\ t_1\in[0,1-\delta_2].
\end{aligned}$$

In the same way, we can show that

$$\begin{aligned}
\mathfrak{W} &= \left|\varphi_p\left(\mathbf{T}^{\alpha}_{0+}(Au)\right)(t_2) - \varphi_p\left(\mathbf{T}^{\alpha}_{0+}(Au)\right)(t_1)\right| \\
&\leq \int_0^1 \left|G_\beta(t_2,\tau) - G_\beta(t_1,\tau)\right| F(\tau,u(\tau),-\mathbf{T}^{\alpha}_{0+}u)\mathrm{d}\tau \\
&\leq \int_0^{1-\delta_2} \left|G_\beta(t_2,\tau) - G_\beta(t_1,\tau)\right| F_u(\tau)\mathrm{d}\tau + \int_{1-\delta_2}^1 \left|G_\beta(t_2,\tau) - G_\beta(t_1,\tau)\right| F_u(\tau)\mathrm{d}\tau \\
&\leq \int_0^{1-\delta_2} |t_2 - t_1| F_u(\tau)\mathrm{d}\tau + \int_{1-\delta_2}^1 |G_\beta(t_2,\tau)| F_u(\tau)\mathrm{d}\tau + \int_{1-\delta_2}^1 |G_\beta(t_1,\tau)| F_u(\tau)\mathrm{d}\tau \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \varepsilon,\ t_1\in[1-\delta_2,1].
\end{aligned} \tag{5.99}$$

Then $\varphi_p\left(\mathbf{T}^{\alpha}_{0+}(Au)\right)(t)$ is equi-continuous on $[0,1]$. Since φ is uniformly continuous on arbitrary closed interval of $\mathbb{R}$. $\left(\mathbf{T}^{\alpha}_{0+}(Au)\right)(t)$ is also equi-continuous on $[0,1]$.

($\mathbf{S_{13}}$) A is continuous in P: Let $u_n \longrightarrow u$ as $n \longrightarrow \infty$ in P. Then there exists $\bar{\rho} > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n\| \leq \bar{\rho}$. Then by (5.62), there holds

$$\begin{aligned}
(Au_n)(t) &= \int_0^1 G_\alpha(t,s)\varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau,u_n(\tau),-\mathbf{T}^{\alpha}_{0+}u_n)\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq \int_0^1 G_\alpha(s)\varphi_q\left(\int_0^1 G_\beta(\tau)F(\tau,u_n(\tau),-\mathbf{T}^{\alpha}_{0+}u_n)\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq \int_0^1 G_\alpha(s)\varphi_q\left(\int_0^1 G_\beta(\tau)f(\tau,u_n(\tau),-\mathbf{T}^{\alpha}_{0+}u_n(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&\leq \left(\theta_{\bar{\rho}}^{\lambda_2-\mu_1-\mu_2}M^{\lambda_2}k_2\right)^{\frac{1}{p-1}}\left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\left(\int_0^{1-\delta_2} G_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right) \\
&\leq \mathcal{L}_{\bar{\rho}}\left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right) < \infty,
\end{aligned} \tag{5.100}$$

where $\mathcal{L}_{\bar{\rho}}$ is given by (5.92) with ρ replaced by $\bar{\rho}$. By Lebesgue's dominated convergence theorem,

$$(Au_n)(t) = \int_0^1 G_\alpha(t,s)\,\varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau,u_n(\tau),-\mathbf{T}^\alpha_{0+}u_n)\mathrm{d}\tau\right)\mathrm{d}s \tag{5.101}$$
$$\longrightarrow \int_0^1 G_\alpha(t,s)\,\varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau,u(\tau),-\mathbf{T}^\alpha_{0+}u)\mathrm{d}\tau\right)$$
$$= (Au)(t) \text{ as } n \longrightarrow \infty, t \in [0,1].$$

which means $(Au_n) \longrightarrow (Au)$ as $n \longrightarrow \infty$.

Similarly, $\mathbf{T}^\alpha_{0+}(Au_n) \longrightarrow \mathbf{T}^\alpha_{0+}(Au)$ as $n \longrightarrow \infty$. Therefore, by Arzelá Ascoli Theorem, $A : P \longrightarrow P$ is completely continuous.

($\mathbf{S_2}$) If $\Omega_1 = \{u \in X : \|u\| < r_1\}$, then

$$\|Au\| \le \|u\| \text{ for } u \in P \cap \partial\Omega_1, \tag{5.102}$$

where $r_1 > 0$ satisfies

$$r_1 < \min\left\{1, \overline{r}^{\frac{p-1}{p-(\lambda_1+\lambda_2+1)}}\right\}, \tag{5.103}$$

where

$$\overline{r} = \left(\left(\frac{k_2}{(3-\alpha)^{\lambda_1}}\right)^{\frac{1}{p-1}}\left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\left(\frac{1+b_0(1-\xi_2)}{1-b_0\xi_2}\right)^{\frac{1}{p-1}}\varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right)\right).$$

If $u \in P \cap \partial\Omega_1$, then from (H), (5.63), (5.69) and (5.103), we have

$$F(t,u(t),-\mathbf{T}^\alpha_{0+}u(t)) \le F(t,u(t),t^{2-\alpha}\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(t)) \tag{5.104}$$
$$= k_2 f(t,v(t),t^{2-\alpha}\frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t))$$
$$\le k_2 f(t,\frac{1}{3-\alpha}G_\alpha(t)\|v''\|_0, t^{2-\alpha}\|v''\|_0)$$
$$\le k_2\left(\frac{1}{3-\alpha}\right)^{\lambda_1}(\|v''\|_0)^{\lambda_1+\lambda_2} f(t,G_\alpha(t),t^{2-\alpha})$$
$$\le k_2\left(\frac{1}{3-\alpha}\right)^{\lambda_1}\|v\|^{\lambda_1+\lambda_2} f(t,G_\alpha(t),t^{2-\alpha})$$
$$\le k_2\left(\frac{1}{3-\alpha}\right)^{\lambda_1} r_1^{\lambda_1+\lambda_2} f(t,G_\alpha(t),t^{2-\alpha}), \text{ for } t \in (0,1),$$

where k_2 is defined in (5.70). From Lemmas 29,30, (5.103) and (5.104), we obtain

$$\begin{aligned}
(Au)(t) &= \int_0^1 G_\alpha(t,s)\,\varphi_q\left(\int_0^1 G_\beta(\tau)F(\tau,u,-\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau\right)\mathrm{d}s \qquad (5.105)\\
&\le \left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\varphi_q\left(\int_0^1 G_\beta(\tau)F(\tau,u,-\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau\right)\\
&\le \left(\left(\frac{1}{3-\alpha}\right)^{\lambda_1} r_1^{\lambda_1+\lambda_2}k_2\right)^{\frac{1}{p-1}}\left(\frac{1+b_0(1-\xi_2)}{1-b_0\xi_2}\right)^{\frac{1}{p-1}}\\
&\times\left(\int_0^1 G_\alpha(s)\,\mathrm{d}s\right)\varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right)\\
&\le r_1=\|u\|
\end{aligned}$$

and

$$\begin{aligned}
\left|\mathbf{T}_{0+}^{\alpha}(Au)(t)\right| &= \left|\varphi_q\left(\int_0^1 G_\beta(t,\tau)F(\tau,u,-\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau\right)\right|\\
&\le \varphi_q\left(\int_0^1 G_\beta(\tau)F(\tau,u,-\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau\right)\\
&\le \left(\left(\frac{1}{3-\alpha}\right)^{\lambda_1} r_1^{\lambda_1+\lambda_2}k_2\right)^{\frac{1}{p-1}}\left(\frac{1+b_0(1-\xi_2)}{1-b_0\xi_2}\right)^{\frac{1}{p-1}}\varphi_q\left(\int_0^1 \mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha})\mathrm{d}\tau\right)\\
&\le r_1=\|u\|,
\end{aligned}$$

for $t\in[0,1]$ and thus .(Au) satisfies (5.102).

(S₃) If $\Omega_2=\{u\in X:\|u\|<r_2\}$, then

$$\|Au\|\ge\|u\| \text{ for } u\in P\cap\partial\Omega_2, \qquad (5.106)$$

where $r_2>0$ satisfies

$$r_2>\max\left\{1,\frac{1}{\varepsilon m_1},\frac{1}{d},\frac{1}{(\delta^{2\alpha})^{\frac{1}{p-1}}}\right\}, \qquad (5.107)$$

m_1 is as defined in (5.85), $\epsilon\in(0,1)$ is a fixed number small enough such that

$$\delta\left(c\left(1-\delta\right)+1\right)\le G_\alpha(t)\le(1-\delta)\left(c\delta+1\right),c<0$$

and

$$\psi_1(t)=t\,(ct+1)\ge\epsilon G_\alpha(t) \text{ for } t\in[\delta,1-\delta]. \qquad (5.108)$$

Setting

$$\begin{aligned}
d &= k_1^{\frac{1}{p-1}}(1-2\delta)^{p-1}\left(\delta^{2\alpha}\right)^{\frac{1}{\lambda_1+\lambda_2+p-1}}(\epsilon m_1)^{\lambda_1}\left(\delta\left(c\left(1-\delta\right)+1\right)\right)^{\frac{\lambda_2}{p-1}} \qquad (5.109)\\
&\times\left(\int_\delta^{1-\delta}\mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha}\mathrm{d}\tau\right)^{\frac{1}{\lambda_1+\lambda_2+p-1}}.
\end{aligned}$$

Now if $u \in P \cap \partial\Omega_2$, from (5.79), 5.81) and (5.107), we have

$$-\left(\mathbf{T}_{0+}^{\alpha} u\right)(t) \geq \left(\psi_1(t)\right)^{\frac{1}{p-1}} \|u''\|_0 \geq \left(\delta\left(c(1-\delta)+1\right)\right)^{\frac{1}{p-1}} \|u''\|_0 \geq \left(\delta\left(c(1-\delta)+1\right)\right)^{\frac{1}{p-1}} \|u\| > 1 \text{ for } t \in [\delta, 1-\delta]. \tag{5.110}$$

Similarly, from (5.84) and (5.107), we have

$$\epsilon x(t) \geq \epsilon m_1 \|u''\|_0 = \epsilon m_1 \|u\| > 1 \text{ for } t \in [\delta, 1-\delta]. \tag{5.111}$$

Moreover, for $t \in [\delta, 1-\delta]$ by Lemma 26 and (5.108),

$$u(t) \geq \psi(t) \|u\|_0 \geq \epsilon G_\alpha(t) \|u\|_0. \tag{5.112}$$

Hence, for $t \in [\delta, 1-\delta]$, from (H), (5.110), (5.111) and (5.112) imply

$$\begin{aligned} F(t, u, -\mathbf{T}_{0+}^{\alpha} u) &\geq f(t, \underline{u}, -\mathbf{T}_{0+}^{\alpha} \underline{u}) = f(t, k_1 v, -k_1 \mathbf{T}_{0+}^{\alpha} v) \\ &\geq k_1 f(t, v(t), -t^{2-\alpha} \frac{\mathrm{d}^2}{\mathrm{d}t^2} v(t)) \\ &\geq k_1 f(t, \epsilon G_\alpha(t) \|v\|_0, (\delta(c(1-\delta)+1))^{\frac{1}{p-1}} t^{2-\alpha} \|v\|) \\ &\geq k_1 (\epsilon \|v\|_0)^{\lambda_1} \left((\delta(c(1-\delta)+1))^{\frac{1}{p-1}} \|v\|\right)^{\lambda_2} f(t, G_\alpha(t), t^{2-\alpha}) \\ &\geq k_1 (\epsilon m_1)^{\lambda_1} (\delta(c(1-\delta)+1))^{\frac{\lambda_2}{p-1}} \|v\|^{\lambda_1+\lambda_2} f(t, G_\alpha(t), t^{2-\alpha}) \\ &= k_1 (\epsilon m_1)^{\lambda_1} (\delta(c(1-\delta)+1))^{\frac{\lambda_2}{p-1}} r_2^{\lambda_1+\lambda_2} f(t, G_\alpha(t), t^{2-\alpha}),\ m_1 \in [0,1], \end{aligned} \tag{5.113}$$

where k_1 is defined in (5.69).

Since $G_\alpha(t,s) \geq \mathcal{G}_\alpha(t,s) \geq \delta\delta^{\alpha-1} = \delta^\alpha \geq \delta^\alpha G_\alpha(s)$ for $t, s \in [\delta, 1-\delta]$, it then follows from (5.88), (5.107) and (5.113), we have

$$\begin{aligned} (Au)(t) &= \int_0^1 G_\alpha(t,s) \varphi_q\left(\int_0^1 G_\beta(s,\tau) F(\tau, u(\tau), -\mathbf{T}_{0+}^{\alpha} u) \mathrm{d}\tau\right) \mathrm{d}s \\ &\geq \delta^\alpha \int_\delta^{1-\delta} \varphi_q\left(\int_\delta^{1-\delta} G_\beta(s,\tau) F(\tau, u(\tau), -\mathbf{T}_{0+}^{\alpha} u) \mathrm{d}\tau\right) \mathrm{d}s \\ &\geq \delta^\alpha (\delta^\alpha)^{\frac{1}{p-1}} \left(\int_\delta^{1-\delta} \mathrm{d}s\right) \left(\varphi_q\left(\int_\delta^{1-\delta} G_\beta(\tau) f(\tau, \underline{u}(\tau), -\mathbf{T}_{0+}^{\alpha} \underline{u}(\tau)) \mathrm{d}\tau\right)\right) \\ &\geq k_1 (1-2\delta) \delta^\alpha (\delta^\alpha)^{\frac{1}{p-1}} (\epsilon m_1)^{\frac{\lambda_1}{p-1}} (\delta(c(1-\delta)+1))^{\frac{\lambda_2}{(p-1)^2}} r_2^{\frac{\lambda_1+\lambda_2}{p-1}} \\ &\times \varphi_q\left(\int_\delta^{1-\delta} \mathcal{G}_\beta(\tau) f(\tau, G_\alpha(\tau), \tau^{2-\alpha} \mathrm{d}\tau\right)) \\ &\geq r_2 = \|u\|. \end{aligned} \tag{5.114}$$

Hence,

$$\|Au\|_0 \geq \|u\| . \tag{5.115}$$

From (5.89), (5.107), (5.109) and (5.113), we have

$$\begin{aligned}
\mathbf{T}_{0+}^{\alpha}(Au)(t) &= \varphi_q\left(\int_0^1 G_\beta(s,\tau)F(\tau,u(\tau),-\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau\right) \\
&\geq (\delta^\alpha)^{\frac{1}{p-1}}\left(\varphi_q\left(\int_\delta^{1-\delta} G_\beta(\tau)F(\tau,u(\tau),-\mathbf{T}_{0+}^{\alpha}u)\mathrm{d}\tau\right)\right) \\
&\geq (\delta^\alpha)^{\frac{1}{p-1}}\left(\varphi_q\left(\int_\delta^{1-\delta} G_\beta(\tau)f(\tau,\underline{u}(\tau),-\mathbf{T}_{0+}^{\alpha}\underline{u}(\tau))\mathrm{d}\tau\right)\right) \\
&\geq k_1(\delta^\alpha)^{\frac{1}{p-1}}(\epsilon m_1)^{\frac{\lambda_1}{p-1}}(\delta(c(1-\delta)+1))^{\frac{\lambda_2}{(p-1)^2}}r_2^{\frac{\lambda_1+\lambda_2}{p-1}} \\
&\times\varphi_q\left(\int_\delta^{1-\delta}\mathcal{G}_\beta(\tau)f(\tau,G_\alpha(\tau),\tau^{2-\alpha}\mathrm{d}\tau\right)) \\
&\geq r_2 = \|u\| .
\end{aligned} \tag{5.116}$$

Hence,

$$\left\|\mathbf{T}_{0+}^{\alpha}(Au)\right\| \geq \|u\| . \tag{5.117}$$

In view of (5.115) and (5.117), we see that (5.106) holds.

Therefore, by steps one to three, and Lemma 7, we see that A has at least one fixed point $u \in P \cap \bar{\Omega}_2 \diagdown \Omega_1$. It can be verified that for $u \in P$, there holds $u(t) \geq G_\alpha(t)\|u\|_0$. Thus, $u(t)$ is a solution of problem (5.63).

Step 2. Finally, we will prove that the CBVP (E_2-C_3) has at least one positive solution. Suppose that $u^*(t)$ is a solution of (E_2-C_3), we only need to prove that $\underline{u}(t) \leq u^*(t) \leq \overline{u}(t)$, $t \in [0,1]$. The method is similar for the two inequalities. We only prove $u^*(t) \leq \overline{u}(t)$ for $t \in [0,1]$.

In fact, since u^* is fixed point of A and (5.90), we get

$$\begin{aligned}
&u^*(0)=0 \quad u^*(1)=b_1u^*(\xi_1) \quad \mathbf{T}_{0+}^{\alpha}u^*(0)=0 \quad \mathbf{T}_{0+}^{\alpha}u^*(1)=b_2\mathbf{T}_{0+}^{\alpha}u^*(\xi_2), \\
&\overline{u}(0)=0 \quad \overline{u}(1)=b_1\overline{u}(\xi_1) \quad \mathbf{T}_{0+}^{\alpha}\overline{u}(0)=0 \quad \mathbf{T}_{0+}^{\alpha}\overline{u}(1)=b_2\mathbf{T}_{0+}^{\alpha}\overline{u}(\xi_2).
\end{aligned} \tag{5.118}$$

Otherwise, suppose by contradiction that $u^*(t) > \overline{u}(t)$. According to the definition of F, one verifies that

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u^*\right)\right)(t) = F\left(t,u^*(t),-\mathbf{T}_{0+}^{\alpha}u^*(t)\right) = f\left(t,\overline{u}(t),-\mathbf{T}_{0+}^{\alpha}u(t)\right), t \in (0,1). \tag{5.119}$$

On the other hand, since $\overline{u}$ is an upper solution to (E_2), we obviously have

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\overline{u}\right)\right)(t) \geq f\left(t, \overline{u}(t), -\mathbf{T}_{0+}^{\alpha}u(t)\right), \ t \in (0,1). \tag{5.120}$$

Setting

$$z(t) = \varphi_p\left(\mathbf{T}_{0+}^{\alpha}\overline{u}(t)\right) - \varphi_p\left(\mathbf{T}_{0+}^{\alpha}u^*(t)\right), \ t \in (0,1). \tag{5.121}$$

From (5.119) and (5.120), we can get

$$\begin{aligned}\mathbf{T}_{0+}^{\beta}z(t) &= \mathbf{T}_{0+}^{\beta}\left(\phi_p\left(\mathbf{T}_{0+}^{\alpha}\overline{u}\right)\right)(t) - \mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u^*\right)\right)(t) \\ &\geq f\left(t, \overline{u}(t), -\mathbf{T}_{0+}^{\alpha}u(t)\right) - f\left(t, \overline{u}(t), -\mathbf{T}_{0+}^{\alpha}u(t)\right) = 0, \ t \in (0,1),\end{aligned}$$

with

$$z(0) = 0 \quad \text{and} \quad z(1) = \varphi_p(b_2)\, z(\zeta_1).$$

Thus, by Lemma 31, we have $z(t) \leq 0$, $t \in [0,1]$, which implies that

$$\varphi_p\left(\mathbf{T}_{0+}^{\alpha}\overline{u}\right)(t) \leq \varphi_p\left(\mathbf{T}_{0+}^{\alpha}u^*\right)(t), t \in [0,1].$$

Since φ_p is monotone increasing, we obtain

$$\mathbf{T}_{0+}^{\alpha}\overline{u}(t) \leq \mathbf{T}_{0+}^{\alpha}u^*(t) \Longrightarrow \mathbf{T}_{0+}^{\alpha}\left(\overline{u}(t) - u^*(t)\right) \leq 0, \ t \in [0,1].$$

Combining Lemma 31 and (5.118), we have $\overline{u}(t) \geq u^*(t)$ This contradiction proves the validity of $\overline{u}(t) < u^*(t)$, $t \in [0,1]$.

Similarly, suppose by contradiction that $\underline{u}(t) > u^*(t)$. By the same way, we also have $u^*(t) \geq \underline{u}(t)$ on $t \in [0,1]$ so,

$$\underline{u}(t) \leq u^*(t) \leq \overline{u}(t), \ t \in [0,1], \tag{5.122}$$

that is, $u^*(t)$ is a positive solution of the conformable boundary value problem (E_2-C_3). Furthermore, $\underline{u}(t)$, $\overline{u}(t) \in P$ implies that there exist two positive constants $0 < a_1 < 1 < a_2$ such that

$$0 < a_1 G_\alpha(t) \leq \underline{u}(t) < u(t) < \overline{u}(t) \leq a_2 G_\alpha(t), \ t \in [0,1].$$

Thus, we have finished the proof of Theorem 7. ■

From the theorem 7, we can easily derive the following corollary.

Corollary 1 *Suppose that condition (H) are satisfied, then the CBVP (E_2-C_3) with the Lidstone boundary conditions*

$$u(0) = 0, \ u(1) = 0, \mathbf{T}_{0+}^{\alpha} u(0) = 0, \ \mathbf{T}_{0+}^{\alpha} u(1) = 0,$$

has at least one positive solution u.

we present the following theorem without proof because the proof are similar to Theorem 7.

Theorem 8 *If $f\left(t, u, \mathbf{T}_{0+}^{\alpha} u(t)\right) \in C([0,1] \times (0,+\infty) \times (-\infty, 0), [0,+\infty))$ is creasing in u and $f(t, G_\alpha(t), t^{2-\alpha}) \neq 0$ for any $G_\alpha(t) > 0$, then the CBVP (E_2-C_3) has at least one positive solution u, and there exist two positive constants $0 < a_1 < 1 < a_2$ such that $a_1 G_\alpha(t) < u(t) < a_2 G_\alpha(t)$.*

5.3.1 Example

We conclude this section with an example as an application of our discussion. Typical functions that satisfy the above sub-linear hypothesis are those taking the form

$$f(t, u, v) = \sum_{l}^{n} \sum_{k}^{m} Q_{l,k}(t) \, u^{\mu_k} v^{\mu_l},$$

here $Q_{l,k}(t) \in C(0,1)$, $Q_{l,k}(t) > 0$ on $(0,1)$, $\mu_k \in \mathbb{R}$, $\mu_l < 1$.

To obtain the approximate solutions of (E_2-C_3), the assumption of $f(t, u, v) \leq N_0(t) N_1(u) N_2(v)$ is usually needed. Now, we give one example to illustrate the above results.

Example 3 *Consider the following p-Laplacian conformable boundary value problem (E_2-C_3) where*

$$f(t, u, v) = t^{\mu_0} (1-t)^{\mu_1} u^{\mu_2} v^{\mu_3}, \ 0 < t < 1, \tag{5.123}$$

where $p > 1$, $\mu_0, \mu_1 \in \mathbb{R}$ and $\mu_2, \mu_3 > 0$, $\mu_2 + \mu_3 > p - 1$.

Clearly, f is nonincreasing relative to u. This shows that (H) holds. Theorem 7 implies that the CBVP (5.123) has at least one positive solution.

Conclusion 1 *A necessary and sufficient condition for problem (5.123) to have at least one $C^2[0,1]$ positive solution is*

$$(\beta - 1) + \mu_0 + (2-\alpha)\mu_3 + (\alpha - 1)\mu_2 > -1 \ \textit{and} \ \ \mu_1 > -2,$$

or

$$(\beta-1)+\mu_0+(2-\alpha)\mu_3+(\alpha-1)\mu_2>-1 \text{ and } \mu_1+\mu_2>-2.$$

5.4 Dependence of solution on the parameters

For f Lipschitz in the second and third variables, the solution's dependence on the order of the differential operator, the boundary values, and the nonlinear term f are also discussed.

In the following, suppose that (A) holds and for any $u \in X$, we let

$$(fu)(t) := f\left(t, u(t), -\mathbf{T}_{0+}^{\alpha} u(t)\right), (fu_\epsilon)(t) := f\left(t, u_\epsilon(t), -\mathbf{T}_{0+}^{\alpha} u_\epsilon(t)\right), \ t \in (0,1). \quad (5.124)$$

5.4.1 The dependence on parameters of the left-hand side of (E_2)

We show that the solutions of two equations with neighboring orders will (under suitable conditions on their right hand sides f) lie close to one another.

Theorem 9 *Suppose that the conditions of Theorem 7 hold. Let $u(t), u_\epsilon(t)$ be the solutions, respectively, of the problems (E_2-C_3) and*

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(T^{\alpha-\epsilon} u\right)\right)(t) = f\left(t, u(t), -\mathbf{T}_{0+}^{\alpha} u(t)\right), \ t \in (0,1), \epsilon > 0, \quad (5.125)$$

with the boundary conditions (C_3), where $1 < \alpha - \epsilon < \alpha \leq 2$. Then $\|u - u_\epsilon\| = \mathcal{O}(\epsilon)$, for ϵ sufficiently small.

Proof. By the above theorems, we can obtain the following results. Let

$$u_\epsilon(t) = \int_0^1 G_{\alpha\epsilon}(t,s)\varphi_q\left(\int_0^1 G_\beta(t,s)(fu_\epsilon)(t)\right)\mathrm{d}s. \quad (5.126)$$

be the solution of (E_2-C_3), where

$$G_{\alpha\epsilon}(t,s) = k_{\alpha-\epsilon}(t,s) + \frac{b_1 t}{1-b_1\xi_1} k_{\alpha-\epsilon}(\xi_1, s). \quad (5.127)$$

On one hand, from (5.27) and (5.126) yields

$$|u(t) - u_\epsilon(t)| = \left|\int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta(t,s)}(fu)\,\mathrm{d}s\right)\mathrm{d}t - \int_0^1 G_{\alpha\epsilon(t,s)}\varphi_q\left(\int_0^1 G_{\beta(t,s)}(fu_\epsilon)\,\mathrm{d}s\right)\mathrm{d}t\right| \quad (5.128)$$

$$= \left| \begin{array}{l} \int_0^1 G_{\alpha(t,s)}\varphi_q \left(\int_0^1 G_{\beta(t,s)} \left(fu\right)(t)\,\mathrm{d}s\right)\mathrm{d}t - \int_0^1 G_{\alpha(t,s)}\varphi_q \left(\int_0^1 G_{\beta(t,s)} \left(fu_\epsilon\right)(t)\,\mathrm{d}s\right)\mathrm{d}t \\ + \int_0^1 G_{\alpha(t,s)}\varphi_q \left(\int_0^1 G_{\beta(t,s)} \left(fu_\epsilon\right)(t)\,\mathrm{d}s\right)\mathrm{d}t - \int_0^1 G_{\alpha\epsilon(t,s)}\varphi_q \left(\int_0^1 G_{\beta(t,s)} \left(fu_\epsilon\right)(t)\,\mathrm{d}s\right)\mathrm{d}t \end{array} \right|$$

$$\begin{aligned} |u(t) - u_\epsilon(t)| \leq & \int_0^1 G_{\alpha(t,s)} \left| \varphi_q \left(\int_0^1 G_{\beta(t,s)} (fu)\,\mathrm{d}s \right) \mathrm{d}t - \varphi_q \left(\int_0^1 G_{\beta(t,s)} (fu_\epsilon) \right) \mathrm{d}s \right| \\ & + \int_0^1 \left| G_{\alpha(t,s)} - G_{\alpha\epsilon(t,s)} \right| \varphi_q \left(\int_0^1 G_{\beta(t,s)} (fu_\epsilon)(t)\,\mathrm{d}s \right) \mathrm{d}t. \end{aligned} \tag{5.129}$$

On the other hand, in a similar manner, we can get

$$\begin{aligned} & \left| \mathbf{T}_{0+}^{\alpha} u(t) - \mathbf{T}_{0+}^{\alpha} u_\epsilon(t) \right| \\ & = \left| t^{1-\alpha} \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)} \varphi_q \left(\int_0^1 G_{\beta(\tau,s)} (fu_\epsilon) \right) \mathrm{d}\tau - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)} \varphi_q \left(\int_0^1 G_{\beta(\tau,s)} (fu_\epsilon) \right) \mathrm{d}\tau \right) \right| \end{aligned}$$

$$= \left| \begin{array}{l} t^{1-\alpha} \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)}\varphi_q \left(\int_0^1 G_{\beta(\tau,s)} (fu_\epsilon) \right) \mathrm{d}\tau - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)}\varphi_q \left(\int_0^1 G_{\beta(\tau,s)} (fu_\epsilon) \right) \mathrm{d}\tau \right) \\ + t^{1-\alpha} \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)}\varphi_q \left(\int_0^1 G_{\beta(\tau,s)} (fu_\epsilon) \right) \mathrm{d}\tau - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha\epsilon(t,s)}\varphi_q \left(\int_0^1 G_{\beta(\tau,s)} (fu_\epsilon) \right) \mathrm{d}\tau \right) \end{array} \right|$$

$$\begin{aligned} & \left| \mathbf{T}_{0+}^{\alpha} u(t) - \mathbf{T}_{0+}^{\alpha} u_\epsilon(t) \right| \\ & \leq \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)} \left| \varphi_q \left(\int_0^1 G_{\beta(t,s)} (fu)(t)\,\mathrm{d}s \right) \mathrm{d}t - \varphi_q \left(\int_0^1 G_{\beta(t,s)} (fu_\epsilon)(t) \right) \mathrm{d}s \right| \\ & + \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)} - \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha\epsilon(t,s)} \right| \varphi_q \left(\int_0^1 G_{\beta(t,s)} (fu_\epsilon)(t)\,\mathrm{d}s \right) \mathrm{d}t. \end{aligned} \tag{5.130}$$

Moreover, from (5.129), (5.130), we have

$$\begin{aligned} \|u - u_\epsilon\| \leq & \varphi_q \left(\int_0^1 G_{\beta(t,s)} \mathrm{d}s \right) \\ & \times \left[\begin{array}{l} \int_0^1 \left| G_{\alpha(t,s)} + \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)} \right| \left| \varphi_q \left((fu)(t)\,\mathrm{d}s \right) \mathrm{d}t - \varphi_q \left((fu_\epsilon)(t) \right) \mathrm{d}s \right| \\ + \int_0^1 \left| \left(G_{\alpha(t,s)} - G_{\alpha\epsilon(t,s)} \right) + \left(\frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)} - \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha\epsilon(t,s)} \right) \right| \varphi_q \left((fu_\epsilon)\,\mathrm{d}s \right) \mathrm{d}t. \end{array} \right] \end{aligned} \tag{5.131}$$

From (5.26), that

$$\int_0^1 \mathcal{G}_\beta(t,s)\mathrm{d}s \leq \int_0^t (1-t)\, s^{\beta-1}\mathrm{d}s + \int_t^1 t\,(1-s)\, s^{\beta-2}\mathrm{d}s \leq \tfrac{1-t}{\beta} + \tfrac{t}{\beta(\beta-1)} \leq \tfrac{1}{\beta-1},$$

by using the analogous argument it holds that

$$\int_0^1 \left| G_{\alpha(t,s)} \right| \mathrm{d}s \leq \tfrac{1}{\alpha-1} \left(1 + \tfrac{b_1}{1-b_1\xi_1} \right) \text{ and } \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}t} G_{\alpha(t,s)} \right| \mathrm{d}s \leq \frac{1}{\alpha-1} \left(1 + \tfrac{b_1}{1-b_1\xi_1} \right), \tag{5.132}$$

similarly, it holds that

$$\int_0^1 \left|G_{\alpha(t,s)} - G_{\alpha\epsilon}(t,s)\right| \mathrm{d}s \leq \tfrac{\epsilon}{(\alpha-1)(\alpha-\epsilon-1)} \left(1 + \tfrac{b_1}{1-b_1\xi_1}\right)$$

and

$$\int_0^1 G_{\beta(t,s)} \mathrm{d}s \leq \tfrac{1}{\beta-1}\left[1 + \tfrac{b_0}{1-b_0\xi_2}\right] \text{ and } \int_0^1 G_{\beta(\xi_2,s)} \mathrm{d}s \leq \tfrac{1}{\beta-1}\left[1 + \tfrac{b_0\xi_2}{1-b_0\xi_2}\right]. \tag{5.133}$$

i) In case $1 < q \leq 2$ we apply (3.6). From (5.129),we have

$$\begin{aligned} &|\varphi_q((fu)(t))\,\mathrm{d}s - \varphi_q((fu_\epsilon)(t))| \\ &\leq 2^{2-q}|(fu)(t) - (fu_\epsilon)(t)|^{q-1} \\ &\leq 2^{2-q}L^{q-1}\|u-u_\epsilon\|^{q-1}, \end{aligned} \tag{5.134}$$

where

$$|(fu)(t) - (fu_\epsilon)(t)| \leq L\|u-u_\epsilon\|.$$

From (5.129), we get

$$\begin{aligned} \|u-u_\epsilon\| \leq{}& \tfrac{2^{2-q}L^{q-1}}{(\alpha-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)\left(\tfrac{1}{(\beta-1)}\left(1+\tfrac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\|u-u_\epsilon\|^{q-1} \\ &+ 2^{2-q}\left(\|f\|\tfrac{1}{(\beta-1)}\left(1+\tfrac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\tfrac{\epsilon}{(\alpha-1)(\alpha-\epsilon-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right). \end{aligned} \tag{5.135}$$

Consequently, we obtain

$$\|u-u_\epsilon\| \leq \frac{2^{2-q}\left(\frac{|||f|||}{(\beta-1)}\left(1+\frac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\left(1+\frac{b_1}{1-b_1\xi_1}\right)}{\left[1-\frac{2^{2-q}L^{q-1}}{(\alpha-1)}\left(1+\frac{b_1}{1-b_1\xi_1}\right)\left(\frac{1}{(\beta-1)}\left(1+\frac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\right]}\frac{\epsilon}{(\alpha-1)(\alpha-\epsilon-1)}, \tag{5.136}$$

where

$$|||f||| = \sup_{0<\epsilon<\alpha-1}\left\{\max\left|f\left(t, u_\epsilon, -\mathbf{T}^{\alpha}_{0+}u_\epsilon\right)\right| : t \in (0,1)\right\}$$

and

$$0 < \left[1 - \tfrac{2^{2-q}L^{q-1}}{(\alpha-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)\left(\tfrac{1}{(\beta-1)}\left(1+\tfrac{b_0}{1-b_0\xi_2}\right)\right)^{q-1}\right] \leq 1.$$

ii) In case $q > 2$, we apply (3.7).and the inequality (3.8), we obtain

$$|\varphi_q((fu)(t))\,\mathrm{d}s - \varphi_q((fu_\epsilon))| \leq (q-1)\left[|(fu)-(fu_\epsilon)| + |(fu_\epsilon)|^{q-2}(|(fu)-(fu_\epsilon)|)\right]$$

$$\begin{aligned} &\leq (q-1)\left[|(fu)(t)-(fu_\epsilon)(t)| + |(fu_\epsilon)(t)|^{q-2}(|(fu)(t)-(fu_\epsilon)(t)|)\right] \\ &\leq (q-1)\left[L\|u-u_\epsilon\| + \|f\|\right]^{q-2}(L\|u-u_\epsilon\|) \\ &\leq (q-1)\left(\lambda^{3-q}L^{q-2}\|u-u_\epsilon\|^{q-2} + \mu^{3-q}\|f\|^{q-2}\right)(L\|u-u_\epsilon\|). \end{aligned}$$

Hence

$$\left|\varphi_q\left((fu)(t)\right)\mathrm{d}s-\varphi_q\left((fu_\epsilon)(t)\right)\right|\leq(q-1)\left(\lambda^{3-q}L^{q-2}+\mu^{3-q}\left\|f\right\|^{q-2}\right)\left(L\left\|u-u_\epsilon\right\|\right).$$

Thus, by (5.129), we have

$$\begin{aligned}\left\|u-u_\epsilon\right\| &\leq C_1\left(\lambda^{3-q}L^{q-2}+\mu^{3-q}\left\|f\right\|^{q-2}\right)\\ &+\tfrac{\epsilon}{(\alpha-1)(\alpha-\epsilon-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)\left(\left\|f\right\|\tfrac{1}{(\beta-1)}\left(1+\tfrac{b_0}{1-b_0\xi_2}\right)\right)^{q-1},\end{aligned}\tag{5.137}$$

where

$$C_1=\tfrac{(q-1)}{(\alpha-1)}\left(1+\tfrac{b_1}{1-b_1\xi_1}\right)\left(\frac{1}{(\beta-1)}\left(1+\frac{b_0}{1-b_0\xi_2}\right)\right)^{q-2}.$$

Thus, in accordance with (5.136) and (5.137) we obtain $\left\|u-u_\epsilon\right\|=O\left(\epsilon\right).$ ■

Theorem 10 *Suppose that the conditions of Theorem 7 hold. Let $u\left(t\right),u_\epsilon\left(t\right)$ be the solutions, respectively, of the problems (E_2-C_3) and*

$$T_t^{\beta-\epsilon}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)(t)=f\left(t,u\left(t\right),-\mathbf{T}_{0+}^{\alpha}u\left(t\right)\right),\ t\in\left(0,1\right),\ \epsilon>0,\tag{5.138}$$

with the boundary conditions (C_3), where $1<\beta-\epsilon<\beta<2$. Then $\left\|u-u_\epsilon\right\|=O\left(\epsilon\right).$

Proof. Let

$$u_\epsilon\left(t\right)=\int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta\epsilon(t,s)}\left(fu_\epsilon\right)(t)\right)\mathrm{d}s,\tag{5.139}$$

be the solution of (E_2-C_3), where

$$G_{\beta\epsilon}(t,s)=k_{\beta-\epsilon}(t,s)+\frac{b_0t}{1-b_0\xi_2}k_{\beta-\epsilon}(\xi_2,s),\ b_0=b_2^{p-1}.\tag{5.140}$$

Then (5.139) and (5.140) yields

Observing that

$$\int_0^1 G_{\beta(t,s)}\mathrm{d}s\leq\tfrac{1}{\beta-1}\left[1+\tfrac{b_0}{1-b_0\xi_2}\right]\text{ and }\int_0^1 G_{\beta(\xi_2,s)}\mathrm{d}s\leq\tfrac{1}{\beta-1}\left[1+\tfrac{b_0\xi_2}{1-b_0\xi_2}\right].\tag{5.141}$$

Similarly of Theorem 9, we also have $\left\|u-u_\epsilon\right\|=O\left(\epsilon\right).$ ■

5.4.2 The dependence on parameters of initial conditions

Let us introduce small changes in the initial conditions of (E$_2$-C$_3$) and consider (E$_2$) with boundary conditions

$$u(0)=0,\ u(1)=b_1u(\xi_1),T_{0+}^{\alpha-\epsilon}u(0)=0,\ T_{0+}^{\alpha-\epsilon}u(1)=b_2T_{0+}^{\alpha-\epsilon}u(\xi_2),\ 1<\alpha-\epsilon<\alpha<2. \tag{5.142}$$

Theorem 11 *Assume the conditions of Theorem 7 hold. Let $u(t), u_\epsilon(t)$ be respective solutions, of the problems (E$_2$-C$_3$) and the boundary conditions (E$_2$-5.142). Then $\|u-u_\epsilon\| = O(\epsilon)$.*

Proof. Let

$$u_\epsilon(t)=\int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta\epsilon}(t,s)(fu_\epsilon)(t)\right)\mathrm{d}s, \tag{5.143}$$

solutions of the problem (E$_2$-5.142), where

$$G_{\beta\epsilon}(t,s)=\mathcal{G}_\beta(t,s)+\tfrac{b_0t}{1-b_0\xi_2}\mathcal{G}_\beta(\xi_2,s), b_0=\varphi_p\left(b_2\xi_2^\epsilon\right). \tag{5.144}$$

It is easy to see that $\|u-u_\epsilon\|=O(\epsilon)$. ■

5.4.3 The dependence on parameters of the right-hand side of (E$_2$)

Theorem 12 *Suppose that the conditions of Theorem 7 hold. Let $u_1(t), u_2(t)$ be the solutions, respectively, of the problems (E$_2$-C$_3$) and*

$$\mathbf{T}_{0+}^{\beta}\left(\varphi_p\left(\mathbf{T}_{0+}^{\alpha}u\right)\right)(t)=f\left(t,u(t),-\mathbf{T}_{0+}^{\alpha}u(t)\right)+\epsilon,\ t\in(0,1), \tag{5.145}$$

with boundary conditions (C$_3$), where $1<\alpha\le 2$. Then $\|u-u_\epsilon\|=O(\epsilon)$.

Proof. In accordance with Lemma 26, we have

$$u_\epsilon(t)=\int_0^1 G_{\alpha(t,s)}\varphi_q\left(\int_0^1 G_{\beta(t,s)}\left[(fu_\epsilon)(t)+\epsilon\right]\mathrm{d}s\right)\mathrm{d}t. \tag{5.146}$$

i) In case $1<q\le 2$ we apply (3.6). From (5.129), we have

$$\begin{aligned}
&|\varphi_q((fu)(t))-\varphi_q((fu_\epsilon)(t)+\epsilon)|\\
&\qquad\le 2^{2-q}\left(|(fu)(t)-(fu_\epsilon)(t)|+\epsilon\right)^{q-1}\\
&\qquad\le 2^{2-q}\left(\lambda^{2-q}L^{q-1}\|u-u_\epsilon\|^{q-1}+\mu^{2-q}\epsilon^{q-1}\right).
\end{aligned} \tag{5.147}$$

Therefore, by taking

$$R_{01} = (q-1)\left[(\lambda_1\mu_1)^{q-3} + (\lambda_1\mu_2)^{q-3}\epsilon^{q-2} + \lambda_2^{q-3}\epsilon + \lambda_2^{q-3}\left|(fu_\epsilon)(t)\right|\right],$$
$$R_{02} = (q-1)\left[(\lambda_1\mu_1)^{q-3} + (\lambda_1\mu_2)^{q-3}\epsilon^{q-2} + \lambda_2^{q-3}\epsilon + \lambda_2^{q-3}\left|(fu_\epsilon)(t)\right|\right],$$

we get

$$\left|\varphi_q\left((fu)(t)\right) - \varphi_q\left((fu_\epsilon)(t)+\epsilon\right)\right| \le R_{01}\left\|u-u_\epsilon\right\| + R_{02}.$$

ii) When $q > 2$, we apply (3.7), we have

$$\begin{aligned}
&\left|\varphi_q\left((fu)(t)\right) - \varphi_q\left((fu_\epsilon)(t)+\epsilon\right)\right| \\
&\le (q-1)\left(\left|(fu)-(fu_\epsilon)-\epsilon\right| + \left|(fu_\epsilon)+\epsilon\right|\right)^{q-2} \times \left(\left|(fu)-(fu_\epsilon)-\epsilon\right|\right) \\
&\le (q-1)\left(\lambda^{3-q}\left|(fu)-(fu_\epsilon)-\epsilon\right|^{q-2} + \mu^{3-q}\left|(fu_\epsilon)+\epsilon\right|^{q-2}\right) \times \left(\left|(fu)-(fu_\epsilon)-\epsilon\right|\right) \\
&\le (q-1)\left(\lambda^{3-q}\left|(fu)-(fu_\epsilon)-\epsilon\right|^{q-2} + \mu^{3-q}\left|(fu_\epsilon)+\epsilon\right|^{q-2}\right) \times \left(\left|(fu)-(fu_\epsilon)-\epsilon\right|\right) \\
&\qquad \le R_{11}\left\|u-u_\epsilon\right\| + R_{12}.
\end{aligned} \tag{5.148}$$

Therefore, by taking

$$R_{11} = (q-1)\left[(\lambda_1\mu_1)^{q-3}(1+\epsilon) + (\lambda_1\mu_2)^{q-3}\epsilon^{q-2} + \lambda_2^{q-3}\epsilon + \lambda_2^{q-3}\left|(fu_\epsilon)(t)\right|\right]$$
$$R_{12} = (q-1)\left[(\lambda_1\mu_2)^{q-3}\epsilon^{q-1} + \lambda_2^{q-3}\epsilon^2 + \lambda_2^{q-3}\left|(fu_\epsilon)(t)\right|\epsilon\right],$$

we get

$$\left|\varphi_q\left((fu)(t)\right) - \varphi_q\left((fu_\epsilon)(t)+\epsilon\right)\right| \le R_{11}\left\|u-u_\epsilon\right\| + R_{12}.$$

It is easy to see that $\left\|u-u_\epsilon\right\| = O(\epsilon)$. ■

Remark 11 *The contents of this chapter is published in the form of single paper as mentioned below:*

- Nonlinear singular p-Laplacian boundary value problems in the frame of conformable derivative Discrete & Continuous Dynamical Systems - S doi: 10.3934/dcdss.2020442

CONCLUSION AND FUTURE WORK

Contents

Conclusion

The p- Laplacian equation has been proposed to describe non-Newtonian fluid theory, nonlinear elastic mechanics, and so forth. However, it has been realized that the classical PLE failed to describe such a complex systems. Thus, the consideration of PLE in the frame of fractional derivatives became compulsory. As a result of this interest, several results have been revealed and different versions of PLE have been under study.

Throughout this doctoral dissertation, mathematical analysis of generalized PLEs with CFD was presented. Techniques used here showcased the well-posedness of the extended PLE with CFD. Illustrative examples of applications validated this extension.

We have considered the following nonlinear fractional differential equations (E_1) and (E_2) subjected to different boundary conditions (C_1), (C_2) and (C_3) respectively, we plan to study the same question such as existence, uniqueness and the dependence of the solution in the case where the derivatives are of type LFD and NLFD in the sense of Caputo.

Firstly, we provided a literature review on fractional calculus. Basic fractional derivatives such as Riemann–Liouville, Caputo, Hadamard, Grünwald–Letnikov and CFD were presented. Related properties governing these derivatives were also presented. The concept of fixed point theorems, upper and lower solutions were introduced.

Secondly, the concept of well-posedness as applied to generalized PLEs has been elaborated in this thesis. This included the demonstration of existence, uniqueness and the dependence of the solution of the CFE and existence of the CFE in the sense of Caputo types of PLEs. Under necessary and sufficient conditions proceeding from various definitions and theorems, the well-posedness of FPLE with CFD was established.

Thirdly, the generalized PLE was defined with the CFD and its well-posedness investigated. It was demonstrated that the generalized PLE admits a solution by proving the existence and uniqueness. Analysis results in each applications were provided.

Lastly, the mathematical analysis conducted in this thesis has demonstrated that PLEs are certainly extendable with the new CFD. The results obtained in Chapters 4 and 5 justified that a generalized PLE has a solution, which is unique with respect to associated parameters.

In Chapter 4, the FBVP for PLE at resonance is investigated. In view of the FBVP (4.85-

4.86) is equivalent to the operator equation (4.94); we only need to find a fixed point of the operator equation (4.94). Firstly, we established the sufficient conditions of existence of FBVP for p-Laplacian equation. Then, by using the extension of Mawhin's continuation theorem due to Ge, we got the fixed point of operator equation (4.94). This result extends many existent results and generalizes many related problems in the literature.

In Chapter 5, we have studied a class of fourth point singular BVP of p-Laplacian operator in the setting of a LFD, namely a newly defined CD. By using the upper and lower solutions method and FPTs on cones, necessary and sufficient conditions for the existence of positive solutions were obtained. We present an example to demonstrate the consistency to the theoretical findings. We have also investigated the continuous dependence of solutions all on its right side function, initial value condition. Using these results, the properties of the solution process can be discussed through numerical simulation.

We claim that the results of this thesis is new and generalize some earlier results. For example, by taking $p = \alpha = \beta = 2$ and $b_2 = 0$, in the results of chapiter 5 which can be considered a special case of a simple Jerk Chaotic circuit equation see [107].

Reported results in this thesis can be considered as a promising contribution to the theory of fractional integral equations. These results can be used to study and develop further quantitative and qualitative properties of generalized fractional differential equations.

Future work

Although it was shown that an application of a nonlinear p-Laplacian equation also complied with the analysis approach presented for PLE, one could explore the concept of well-posedness for generalized nonlinear PLEs. Since the proposed analysis only assumed for linear operators, then an investigation of the well-posedness for non-homogeneous fractional p- Laplacian with CFD can be subject to future work.

Results obtained in this thesis can be considered as a contribution to the developing field of FC with generalized fractional derivative operators. We also remark that the extension of the previous results to the nonlinearity depending on the time delayed differential system or inclusion differential equation taking into account that sometimes the corresponding research when the FD with non-singular kernels are considered is interesting, in the future work, we will focus our concentration on the Caputo-Fabrizio derivative, Atangana-Baleanu, fractional derivatives of a function with respect to another function and try to mix idea of this work with q-fractional derivatives.

Also, the reader can find some new methods for approximate solutions of fractional integro-differential equations involving the Caputo-Fabrizio derivative or extended fractional Caputo-Fabrizio derivative. The approximation solutions are interesting and need more concentration.

At the end of this work we also anticipate that the methods and concepts here can be extended to the systems with economic processes such as risk model, the CIR model, and the Gaussian model. We hope to consider these problems in future works.

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