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Neighbourhood Star Selection Properties in Bitopological Spaces

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Abstract. In this paper we introduce and study some new types of star-selection principles ((i, j)-NSM, (i, j)-NSR and (i, j)-NSH) in bitopologivcal spaces. Various properties of these selection properties are established and their relations with known selection principles are discussed. Several examples are given.

1. Introduction

Selection principles theory is one of most active research areas of topology in the last two-three decades. Classical concepts and results in this theory appeared in 1920s and 1930s years in works by Menger, Hurewicz and Rothberger. A systematic study in this field began in 1996 by Scheepers [24]. In 1999, Kočinac [14] introduced star selection principles, and (under different name) neighbourhood star selection principles [15] which have been studied in details in [2]. In this paper we extend this investigation and introduce and study neighbourhood star selection (covering) properties in bitopological spaces and so complement research in bitopological context. Let us mention that bitopological selection principles have been discussed in a number of papers [16, 17, 20–22].

The paper is organized in the following way. After this short introduction, in Section 2 we give necessary information about selection principles and bitopological spaces. In Section 3 we consider neighbourhood star-Lindelöf bitopological spaces, and in Section 4 we introduce neighbourhood star selection properties, which are the main subject of our article, and study neighbourhood star-Menger, star-Hurewicz and star-Rorhberger properties in bitopological context. Their behavior under known topological operations and constructions are discussed. In Section 5 we investigate weaker forms of neighbourhood star selection properties. In particular, we discuss preservation of these properties under certain kinds of mappings.

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and the set of real numbers. Let X be a topological space, $\mathcal{U}$ a collection of subsets of X, $A \subset X$. Then $\bigcup \mathcal{U} = \bigcup\{U : U \in \mathcal{U}\}$. The set $\mathrm{St}(A, \mathcal{U}) := \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ is called the *star* of A with respect to $\mathcal{U}$. If $x \in X$, we write $\mathrm{St}(x, \mathcal{U})$

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instead of $\mathrm{St}(\{x\}, \mathcal{U})$. If $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of a space X, then the symbol $\mathcal{B} < \mathcal{A}$ denotes the fact that for each $B \in \mathcal{B}$ there is $A \in \mathcal{A}$ with $B \subset A$.

Our notation and terminology follow [6] (for topological spaces), [5] (for bitopological spaces), [4] (for star covering properties).

A. Selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of sets (in this paper they will be mainly collections of covers of a (bi)topological space X). Then the symbol $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(a_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $a_n \in A_n$ and $\{a_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The symbol $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis that for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(B_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, B_n is a finite subset of A_n and $\bigcup_{n\in\mathbb{N}} B_n$ is an element of $\mathcal{B}$ ([24]).

In [14] (see also [15]), Kočinac introduced star selection hypothesis similar to the previous ones. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of a space X. Then:

(1) The symbol $\mathsf{S}_{\mathrm{fin}}{}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(V, \mathcal{U}_n) : V \in \mathcal{V}_n)\}$ is an element of $\mathcal{B}$.

(2) The symbol $\mathsf{SS}^*_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(F_n : n \in \mathbb{N})$ of finite) subsets of X such that $\{\mathrm{St}(F_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

(3) The symbol $\mathsf{S}_1{}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $\bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n, \mathcal{U}_n)$ is an element of $\mathcal{B}$.

(4) The symbol $\mathsf{SS}^*_1(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis: for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there exists a sequence $(x_n : n \in \mathbb{N})$ of elements of X such that $\{\mathrm{St}(x_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

Let $\mathcal{O}$ denote the collection of all open covers of a space X.

Definition 2.1. ([14]) A space X is said to be *star-Menger* [resp., *star-Rothberger*] if it satisfies the selection hypothesis $\mathsf{S}_{\mathrm{fin}}{}^*(\mathcal{O}, \mathcal{O})$ [resp., $\mathsf{S}_1{}^*(\mathcal{O}, \mathcal{O})$]. X is *strongly star-Menger* (resp., *strongly star-Rothberger*) if it satisfies $\mathsf{SS}^*_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ (resp., $\mathsf{SS}^*_1(\mathcal{O}, \mathcal{O})$).

Definition 2.2. ([14], [1]) A space X is said to be *star-Hurewicz* (resp., *strongly star-Hurewicz*) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is for each n a finite set $\mathcal{V}_n \subset \mathcal{U}_n$ (resp., a finite $F_n \subset X$) so that each $x \in X$ belongs to $\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ (resp., $\mathrm{St}(F_n, \mathcal{U}_n)$) for all but finitely many n.

The following three generalizations of star selection properties have been introduced (in a general form and under different names) in [15] and studied in details in [2].

Definition 2.3. ([2]) A space X is said to be *neighbourhood star-Menger* (NSM) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, one can choose finite sets $F_n \subset X$, $n \in \mathbb{N}$, so that for every open set $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\bigcup_n\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\} = X$.

Definition 2.4. ([2]) A space X is said to be *neighbourhood star-Rothberger*(NSR) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, one can choose a sequence $(x_n \in X : n \in \mathbb{N})$ so that for every open $O_n \ni x_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n) = X$.

Definition 2.5. ([2]) A space X is said to be *neighbourhood star-Hurewicz* (NSH) if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, so that for every open $O_n \supset F_n$, $n \in \mathbb{N}$, each $x \in X$ belongs to $\mathrm{St}(O_n, \mathcal{U}_n)$ for all but finitely many n.

For investigation of star selection principles related to this paper see also [25–27].

In this article we define and study neighbourhood star selection properties in bitopological spaces.

B. Bitopological spaces. A set X endowed with two, in general unrelated, topologies τ_1 and τ_2 is called a *bitopological space* (or shortly, *bispace*) and is denoted by (X, τ_1, τ_2) (and sometimes simply by X).

For a subset A of X, $\mathrm{Cl}_{\tau_i}(A)$ and $\mathrm{Int}_{\tau_i}(A)$ will denote the closure of A and the interior of A in (X, τ_i), $i = 1, 2$, respectively. If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a mapping between bispaces, then f_i denotes the mapping $f : (X, \tau_i) \to (Y, \sigma_i)$. If $\mathcal{P}$ is a topological property, then a bispace (X, τ_1, τ_2) is *double* $\mathcal{P}$ (or shortly d-$\mathcal{P}$) if both (X, τ_1) and (X, τ_2) have the property $\mathcal{P}$. For example, in [3], d-separability has been defined in this way.

Definition 2.6. ([23]) Let (X, τ_1, τ_2) be a bispace. A subset A of X is said to be:

1. *open* in X if it is both τ_1-open and τ_2-open;
2. *closed* in X if it is τ_1-closed and τ_2-closed;
3. *(i, j)-clopen* if A is τ_i-closed and τ_j-open; F is *clopen* if it is both (i, j)-clopen and (j, i)-clopen in X.

Selective properties in bitopological spaces have been studied in [17, 20, 21], and for weak covering properties in the bitopological context the study began with the paper [22] on the almost Menger property and continued in [7, 8].

3. About (i, j)-NSL bispaces

In this section we consider the class of neighbourhood star-Lindelöf bispaces which is strongly related to the main topic of this article.

Definition 3.1. A bitopological space (X, τ_1, τ_2) is said to be:
(1) *(i, j)-neighbourhood star-Lindelöf* (shortly, (i, j)-NSL), $i, j = 1, 2$, if for every τ_i-open cover $\mathcal{U}$ of X one can choose a countable set $F \subset X$ so that for every τ_j-open set $O \supset F$, we have $\mathrm{St}(O, \mathcal{U}) = X$;
(2) *(i, j)-weakly neighbourhood star-Lindelöf* (shortly, (i, j)-wNSL), $i, j = 1, 2$, if for every τ_i-open cover $\mathcal{U}$ of X one can choose a countable set $F \subset X$ so that for every τ_j-open set $O \supset F$, we have $\mathrm{Cl}_{\tau_j}(\mathrm{St}(O, \mathcal{U})) = X$.

The proof of the following theorem is omitted because it is similar to the proof of Theorem 4.7 below.

Theorem 3.2. *A bispace X is (i, j)-NSL, $i, j = 1, 2$, if and only if for every τ_i-open cover $\mathcal{U}$ of X there is a countable $F \subset X$ such that for every $x \in X$ we have $\mathrm{Cl}_{\tau_j}(\mathrm{St}(\{x\}, \mathcal{U})) \bigcap F \neq \emptyset$.*

Example 3.3. Endow the real line $\mathbb{R}$ by the two topologies: τ_1 is the usual topology on $\mathbb{R}$, and τ_2 is the Sorgenfrey topology [6]. Then the bispace $(\mathbb{R}, \tau_1, \tau_2)$ is $(1, 2)$-NSL. Indeed, let $\mathcal{U}$ be an open cover of $(\mathbb{R}, \tau_1)$. Then $\mathcal{U}$ is also an open cover of $(\mathbb{R}, \tau_2)$. Since $(\mathbb{R}, \tau_2)$ is separable, for any countable set A dense in $(\mathbb{R}, \tau_2)$ we have $\mathbb{R} = \mathrm{St}(A, \mathcal{U})$. Then, clearly, for any τ_2-neighbourhood O of A it holds $\mathrm{St}(O, \mathcal{U}) = \mathbb{R}$ which means that $(\mathbb{R}, \tau_1, \tau_2)$ is $(1, 2)$-NSL.

Remark 3.4. The bispace in the previous example is also (2,1)-NSL. Let $\mathcal{V}$ be a τ_2-open cover of $\mathbb{R}$. Take a dense countable subset C of $(\mathbb{R}, \tau_2)$. Then $\mathrm{St}(C, \mathcal{V}) = \mathbb{R}$. Clearly, for any τ_1-open set O containing C, it holds $\mathbb{R} = \mathrm{St}(O, \mathcal{V})$, which means that $(\mathbb{R}, \tau_1, \tau_2)$ is (2,1)-NSL.

We prove now a few properties of (i, j)-NSL bispaces.

Theorem 3.5. *Every clopen subspace $(Y, \tau_1|Y, \tau_2|Y)$ of an (i, j)-NSL bispace (X, τ_1, τ_2) is also (i, j)-NSL.*

Proof. Let $\mathcal{U}$ be a $\tau_i|Y$-open cover of Y. As Y is τ_i-closed, $\mathcal{V} = \mathcal{U} \cup (X \setminus Y)$ is a τ_i-open cover of X, and since X is (i, j)-NSL, there is a countable set $A \subset X$ such that for every τ_j-open neighbourhood O of A, $\mathrm{St}(O, \mathcal{V}) = X$. The set $B = Y \cap A$ is a countable subset of Y. Take any $\tau_j|Y$-open set G with $G \supset B$. As Y is clopen, the set $H = G \cup (X \setminus Y)$ is a τ_j-open set containing A so that $\mathrm{St}(H, \mathcal{V}) = X$. Since $G \cap (X \setminus Y) = \emptyset$ one concludes that $\mathrm{St}(G, \mathcal{U}) = Y$, i.e. $(Y, \tau_1|Y, \tau_2|Y)$ is (i, j)-NSL. □

Definition 3.6. Let Y be a subspace of a bispace (X, τ_1, τ_2). Then Y is *relatively* (i, j)-NSL in X if for each τ_i-open cover $\mathcal{U}$ of X, there is a countable set $F \subset X$, such that for every τ_j-open $O \supset F$ we have $Y \subset \mathrm{St}(O, \mathcal{U})$.

Proposition 3.7. *If $X = \bigcup\{Y_k : k \in \mathbb{N}\}$ and every Y_k is relatively (i, j)-NSL in X, then X is (i, j)-NSL.*

Proof. Let $\mathcal{U}$ be a τ_i-open cover of X. Then for each $k \in \mathbb{N}$, $\mathcal{U}$ covers Y_k, and since Y_k is relatively (i,j)-NSL, there is a countable set $F_k \subset X$, such that for each τ_j-open set $O \supset F_k$ we have $\mathrm{St}(O,\mathcal{U}) \supset Y_k$. Put $F = \bigcup_{k\in\mathbb{N}} F_k$. Then F is a countable subset of X. Let O be any τ_j-open set containing F. Using the fact that O contains all F_k, $k \in \mathbb{N}$, we conclude that $\mathrm{St}(O,\mathcal{U}) \supset \bigcup_{k\in\mathbb{N}} Y_k = X$, which means that X is (i,j)-NSL. □

In the literature there is the following wrong definition of (i,j)-Lindelöf bispaces: a bispace (X,τ_1,τ_2) is (i,j)-Lindelöf if for each τ_i-open cover $\mathcal{U}$ of X there is a countable τ_j-subcover. We give another definition.

Definition 3.8. A bitopological space (X,τ_1,τ_2) is called (i,j)-*Lindelöf* if for every τ_i-open cover of X there is a countable τ_j-open refinement.

Definition 3.9. A bispace (X,τ_1,τ_2) is said to be:

(1) (i,j)-*para-Lindelöf* if each τ_i-open cover of X has a τ_i-open refinement which is τ_j-locally countable [5];

(2) (i,j)-*weakly Lindelöf* if each τ_i-open cover $\mathcal{U}$ of X has a τ_i-open countable collection $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$ and $\mathrm{Cl}_{\tau_j}(\cup\mathcal{V}) = X$.

Theorem 3.10. *If (X,τ_1,τ_2) is an (i,j)-para-Lindelöf (i,j)-NSL bispace, then (X,τ_i) is Lindelöf.*

Proof. Let $\mathcal{U}$ be a τ_i-open cover of X. As X is (i,j)-para-Lindelöf, there is a τ_i-open refinement $\mathcal{V}$ of $\mathcal{U}$ so that $\mathcal{V}$ is τ_j-locally countable. Since X is (i,j)-NSL, there is a countable set $A \subset X$ such that for each τ_j-open set $G \supset A$ we have $\mathrm{St}(G,\mathcal{V}) = X$. For each $a \in A$ choose a τ_j-open set W_a intersecting the most countably many elements in $\mathcal{V}$. Then $G(A) = \cup\{W_a : a \in A\}$ is a τ_j-open neighbourhood of A and thus $\mathrm{St}(G(A),\mathcal{V}) = X$. The set $\mathcal{V}' = \{V \in \mathcal{V} : V \cap G(A) \neq \emptyset\}$ is a countable subset of $\mathcal{V}$ and satisfies $\cup\{V : V \in \mathcal{V}'\} = X$. For each $V \in \mathcal{V}'$ pick a set $U(V) \in \mathcal{U}$ containing V. Then the subset $\mathcal{U}' = \{U(V) : V \in \mathcal{V}'\}$ witnesses for $\mathcal{U}$ that (X,τ_i) is Lindelöf. □

Theorem 3.11. *Every (i,j)-para-Lindelöf, (i,j)-wNSL bispace is (i,j)-weakly Lindelöf.*

Proof. Let $\mathcal{U}$ be a τ_i-open cover of X. There exists a τ_j-locally countable τ_i-open refinement $\mathcal{V}$ of $\mathcal{U}$. For each $x \in X$, there exists a τ_j-open neighbourhood W_x of x such that $\{V \in \mathcal{V} : W_x \cap V \neq \emptyset\}$ is countable.

Since X is (i,j)-wNSL, there exists a countable subset A of X such that for every τ_j-open set $O \supset A$, $X = \mathrm{Cl}_{\tau_j}(\mathrm{St}(O,\mathcal{V}))$. Especially, it is true for τ_j-open set $O_A = \bigcup\{W_x : x \in A\} \supset A$, i.e. $\mathrm{Cl}_{\tau_j}(\mathrm{St}(O_A,\mathcal{V})) = X$. Set $\tilde{\mathcal{V}} = \{V \in \mathcal{V} : V \cap O_A \neq \emptyset\}$. Then $\tilde{\mathcal{V}}$ is a countable subset of $\mathcal{V}$, and clearly we have $\mathrm{Cl}_{\tau_j}(\bigcup\tilde{\mathcal{V}}) = X$.

For each $V \in \tilde{\mathcal{V}}$, choose $U_V \in \mathcal{U}$ with $V \subseteq U_V$. Then $\{U_V : V \in \tilde{\mathcal{V}}\}$ is a countable subcover of $\mathcal{U}$, and $X = \mathrm{Cl}_{\tau_j}(\bigcup_{V\in\tilde{\mathcal{V}}} U_V)$ which shows that X is (i,j)-weakly Lindelöf. □

Definition 3.12. ([5]) A mapping $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ between bispaces is said to be *double continuous* (shortly *d-continuous*) if the induced mappings $f_i : (X,\tau_i) \to (Y,\sigma_i)$ are continuous for $i = 1,2$.

Theorem 3.13. *Let (X,τ_1,τ_2) be an (i,j)-NSL bispace and let (Y,σ_1,σ_2) be a bispace. If $f : X \to Y$ is a d-continuous surjection, then Y is an (i,j)-NSL bispace.*

Proof. Let $\mathcal{V}$ be a σ_i-open cover of Y. Then $\mathcal{U} = f^{\leftarrow}(\mathcal{V}) = \{f^{\leftarrow}(V) : V \in \mathcal{V}\}$ is a τ_i-open cover of X. Since X is (i,j)-NSL, there is a countable $F \subset X$ such that for each τ_j-open O containing F we have $X = \mathrm{St}(O,\mathcal{U})$. Let $K = f(F)$ and let G be a σ_j-open neighbourhood of K. Then $f^{\leftarrow}(G)$ is a τ_j-open neighbourhood of F so that $X = \mathrm{St}(f^{\leftarrow}(G),\mathcal{U})$. We prove $Y = \mathrm{St}(G,\mathcal{V})$.

Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. Then $x \in \mathrm{St}(f^{\leftarrow}(G),\mathcal{U})$. It follows

$$y = f(x) \in f(\mathrm{St}(f^{\leftarrow}(G),\mathcal{U})) \subset \mathrm{St}(G,\mathcal{V}).$$

Therefore, K and G witness for $\mathcal{V}$ that Y is (i,j)-NSL. □

Proposition 3.14. *If (X,τ_1,τ_2) is a bispace such that $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(Y_k)$ and each Y_k is relatively (i,j)-NSL in X, then X is (i,j)-wNSL.*

Proof. Let $\mathcal{U}$ be a τ_i-open cover of X. Each Y_k is covered by $\mathcal{U}$. As Y_k is relatively (i,j)-NSL in X, there is for each $k \in \mathbb{N}$ a countable $F_k \subset X$ such that for each τ_j-open O containing F_k, $Y_k \subset \mathrm{St}(O,\mathcal{U})$. Let $F = \bigcup_{k\in\mathbb{N}} F_k$ and let G be a τ_j-open set containing F. Then

$$X = \bigcup_{k\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(Y_k) \subset \mathrm{Cl}_{\tau_j}(\mathrm{St}(G,\mathcal{U}))$$

i.e. X is (i,j)-wNSL. □

Theorem 3.15. *Let a bispace (X,τ_1,τ_2) be (i,j)-wNSL and let (Y,σ_1,σ_2) be a bispace. If $f : X \to Y$ is a d-continuous surjection, then Y is also (i,j)-wNSL.*

Proof. Let $\mathcal{V}$ be a σ_i-open cover of Y. Then $\mathcal{U} = f^{\leftarrow}(\mathcal{V}) = \{f^{\leftarrow}(V) : V \in \mathcal{V}\}$ is a τ_i-open cover of X. Since X is (i,j)-wNSL, there is a countable $F \subset X$ such that for each τ_j-open O containing F we have $X = \mathrm{Cl}_{\tau_j}(\mathrm{St}(O,\mathcal{U}))$. Let $K = f(F)$ and let G be a σ_j-open neighbourhood of K. Then $f^{\leftarrow}(G)$ is a τ_j-open neighbourhood of F so that $X = \mathrm{Cl}_{\tau_j}(\mathrm{St}(f^{\leftarrow}(G),\mathcal{U}))$. We prove $Y = \mathrm{Cl}_{\tau_j}(\mathrm{St}(G,\mathcal{V}))$.

Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. Then $x \in \mathrm{Cl}_{\tau_j}(\mathrm{St}(f^{\leftarrow}(G),\mathcal{U}))$. It follows,

$$y = f(x) \in \mathrm{Cl}_{\tau_j}(f(\mathrm{St}(f^{\leftarrow}(G),\mathcal{U}))) \subset \mathrm{Cl}_{\tau_j}(\mathrm{St}(G,\mathcal{V})).$$

Therefore, K and G witness for $\mathcal{V}$ that Y is (i,j)-wNSL. □

4. Neighbourhood star selection principles in bispaces

In this section we introduce and study (i,j)-NSM, (i,j)-NSR and (i,j)-NSH bitopological spaces.

Definition 4.1. A bitopological space (X,τ_1,τ_2) is said to be:

(1) *(i,j)-neighbourhood star-Menger* (shortly, (i,j)-NSM), $i,j = 1,2$, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X one can choose finite sets $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open set $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(O_n,\mathcal{U}_n)\} = X$;

(2) *(i,j)-neighbourhood star-Rothberger* (shortly, (i,j)-NSR), $i,j = 1,2$, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, one can choose $x_n \in X$, $n \in \mathbb{N}$, so that for every τ_j-open set $O_n \supset x_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(O_n,\mathcal{U}_n)\} = X$;

(3) *(i,j)-neighbourhood star-Hurewicz* (shortly, (i,j)-NSH), $i,j = 1,2$, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open set $O_n \supset F_n$, $n \in \mathbb{N}$, each $x \in X$ belongs to $\mathrm{St}(O_n,\mathcal{U}_n)$ for all but finitely many n.

Remark 4.2. Of course, every (i,j)-NSR and every (i,j)-NSH bispace is (i,j)-NSM, and every (i,j)-NSM bispace is (i,j)-NSL.

The following proposition is evident (from the definitions), but useful for the following examples.

Proposition 4.3. *If (X,τ_1,τ_2) is a bispace such that $\tau_1 \leq \tau_2$, then:*

(1) *If (X,τ_2) is* NSM *(resp.,* NSH, NSR*), then (X,τ_1,τ_2) is $(1,2)$-*NSM *(resp., $(1,2)$-*NSH*, $(1,2)$-*NSR*).*

(2) *If (X,τ_1,τ_2) is $(1,2)$-*NSM *(resp., $(1,2)$-*NSH*, $(1,2)$-*NSR*), then (X,τ_1) is* NSM *(resp.,* NSH, NSR*).*

Example 4.4. Let τ_1 be the cofinite topology on $\mathbb{R}$ and τ_2 the usual metric topology on $\mathbb{R}$. Then $\tau_1 \leq \tau_2$ and $(\mathbb{R},\tau_2)$ is an NSH bispace. Therefore, by Proposition 4.3, $(\mathbb{R},\tau_1,\tau_2)$ is $(1,2)$-NSH and thus $(1,2)$-NSM.

Recall that $\mathfrak{d}$, $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{M})$ denote the following small combinatorial cardinals: the dominating number, the unbounded number, and the minimal cardinality of a cover of the real line by meager sets.

We have the following consistent examples.

Example 4.5. Endow the real line $\mathbb{R}$ with the usual metric topology. Let Y be the subspace of $\mathbb{R}$ such that $|Y \cap U| = \omega_1$ for each open set U in $\mathbb{R}$, and let $[0, \omega]$ be the ordinal space. Consider the space $X = Y \times [0, \omega]$ with the following two topologies:

(i) τ_1 is the product topology.

(ii) τ_2 is the topology in which a basic neighbourhood of a point $< x, n >$, $x \in \mathbb{Y}$, $n < \omega$, is of the form $((Y \cap U) \setminus C) \times \{n\}$, where U is a neighbourhood of x in Y, and C is a countable set with $x \notin C$, while a basic neighbouhood of a point $< x, \omega >$, is of the form $((Y \cap U) \setminus C) \times (n, \omega) \cup \{< x, \omega >\}$. Notice that $\tau_1 \leq \tau_2$.

1. It is proved in [2] that under assumption $\omega_1 < \mathfrak{d}$, the space (X, τ_2) is NSM. By Proposition 4.3, the bispace (X, τ_1, τ_2) is $(1, 2)$-NSM.

2. Under assumption $\omega_1 < \mathfrak{b}$, the bispace (X, τ_1, τ_2) is $(1, 2)$-NSH. It follows from [2] and Proposition 4.3.

3. Under $\omega_1 < \mathsf{cov}(\mathcal{M})$, $(X, \tau_1 < \tau_2)$ is $(1, 2)$-NSR (see again Proposition 4.3 and [2]).

Example 4.6. Let $\alpha D(\kappa)$ be the Alexandroff one-point compactification of the discrete space of uncountable cardinality κ. Consider the set $X = \alpha D(\kappa) \times [0, \kappa^+) \cup D(\kappa) \times \{k^+\}$ equipped with the following two topologies: τ_1 is the subspace topology of the space $\alpha D(\kappa) \times [0, \kappa^+]$ with the product topology, and τ_2 is the discrete topology on X. Then (X, τ_1, τ_2) is not $(1, 2)$-NSM. Otherwise, by Proposition 4.3(2), (X, τ_1) must be NSM. However, it is not the case because (X, τ_1) is not an NSL (see [2, Example 3.7]) and thus cannot be NSM

Theorem 4.7. *A bispace (X, τ_1, τ_2) is (i, j)-NSM, $i, j = 1, 2$, if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$ there is $n \in \mathbb{N}$ such that each τ_j-neighbourhood of A_n meets $\mathrm{St}(x, \mathcal{U}_n)$.*

Proof. Let a bispace X be (i, j)-NSM, $i, j = 1, 2$, and $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. For each $n \in \mathbb{N}$ there exists a finite set $A_n \subset X$ such that for every τ_j-open set $O_n \supset A_n$, $n \in \mathbb{N}$, we have $\bigcup_{n \in \mathbb{N}} \{\mathrm{St}(O_n, \mathcal{U}_n)\} = X$. Let $x \in X$. Then there exists $k \in \mathbb{N}$ fulfilling $x \in \mathrm{St}(O_k, \mathcal{U}_k)$. In other words, x belongs to some $U \in \mathcal{U}_k$ which intersects O_k. This means $O_k \cap \mathrm{St}(x, \mathcal{U}_k) \neq \emptyset$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. By assumption there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X fulfilling that for every $x \in X$ there exists $n \in \mathbb{N}$ such that each τ_j-neighbourhood O_n of A_n intersects $\mathrm{St}(x, \mathcal{U}_n)$. Therefore, for some $U \in \mathcal{U}_n$ containing x we have $O_n \cap U \neq \emptyset$ which implies $x \in \mathrm{St}(O_n, \mathcal{U}_n)$. This implies that for every τ_j-open $O_n \supset A_n$ we have: $(St(\{x\}, \mathcal{U}_n)) \bigcap O_n \neq \emptyset$. Because O_n was an arbitrary τ_j-neighbourhood of A_n one concludes that (X, τ_1, τ_2) is (i, j)-NSM. □

In a similar way one can prove the following two theorems.

Theorem 4.8. *A bispace X is (i, j)-NSR, $i, j = 1, 2$, if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X there is a sequence $(x_n : n \in \mathbb{N})$ of points of X such that for every $x \in X$ there is $n \in \mathbb{N}$ for which we have $x_n \in \mathrm{Cl}_{\tau_j}(\mathrm{St}(x, \mathcal{U}_n))$.*

Theorem 4.9. *A bispace X is (i, j)-NSH, $i, j = 1, 2$, if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$ we have that every τ_j-neighbourhood of F_n meets $\mathrm{St}(x, \mathcal{U}_n)$ for all but finitely many n.*

Definition 4.10. Let Y be a subspace of a bispace X. Then:

1) Y is *relatively* (i, j)-NSM (resp., *relatively* (i, j)-NSH) in X if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $Y \subset \bigcup_{n \in \mathbb{N}} \{\mathrm{St}(O_n, \mathcal{U}_n)\}$ (resp., for each $y \in Y$, $y \in \mathrm{St}(O_n, \mathcal{U}_n)$ for all but finitely many n);

2) Y is *relatively* (i, j)-NSR in X if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, one can choose $x_n \in X$, $n \in \mathbb{N}$, so that for every τ_j-open $O_n \ni x_n$, $n \in \mathbb{N}$, we have $Y \subset \bigcup_{n \in \mathbb{N}} \{\mathrm{St}(O_n, \mathcal{U}_n)\}$.

Proposition 4.11. *If $X = \bigcup \{Y_k : k \in \mathbb{N}\}$ and Y_k is relatively (i, j)-NSM(resp., relatively (i, j)-NSH, relatively (i, j)-NSR) in X, then X is (i, j)-NSM(resp., (i, j)-NSH, (i, j)-NSR)*

Proof. We prove the NSM case; the other two cases can be proved similarly. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. Then for each $k, n \in \mathbb{N}$, $\mathcal{U}_n$ covers Y_k and since Y_k is relatively (i, j)-NSM, there are finite $F_{k,n} \subset X$, such that for each τ_j-open $O_{k,n} \supset F_{k,n}$, $n \in \mathbb{N}$ we have $Y_k \subset \bigcup_{n\in\mathbb{N}}\{\mathrm{St}(O_{k,n}, \mathcal{U}_n)\}$ Consider the sequence $(F_{k,n} : k, n \in \mathbb{N})$ and τ_j-open $(G_{k,n} : k, n \in \mathbb{N})$ of neighbourhoods of $F_{k,n}$. It is easy to conclude that

$$\bigcup_{k\in\mathbb{N}} \mathrm{St}(G_{k,n}, \mathcal{U}_n) \supset \bigcup_{k\in\mathbb{N}} Y_k = X$$

which means that X is (i, j)-NSM. □

Theorem 4.12. *Let a bispace* (X, τ_1, τ_2) *be* (i, j)-NSM *(resp.* (i, j)-NSH, (i, j)-NSR*), and let* (Y, σ_1, σ_2) *be a bispace. If* $f : X \to Y$ *is a d-continuous surjection, then* Y *is also an* (i, j)-NSM *(resp.,* (i, j)-NSH, (i, j)-NSR*) bispace.*

Proof. We prove only the (i, j)-NSM case. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of σ_i-open covers of Y. For each $n \in \mathbb{N}$, the set $\mathcal{U}_n := \{f^{\leftarrow}(V) : V \in \mathcal{V}_n\}$ is a τ_i-open cover of X. Since X is (i, j)-NSM, there are finite sets $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open $O_n \supset F_n$, $n \in \mathbb{N}$, $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a cover of X. The sets $f(F_n)$, $n \in \mathbb{N}$, are finite in Y. For each n, let G_n be a σ_j-open neighbourhood of $f(F_n)$. Then $f^{\leftarrow}(G_n) = H_n$ is a τ_j-open subset of X for each $n \in \mathbb{N}$ and $H_n \supset F_n$. Thus $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(H_n, \mathcal{U}_n)$. We prove that $Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(G_n, \mathcal{V}_n)$.

Let $y \in Y$ and let $x \in X$ such that $y = f(x)$. Then there is $k \in \mathbb{N}$ such that $x \in \mathrm{St}(H_k, \mathcal{U}_k)$. Then $y = f(x) \in f(\mathrm{St}(H_k, \mathcal{U}_k))$. Because $f(\mathrm{St}(H_k, \mathcal{U}_k)) \subset f(\mathrm{St}(f^{\leftarrow}(G_k), \mathcal{U}_k)) \subset \mathrm{St}(G_k, \mathcal{V}_k)$ we have $y \in \mathrm{St}(G_k, \mathcal{V}_k)$. Therefore $Y = \bigcup_{k\in\mathbb{N}} \mathrm{St}(G_k, \mathcal{V}_k)$, i.e. Y is (i, j)-NSM. □

Theorem 4.13. *Let* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *be an open and closed, finite-to-one continuous mapping from a bispace* X *onto an* $(1, 2)$-NSH *bispace* Y*. Then* X *is* $(1, 2)$-NSH.

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of (X, τ_1) and let $y \in Y$. Since $f^{\leftarrow}(y)$ is finite, for each $n \in \mathbb{N}$ there exists a finite $\mathcal{U}_n(y) \subset \mathcal{U}_n$ such that

$$f^{\leftarrow}(y) \subset \cup\mathcal{U}_n(y) \text{ and } U \cap f^{\leftarrow}(y) \neq \emptyset \text{ for each } U \in \mathcal{U}_n(y).$$

Since $f : (X, \tau_1) \to (Y, \sigma_1)$ is closed, there exists a σ_1-open neighbourhood $V_n(y)$ of y such that $f^{\leftarrow}(V_n(y)) \subset \cup\mathcal{U}_n(y)$. Because $f : (X, \tau_1) \to (Y, \sigma_1)$ is open, one can assume that $V_n(y) \subset f(U)$ for each $U \in \mathcal{U}_n(y)$. For each $n \in \mathbb{N}$ set $\mathcal{V}_n = \{V_n(y) : y \in Y\}$. In this way we have a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of open covers of (Y, σ_1). As (Y, σ_1, σ_2) is $(1, 2)$-NSH, there is a sequence $(B_n : n \in \mathbb{N})$ of finite subsets of Y such that for all σ_2-open $O_n \supset B_n$, $n \in \mathbb{N}$, for each $y \in Y$, $y \in \mathrm{St}(O_n, \mathcal{V}_n)$ for all but finitely many n. Since f is finite-to-one, $(A_n = f^{\leftarrow}(B_n) : n \in \mathbb{N})$ is a sequence of finite subsets of X.

We prove that the sequence $(A_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X is $(1, 2)$-NSH. Let for each $n \in \mathbb{N}$, G_n be a τ_2-neighbourhood of A_n, $x \in X$ and $y = f(x)$. Since $f : (X, \tau_2) \to (Y, \sigma_2)$ is closed there exists a σ_2-open set O_n containing B_n such that $f^{\leftarrow}(O_n) \subset G_n$ for each $n \in \mathbb{N}$. There is $n_y \in \mathbb{N}$ such that $y \in \mathrm{St}(O_n, \mathcal{V}_n)$ for all $n \geq n_y$. Also, for all $n \geq n_y$, there exists $V_n(y) \in \mathcal{V}_n$ such that $y = f(x) \in V_n(y)$ and $V_n(y) \cap O_n \neq \emptyset$. As $x \in f^{\leftarrow}(V_n(y)) \subset \cup\mathcal{U}_n(y)$, we can choose $U \in \mathcal{U}_n(y)$ with $x \in U$. Then $V_n(y) \subset f(U)$, and thus $U \cap f^{\leftarrow}(O_n) \neq \emptyset$, hence $U \cap G_n \neq \emptyset$. Thus $x \in \mathrm{St}(G_n, \mathcal{U}_n)$, and as x was arbitrary we conclude that X is $(1, 2)$-NSH. □

5. Weaker versions of neighbourhood star selection properties

In this section we introduce and investigate weaker versions of (i, j)-NSM, (i, j)-NSR and (i, j)-NSH bispaces. We provide a few examples related to the Menger-type properties.

Definition 5.1. A bitopological space (X, τ_1, τ_2) id said to be:

(1) (i, j)*-almost neighbourhood star-Menger* (shortly, (i, j)-aNSM) (resp., (i, j)*-weakly neighbourhood star-Menger* (shortly, (i, j)-wNSM), (i, j)*-faintly neighbourhood star-Menger* (shortly, (i, j)-fNSM)), $i, j = 1, 2$, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X one can choose finite $F_n \subset X$, $n \in \mathbb{N}$, so that for

every τ_j-open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(O_n, \mathcal{U}_n)) = X$ (resp., $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)) = X$, $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\mathrm{Cl}_{\tau_j}(O_n), \mathcal{U}_n) = X$);

(2) (i, j)*-almost neighbourhood star-Rothberger* (shortly, (i, j)-aNSR) (resp., (i, j)*-weakly neighbourhood star-Rothberger* (shortly, (i, j)-wNSR), (i, j)*-faintly neighbourhood star-Rothberger* (shortly, (i, j)-fNSR)), $i, j = 1, 2$, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X one can choose $x_n \in X$, $n \in \mathbb{N}$, so that for every τ_j-open $O_n \supset x_n$, $n \in \mathbb{N}$ we have $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(O_n, \mathcal{U}_n)) = X$, (resp., $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)) = X$, $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\mathrm{Cl}_{\tau_j}(O_n), \mathcal{U}_n) = X$)

(3) (i, j)*-almost neighbourhood star-Hurewicz* (shortly, (i, j)-aNSH), (resp., (i, j)*-faintly neighbourhood star-Hurewicz* (shortly, (i, j)-fNSH)), $i, j = 1, 2$, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X one can choose a finite $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open $O_n \supset F_n$, $n \in \mathbb{N}$, each $x \in X$ belongs to $\mathrm{Cl}_{\tau_j}(\mathrm{St}(O_n, \mathcal{U}_n))$ (resp., to $\mathrm{St}(\mathrm{Cl}_{\tau_j}(O_n), \mathcal{U}_n)$) for all but finitely many n.

Remark 5.2. Every (i, j)-NSM bispace is (i, j)-aNSM, and every (i, j)-aNSM bispace is (i, j)-wNSM. Similarly, for Rothberger-type and Hurewicz-type properties.

In fact, we have the following relations among classes of bispaces defined above

$$\begin{array}{ccccccc}
(i,j)\text{-NSR} & \Longrightarrow & (i,j)\text{-aNSR} & \Longrightarrow & (i,j)\text{-wNSR} & & \\
\Downarrow & & \Downarrow & & \Downarrow & & \\
(i,j)\text{-NSM} & \Longrightarrow & (i,j)\text{-aNSM} & \Longrightarrow & (i,j)\text{-wNSM} & \Longrightarrow & (i,j)\text{-wNSL} \\
\Uparrow & & \Uparrow & & & & \\
(i,j)\text{-NSH} & \Longrightarrow & (i,j)\text{-aNSH} & & & &
\end{array}$$

Diagram 1

Recall that a bispace (X, τ_1, τ_2) is (i, j)*-Menger* if for any sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite collections of τ_j-open sets such that $\mathcal{V}_n < \mathcal{U}_n$, $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n = X$ (see [17, Definition 29], where the authors used the name δ_2-Menger).

Example 5.3. There is an $(1, 2)$-aNSM bispace which is not $(1, 2)$-Menger.

Let X be the Euclidean plane with the following two topologies: τ_1 is the deleted radius topology (see [28, Example 77]), and τ_2 is the usual metric topology.

(1) (X, τ_1, τ_2) is $(1, 2)$-aNSM

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since (X, τ_1) is an almost Menger space (see [12]) there are finite collections $\mathcal{V}_1 \subset \mathcal{U}_1$, $\mathcal{V}_2 \subset \mathcal{U}_2$, …, such that $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_1}(\bigcup \mathcal{V}_n)$. For each $n \in \mathbb{N}$ and each $V \in \mathcal{V}_n$ pick a point $x_{V,n} \in V$ and set $F_n = \{x_{V,n} : V \in \mathcal{V}_n\}$. Then each F_n is a finite subset of X and $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_1}(\mathrm{St}(F_n, \mathcal{U}_n))$. As $\tau_1 \geq \tau_2$, this implies that for any τ_2-open set $O_n \supset F_n$, $n \in \mathbb{N}$, we have $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_2}(\mathrm{St}(O_n, \mathcal{U}_n)$. Therefore, X is an $(1, 2)$-aNSM bispace.

(2) (X, τ_1, τ_2) is not $(1, 2)$-Menger.

Suppose, to the contrary, that (X, τ_1, τ_2) is $(1, 2)$-Menger. We claim that then (X, τ_1) is a Menger space. Indeed, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. As we supposed that (X, τ_1, τ_2) is (1,2)-Menger, there are finite $\mathcal{V}_1, \mathcal{V}_2, \ldots$ such that for each n, $\mathcal{V}_n < \mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$. For each n and each $V \in \mathcal{V}_n$ take $U_V \in \mathcal{U}_n$ with $V \subset U_V$ and put $\mathcal{W}_n = \{U_V; V \in \mathcal{V}_n\}$. Then finite subsets $\mathcal{W}_n$ of $\mathcal{U}_n$, $n \in \mathbb{N}$, witness for $(\mathcal{U}_n : n \in \mathbb{N})$ that (X, τ_1) is a Menger space. However, the space (X, τ_1) is not Lindelöf [28] and thus it cannot be Menger. This contradiction shows that (X, τ_1, τ_2) is not $(1, 2)$-Menger.

Example 5.4. There is a $(1, 2)$-wNSM bispace which is not $(1, 2)$-Menger.

Let X be the real line endowed with the two topologies: τ_1 is the rational sequence topology (see [28, Example 65], and τ_2 is the usual metric topology.

(1) (X, τ_1, τ_2) is $(1, 2)$-wNSM

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. In [13] it was shown that (X, τ_1) is a weakly Menger space. Therefore, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ for each $n \in \mathbb{N}$ and $X = \mathrm{Cl}_{\tau_1}(\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n)$. Take for each $n \in \mathbb{N}$ and each $V \in \mathcal{V}_n$ a point $x_{V,n} \in V$. Then finite sets $F_n = \{x_{V,n} : V \in \mathcal{V}_n\}$, $n \in \mathbb{N}$, satisfy $X = \mathrm{Cl}_{\tau_1}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{U}_n))$. The fact $\tau_1 \geq \tau_2$ implies that for any τ_2-open neighbourhood O_n of F_n, $n \in \mathbb{N}$,it holds $X = \mathrm{Cl}_{\tau_2}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)$. This means that X is an $(1, 2)$-wNSM bispace.

(2) (X, τ_1, τ_2) is not $(1, 2)$-Menger.

Assume, that (X, τ_1, τ_2) is $(1, 2)$-Menger. By the argumentation similar to the proof of (2) in the previous example we prove that in that case (X, τ_1) is a Menger space which is a contradiction, because (X, τ_1) is not Lindelöf, hence not Menger. Therefore, one concludes that (X, τ_1, τ_2) is not $(1, 2)$-Menger.

Example 5.5. There is a $(1, 2)$-fNSM bispace.

Let $X = \mathbb{R}$ equipped with the following two topologies: τ_1 is the Euclidean topology, and τ_2 is the collection of sets of the form $O \setminus C$, where $O \in \tau_1$ and C is a countable subset of X.

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_1-open covers of X. Since (X, τ_1) is a strongly star Menger space there are finite sets $F_1, F_2, \ldots$ in X such that $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(F_n, \mathcal{U}_n)$. Let for each n, $G_n = O_n \setminus C_n$ be a τ_2-neighbourhood of F_n. Since $\mathrm{Cl}_{\tau_2}(G_n) = \mathrm{Cl}_{\tau_1}(O_n)$ we get $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\mathrm{Cl}_{\tau_2}(G_n), \mathcal{U}_n)$, i.e. (X, τ_1, τ_2) is $(1, 2)$-fNSM.

Observe that the last example is $(1, 2)$-Menger.

5.1. (i, j)-almost and weakly neighbourhood star properties

We are going now to give a characterization of (i, j)-wNSM bispaces in terms of (i, j)-regular open sets.

Definition 5.6. ([5, 10]) Let (X, τ_1, τ_2) be a bitopological space. A set $A \in X$ is called *(i, j)-regular open* (resp., *(i, j)-regular closed*) if $A = \mathrm{Int}_{\tau_i}(\mathrm{Cl}_{\tau_j}(A))$ (resp., $A = \mathrm{Cl}_{\tau_i}(\mathrm{Int}_{\tau_j}(A))$. A is said to be *pairwise regular open* (resp., *pairwise regular closed*) if it is both (i, j)-regular open and (j, i)-regular open (resp., (i, j)-regular closed and (j, i)-regular closed).

Clearly, every (i, j)-regular open set in (X, τ_1, τ_2) is τ_i-open.

Theorem 5.7. *A bispace (X, τ_1, τ_2) is (i, j)-wNSM if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by (i, j)-regular open sets there exist finite sets $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open $O_n \supset F_n$, $n \in \mathbb{N}$, it holds $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)) = X$.*

Proof. ($\Rightarrow$): It is obvious because every (i, j)-regular open set in (X, τ_1, τ_2) is τ_i-open.

($\Leftarrow$): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. Putting $\mathcal{V}_n =: \{\mathrm{Int}_{\tau_i}\mathrm{Cl}_{\tau_j}(U) : U \in \mathcal{U}_n\}, n \in \mathbb{N}$, we obtain a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of covers of X by (i, j)-regular open sets. Then, by assumption, there exist finite $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open set $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{V}_n)) = X$. For every $n \in \mathbb{N}$ and every $V \in \mathcal{V}_n$ there exists a $U_V \in \mathcal{U}_n$ such that $V = \mathrm{Int}_{\tau_i}(\mathrm{Cl}_{\tau_j}(U_V))$. Consider the sequence $(\mathcal{W}_n : n \in \mathbb{N})$, where $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_n\}$. We claim that $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)) = X$.

Let $x \in X$ and let G be a neighbourhood of x. There exist $k \in \mathbb{N}$ and $V \in \mathcal{V}_k$ such that $G \cap V \neq \emptyset$ and $V \cap O_k \neq \emptyset$, i.e. there is $U = U_V \in \mathcal{U}_k$ such that $G \cap \mathrm{Int}_{\tau_i}(\mathrm{Cl}_{\tau_j}(U)) \neq \emptyset$ and $O_k \cap \mathrm{Int}_{\tau_i}(Cl_{\tau_j}(U)) \neq \emptyset$. Then $G \cap U \neq \emptyset$ and $O_k \cap U \neq \emptyset$. Therefore, $x \in \mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n))$, that is X is (i, j)-wNSM. □

Theorem 5.8. *Every clopen subset of an (i, j)-aNSM bispace is also (i, j)-aNSM.*

Proof. Let $(Y, \tau_1|Y, \tau_2|Y)$ be a clopen subset of an (i, j)-aNSM bispace (X, τ_i, τ_j) and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of $\tau_i|Y$-open covers of Y. As Y is clopen, $\mathcal{V}_n = \mathcal{U}_n \cup (X \setminus Y)$ is a τ_i-open cover of X for every $n \in \mathbb{N}$. Since X is (i, j)-aNSM, one can choose finite sets $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(O_n, \mathcal{V}_n)) = X$.

Define now $H_n = Y \cap F_n$ if $Y \cap F_n \neq \emptyset$, and H_n = any finite subset of Y, otherwise. We claim that $(H_n : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that Y is (i, j)-aNSM.

Let G_n be a $\tau_j|Y$-open set in Y containing H_n, $n \in \mathbb{N}$. Then $W_n = G_n \cup (X \setminus Y)$ is a τ_j-open set in X containing F_n, $n \in \mathbb{N}$, and thus $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(W_n, \mathcal{V}_n)) = X$. Because Y is closed in X, $H_n \cap (X \setminus Y) = \emptyset$ and $\mathrm{Cl}_{\tau_j}(G_n) \subset Y$. We conclude that $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j|Y}(\mathrm{St}(G_n, \mathcal{U}_n)) = Y$, which means that $(Y, \tau_1|Y, \tau_2|Y)$ is (i, j)-aNSM. □

Proposition 5.9. *Let (X, τ_1, τ_2) be a bispace. Then:*

(1) If $X = \bigcup\{\mathrm{Cl}_{\tau_j}(Y_k) : k \in \mathbb{N}\}$, and each Y_k is relatively (i, j)-NSM (resp., relatively (i, j)-NSR in X), then X is (i, j)-wNSM (resp., (i, j)-wNSR);

(2) If $X = \bigcup\{Y_k : k \in \mathbb{N}\}$ and each Y_k is (i, j)-wNSM (resp., (i, j)-wNSR) in X, then X is (i, j)-aNSM (resp., (i, j)-aNSR)

Proof. We shall prove the (i, j)-NSM case.

(1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. Each Y_k is covered by $\mathcal{U}_n$. As Y_k is relatively (i, j)-NSM in X, there is for each $k \in \mathbb{N}$ a sequence $(F_{k,n} : n \in \mathbb{N})$ of finite subsets of X such that for all τ_j-open $O_{k,n} \supset F_{k,n}$ we have: $Y_k \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(O_{k,n}, \mathcal{U}_n)$. Then $X = \bigcup \mathrm{Cl}_{\tau_j}(Y_k) \subset Cl_{\tau_j}(\mathrm{St}(O_{k,n}, \mathcal{U}_n))$, i.e. X is (i, j)-wNSM.

(2) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. Rearrange this sequence to $(\mathcal{U}_{k,m} : k, m \in \mathbb{N})$. For each $k \in \mathbb{N}$, $(\mathcal{U}_{k,m} : m \in \mathbb{N})$ is a sequence of covers of Y_k by τ_i-open sets in X. For each k, Y_k is (i, j)-wNSM, and thus there are finite sets $F_{k,m} \subset X$, $m \in \mathbb{N}$, so that for every τ_j-open $O_{k,m} \supset F_{k,m}$, $m \in \mathbb{N}$, we have $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_{k,m}, \mathcal{U}_{k,m})) \supset Y_k$. By the assumption $X = \bigcup_{n\in\mathbb{N}} Y_k$. It follows $X = \bigcup_{k\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\bigcup_{m\in\mathbb{N}} \mathrm{St}(O_{k,m}, \mathcal{U}_{k,m}))$,i.e. X is (i, j)-aNSM. □

When an (i, j)-wNSM bispace is (i, j)-aNSM?

Definition 5.10. A topological space (X, τ) is a *P-space* if the intersection of any countable family of open sets is again an open set.

In [11], a bitopological space (X, τ_1, τ_2) is defined to be (i, j)*-weakly P-bispace* if for every countable family $\{U_n : n \in \mathbb{N}\}$ of τ_i-open subsets of X, $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} U_n) = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(U_n)$.

Theorem 5.11. *Let (X, τ_1, τ_2) be a bispace such that (X, τ_j) is a P-space. Then the following statements are equivalent:*

(1) *X is an (i, j)-aNSM bispace;*

(2) *X is an (i, j)-wNSM bispace.*

Proof. (1) ⇒ (2) is always true.

(2) ⇒ (1): Let X be an (i, j)-wNSM bispace and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. As X is (i, j)-wNSM there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for any τ_j-open set $O_n \supset F_n$, $n \in \mathbb{N}$, it holds $X = \mathrm{Cl}_{\tau_j} (\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n))$. The set on the right side of the previous equality is the smallest τ_j-closed set containing $\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)$. Because (X, τ_j) is a P-space, the set $\bigcup_{n\in\mathbb{N}} \mathrm{Cl}(\mathrm{St}(O_n, \mathcal{U}_n)$ is τ_j-closed and thus it contains $\mathrm{Cl}_{\tau_j} (\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n))$. It follows $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(O_n, \mathcal{U}_n))$, i.e. X is (i, j)-aNSM. □

We can prove the following theorem.

Theorem 5.12. *In (i, j)-weakly P-bispaces (i, j)-wNSM and (i, j)-aNSM are equivalent. Similarly for the Rothberger case.*

Proof. Let (X, τ_1, τ_2) be an (i, j)-wNSM bispace and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of τ_i-open covers of X. There is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for any τ_j-open set $O_n \supset F_n$, $n \in \mathbb{N}$, it holds $X = \mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n))$. As $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a countable family of τ_i-open sets and X is (i, j)-weakly P-bispace, we have $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)) = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(O_n, \mathcal{U}_n))$. This means that X is an (i, j)-aNSM bispace. □

Definition 5.13. ([11]) A bitopological space X is said to be (i, j)*-nearly paracompact* if every family $\mathcal{U}$ of τ_i-open sets admits a τ_j-locally finite τ_j-open refinement.

Theorem 5.14. *If an (i, j)-nearly paracompact bispace X is (i, j)-wNSM, then X is (i, j)-aNSM.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. Since X is (i, j)-wNSM one can choose finite $F_n \subset X$, so that for every τ_j-open $O_n \supset F_n$, $n \in \mathbb{N}$, we have $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)) = X$. By the assumption, $\{\mathrm{St}(O_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ has a τ_j-locally finite τ_j-open refinement $\mathcal{W}$. Then $\bigcup \mathcal{W} = (\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n))$ and therefore $\mathrm{Cl}_{\tau_j}(\bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)) = \mathrm{Cl}_{\tau_j}(\bigcup \mathcal{W})$, i.e. $\bigcup \mathcal{W}$ is τ_j-dense in X. As $\mathcal{W}$ is a τ_j-locally finite family, we have that $\mathrm{Cl}_{\tau_j}(\bigcup \mathcal{W}) = \bigcup_{W\in\mathcal{W}} \mathrm{Cl}_{\tau_j}(W)$.

Since for every $W \in \mathcal{W}$ there is $k = k(W) \in \mathbb{N}$ such that $W \subset \mathrm{St}(O_k, \mathcal{U}_k)$, it follows $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(O_n, \mathcal{U}_n))$, i.e. X is (i, j)-aNSM. □

Theorem 5.15. *Let a bispace (X, τ_1, τ_2) be (i, j)-aNSM and let (Y, σ_1, σ_2) be a bispace. If $f : X \to Y$ is a d-continuous surjection, then Y is also (i, j)-aNSM.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of σ_i-open covers of Y. For each $n \in \mathbb{N}$, the set $\mathcal{U}_n := \{f^{\leftarrow}(V) : V \in \mathcal{V}_n\}$ is a τ_i-open cover of X. Since X is (i, j)-aNSM, there are finite sets $F_n \subset X$, $n \in \mathbb{N}$, so that for every τ_j-open $O_n \supset F_n$, $n \in \mathbb{N}$, $\{Cl_{\tau_j}(\mathrm{St}(O_n, \mathcal{U}_n)) : n \in \mathbb{N}\}$ is a cover of X. The sets $f(F_n)$, $n \in \mathbb{N}$, are finite in Y. For each n, let G_n be a σ_j-open neighbourhood of $f(F_n)$. Then $f^{\leftarrow}(G_n) = H_n$ is a τ_j-open subset of X for each $n \in \mathbb{N}$ and $H_n \supset F_n$. Thus $X = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\tau_j}(\mathrm{St}(H_n, \mathcal{U}_n))$. We prove that $Y = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\sigma_j}(\mathrm{St}(G_n, \mathcal{V}_n))$.

Let $y \in Y$ and let $x \in X$ be such that $y = f(x)$. Then there is $k \in \mathbb{N}$ such that $x \in \mathrm{Cl}_{\tau_j}(\mathrm{St}(H_k, \mathcal{U}_k))$. Then $y = f(x) \in \mathrm{Cl}_{\sigma_j}(f(\mathrm{St}(H_k, \mathcal{U}_k))$. Because $f(\mathrm{St}(H_k, \mathcal{U}_k)) \subset f(\mathrm{St}(f^{\leftarrow}(G_k), \mathcal{U}_k)) \subset \mathrm{St}(G_k, \mathcal{V}_k)$, we have $y \in \mathrm{Cl}_{\sigma_j}(\mathrm{St}(G_k, \mathcal{V}_k))$. Therefore $Y = \bigcup_{n\in\mathbb{N}} \mathrm{Cl}_{\sigma_j}(\mathrm{St}(G_n, \mathcal{V}_n))$, i.e. Y is (i, j)-aNSM. □

Theorem 5.16. *Let (X, τ_1, τ_2) be an (i, j)-aNSR bispace and let (Y, σ_1, σ_2) be a bispace. If $f : X \to Y$ is a d-continuous surjection, then Y is also (i, j)-aNSR.*

Theorem 5.17. *Let (X, τ_1, τ_2) be an (i, j)-aNSH bispace and let (Y, σ_1, σ_2) be a bispace. If $f : X \to Y$ is a d-continuous surjection, then Y is also (i, j)-aNSH.*

5.2. *(i, j)-faintly neighbourhood star properties*

In this subsection we consider faintly versions of weaker forms of neighbourhood star properties in bispaces. In particular, we investigate preservation of the properties that we consider in this article under some kinds of mappings.

First, recall some definitions for topological spaces.

Definition 5.18. A mapping f from a topological space X into a topological space Y is called *weakly continuous* [18] (resp., *θ-continuous* [9], *strongly θ-continuous* [19]) if for each $x \in X$ and each open neighbourhood V of $f(x)$ there is an open neighbourhood U of x such that $f(U) \subset \mathrm{Cl}(V)$ (resp., $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$, $f(\mathrm{Cl}(U)) \subset V$).

Theorem 5.19. *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bispaces such that X is (i, j)-fNSR. If $f : X \to Y$ is a surjective mapping such that f_i is weakly continuous and f_j is θ-continuous, then Y is also (i, j)-fNSR.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of σ_i-open covers of Y. Fix $x \in X$. For each $n \in \mathbb{N}$, there is a set $V_n^x \in \mathcal{V}_n$ such that $f(x) \in V_n^x$. As f_i is weakly continuous there is an open set $U_n^x \subset X$ containing x and satisfying $f(U_n^x) \subset \mathrm{Cl}_{\sigma_i}(V_n^x)$. The set $\mathcal{U}_n := \{U_n^x : x \in X\}$ is a τ_i-open cover of X for each $n \in \mathbb{N}$. Since X is (i, j)-fNSR there is a sequence $(a_n : n \in \mathbb{N})$ of points in X such that for any sequence $(S_n : n \in \mathbb{N})$ of τ_j-open neighbouhoods of a_n, $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(\mathrm{Cl}_{\tau_j}(S_n), \mathcal{U}_n) = X$.

Consider the sequence $(b_n = f(a_n) : n \in \mathbb{N})$ of points in Y and a sequence $(T_n : n \in \mathbb{N})$ of σ_j-open neighbourhoods of b_n, $n \in \mathbb{N}$. As f_2 is θ-continuous, for each n there exists a τ_j-open set $O_n \ni a_n$ so that $f(\mathrm{Cl}_{\tau_j}(O_n)) \subset \mathrm{Cl}_{\sigma_j}(T_n)$. Then $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\mathrm{Cl}_{\tau_j}(O_n), \mathcal{U}_n)$ implies $Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\mathrm{Cl}_{\sigma_j}(T_n), \mathcal{V}_n)$. It follows that Y is an (i, j)-fNSR bispace. □

Similarly, we can prove the following.

Theorem 5.20. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a mapping from an (i, j)-fNSM (resp., (i, j)-fNSH) bispace X onto Y such that f_i is weakly continuous and f_j is θ-continuous, then Y is also (i, j)-fNSM (resp., (i, j)-fNSH).*

The following results show relationships between (i, j)-NSM (resp., (i, j)-NSH, (i, j)-NSR) bispaces and (i, j)-fNSM (resp., (i, j)-fNSH, (i, j)fNSR) bispaces.

Theorem 5.21. *If a bispace (Y, σ_1, σ_2) is the image of an (i, j)-NSM bispace (X, τ_1, τ_2) under a d-weakly continuous mapping f, then Y is (i, j)-fNSM.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of σ_i-open covers of Y. Working as in the first part of the proof of Theorem 5.19 one constructs a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of τ_i-open covers of X, where $\mathcal{U}_n = \{U_n^x : x \in X\}$. Apply the fact that X is an (i, j)-NSM bispace to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ and find a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for any τ_j-open sets $G_n \supset A_n$, $n \in \mathbb{N}$, $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(G_n, \mathcal{U}_n)$. Put $B_n = f(A_n)$, $n \in \mathbb{N}$. We have the sequence $(B_n : n \in \mathbb{N})$ of finite subsets of Y. We prove that this sequence witnesses for $(\mathcal{V}_n : n \in \mathbb{N})$ that Y is (i, j)-fNSM.

For each $n \in \mathbb{N}$ take an arbitrary σ_j-neighbourhood H_n of B_n. Since $A_n \subset X$ is finite and f_2 is weakly continuous, there is a τ_j-neighbourhood O_n of A_n such that $f(O_n) \subset \mathrm{Cl}_{\sigma_j}(H_n)$. It is easy now to prove that from construction of the sequences $\mathcal{U}_n$ and $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(O_n, \mathcal{U}_n)$, it follows that $Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(Cl_{\sigma_j}(H_n), \mathcal{V}_n)$. This shows that Y is an (i, j)-fNSM bispace. □

Quite similarly one proves the following.

Theorem 5.22. *If $Y = f(X)$ is the image of an (i, j)-NSH (resp., (i, j)-NSR) bispace (X, τ_1, τ_2) under d-weakly continuous mapping f, then $(Y.\sigma_1, \sigma_2)$ is (i, j)-fNSH (resp., (i, j)-fNSR).*

Theorem 5.23. *If a bispace (Y, σ_1, σ_2) is the image of an (i, j)-fNSM (resp., (i, j)-fNSH, (i, j)-fNSR) bispace (X, τ_1, τ_2), such that f_i is weakly continuous and f_j is strongly θ-continuous, then Y is (i, j)-NSM (resp., (i, j)-NSH, (i, j)-NSR).*

Proof. We prove the Rothberger case; the other two cases are proved similarly. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of σ_i-open covers of Y. As in the proofs of Theorem 5.19 and Theorem 5.21 we obtain τ_i-open covers $\mathcal{U}_n = \{U_n^x : x \in X\}$, $n \in \mathbb{N}$. Then there are points $p_1, p_2, \ldots$ in X so that for arbitrary τ_j-open sets $G_1 \ni p_1, G_2 \ni p_2, \ldots, X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\mathrm{Cl}_{\tau_j}(G_n), \mathcal{U}_n)$.

Set $q_n = f(p_n)$, $n \in \mathbb{N}$, and take for each n a σ_j-open set $H_n \ni q_n$. Next, for each n pick a τ_j-open set $O_n \ni p_n$ such that $f(\mathrm{Cl}(O_n)) \subset H_n$. Then $X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(\mathrm{Cl}_{\tau_j}(O_n), \mathcal{U}_n)$ implies $Y = \bigcup_{n\in\mathbb{N}} \mathrm{St}(H_n, \mathcal{V}_n)$, i.e. Y is an (i, j)-NSR bispace. □

6. Conclusion

We study here classes of bitopological spaces (((i, j)-neighbourhood star-Menger, (i, j)-neighbourhood star-Rothberger and (i, j)-neighbourhood star-Hurewicz and their weaker versions) defined in the standard selection principles manner by using the star operator. We established a number of properties of those classes, and proved that they are different from the classes of known bitopological spaces. This study complements and continues earlier investigations of selective properties in bitopological spaces. We believe that it would be interesting to study k-neighbourhod star selection properties, $k \geq 2$, in bitopological spaces defined in a similar way, but by the iteration of the star operator: for a family $\mathcal{F}$ of subsets of a set X and a subset A of X one defines $\mathrm{St}^0(A, \mathcal{F}) = A$, and for $k \geq 1$, $\mathrm{St}^k(A, \mathcal{F}) = \mathrm{St}(\mathrm{St}^{k-1}(A, \mathcal{F}), \mathcal{F})$. Also, relations of all these selective properties with game theory may be investigated.

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