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Robust Controllability and Observability tests of Linear Time-delay Systems with Parametric Uncertainties using Interval Analysis

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***Abstract*—In this paper, robust controllability and observability problem of linear time-delay systems with parametric uncertainties is considered. A functional-based transformation is first presented, which transforms the system with delayed measurements into a system without delay formally. Based on the transformed system, we develop an effective method for checking the robust controllability and observability tests of the uncertain system using interval analysis.**

***Index Terms*— Functional transformation, Interval analysis, Linear time-delay systems, Parametric uncertainties, Robust controllability test, Robust observability test, Uncertain system.**

I. INTRODUCTION

Solutions have been made to the problem of parametrical uncertainties affecting a linear time invariant (LTI) system without taking into account delays in [1] and [2] for the robust controllability and observability tests using interval analysis. Indeed, in most situations, the uncertainties are poorly known but not totally unknown. In general, the upper and lower limits are known. So it would be interesting to take advantage of this partial knowledge for the structural controllability and observability tests of uncertain systems by using intervals for the reformulation of the parametrical uncertainties of an LTI system. For instance in [1], the controllability and observability tests are verified using a particular geometric form of interval linear systems, known as "complete generalized antisymmetric stepwise configuration" independently of the bounds of the uncertain system. In [2], necessary and sufficient condition has been established for the controllability of single-input/multi-output LTI systems with interval plants and the observability for multi-input/single output LTI systems with interval plants. The robust controllability and observability tests in this case depend completely of the bounds of the uncertainties. In [3] and [4], necessary and sufficient condition based on interval division method has been provided for multi-input/multi-output LTI systems. The robust controllability and un-controllability problems of uncertain interval systems are solved in [5] using linear independency condition of interval vectors.

Actually, in practical engineering, time-delay phenomena are common by the sensor measurement, such as component analysis nearing instruments have biggish delayed measurement and it is well known that robust observability under parametrical uncertainties and delay affecting the output measurements plays a crucial role in stabilization of systems in control theory as well as in observer design.

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To the best of the authors' knowledge, the condition for robust observability where both parametrical uncertainties and measurement time-delay are present has not yet been derived. The case where the uncertain system can be delayed is of our interest.

In this work, where delay can affect the linear uncertain system, we develop a method to overcome this obstacle by using a functional based-transformation which transforms the system with delayed measurements into a system without delay formally. Based on the transformed system, we use the interval arithmetic technique to enclose the parametrical uncertainties using interval matrices with a priori known upper and lower bound. Since an LTI system with parametrical uncertainties can be represented by interval matrices, it can also be considered as a set of column interval vectors. We will see that suggesting a linear independency of these interval vectors contributes thereafter to the verification of the controllability and the observability tests of the uncertain interval system. This paper can be considered as a development of the ideas discussed in [5] within addition the delay affecting the output measurements for the robust observability test of the transformed system.

The paper is organized as follows: In Section II, we give some definitions about interval analysis ([6], [7], [8]) and its application to enclose parametrical uncertainties using interval vectors and interval matrices. In Section III, we consider a nominal delayed system without considering any uncertainties and a functional based-transformation [9] to overcome the delay affecting the output measurements. In section IV, we develop the linear independency condition of interval vectors [5]. To the transformed system, we include parametric uncertainties and we apply the technique of linear independency of interval vectors in order to verify the controllability and observability tests of the uncertain transformed system, then we describe the method through an example of an uncertain delayed system, all is discussed in section V. The method is developed in Matlab environment using Intalab toolbox for interval analysis ([10], [11]). Conclusions are given in Section VI.

II. PRELIMINARIES

Throughout the paper, we need the following definitions.

Definition 2.1: The spectral radius ρ of square matrix F is defined as the maximum of the absolute value of its eigenvalues $\rho(F) = \max(|\lambda(F)|)$

Definition 2.2: An uncertain parameter x is called interval if it can be enclosed between an upper and lower boundary value. So a real interval scalar x^I is defined as $x^I = [\underline{x}, \overline{x}]$, where $\underline{x}$ and $\overline{x}$ are the lower and upper bound respectively. An interval matrix $X^I \in IR^{m\times n}$ is defined as a set of n interval vectors of IR^m, $X^I = (x_1^I, x_2^I, \dots \dots, x_n^I)$ with $x_i^I \in IR^m$ are interval vectors.

Definition 2.3: An interval matrix can also be seen as an uncertain matrix which can be enclosed between an upper and lower bound as follows: $X^I = [\underline{X^I} \quad \overline{X^I}]$ with $\underline{X^I} = [\underline{x_1^I}, \underline{x_2^I}, \dots \dots, \underline{x_n^I}]$and $\overline{X^I} = [\overline{x_1^I}, \overline{x_2^I}, \dots \dots, \overline{x_n^I}]$ are the lower and upper bound of the interval matrix respectively. As we can also define an interval matrix with its center matrix and its radius $X^I = [X_0 - \Delta X, X_0 + \Delta X]$, where $X_0 = mid(X) = \frac{\underline{X^I} + \overline{X^I}}{2}$ and $\Delta X = rad(X) = \frac{\overline{X^I} - \underline{X^I}}{2}$ are the center matrix and the radius matrix respectively.

Definition 2.4: For non empty closed intervals, the addition of two real interval scalars x^I and y^I is defined and calculated as: $x^I \oplus y^I = [\underline{x} + \underline{y}, \bar{x} + \bar{y}]$, the subtraction is defined as: $x^I \ominus y^I = [\underline{x} - \bar{y}, \bar{x} - \underline{y}]$, the multiplication is defined as: $x^I \otimes y^I = \left[\min\left\{\underline{xy}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\right\}, \max\left\{\underline{xy}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}\right\}\right]$. The division of two interval scalars is defined and calculated as: $x^I \oslash y^I = x^I \otimes \frac{1}{y^I}$, where $\frac{1}{y^I}$ is defined in [6] as follows:

$$\begin{aligned}
\frac{1}{y^I} &= \emptyset \text{ iff } y^I = [0,0] \\
&= \left[\frac{1}{\bar{y}}, \frac{1}{\underline{y}}\right] \text{ iff } 0 \notin y^I \\
&= \left[\frac{1}{\bar{y}}, \infty\right) \text{ iff } \underline{y} = 0 \text{ and } \bar{y} > 0 \\
&= \left(-\infty, \frac{1}{\bar{y}}\right] \text{ iff } \underline{y} < 0 \text{ and } \bar{y} = 0 \\
&= (-\infty, \infty) \text{ iff } \underline{y} < 0 \text{ and } \bar{y} > 0
\end{aligned}$$

The intersection between two non empty closed intervals is defined as $x^I \cap y^I := \{z | z \in x^I \text{ and } z \in y^I\}$ and the union is defined as $x^I \cap y^I := \{z | z \in x^I \text{ or } z \in y^I\}$

Definition 2.5: The ratio between two interval vectors $x^I, y^I \in IR^m$ is defined as: $\mu_{xy} = x^I \oslash y^I = \left\{x_1{}^I \oslash y_1{}^I, x_2{}^I \oslash y_2{}^I \dots, x_m{}^I \oslash y_m{}^I\right\}$, where $x_i{}^I \oslash y_i{}^I$ is the division of two interval scalars.

Definition 2.6: Two m dimensional LTI interval vectors $x_1{}^I$ and $x_2{}^I$ are said linearly independent if there exist only trivial solution $a_1 = a_2 = 0$ such that

$$a_1 x_1{}^I \oplus a_2 x_2{}^I = 0^I \tag{1}$$

Definition 2.7: To generalize *definition 2.6*, the m dimensional interval vectors $x_1{}^I, x_2{}^I, \dots, x_n{}^I$ are linearly independent if there exist only trivial solution $a_1 = a_2 = \cdots a_n = 0$ such that

$$a_1 x_1{}^I \oplus a_2 x_2{}^I \oplus \dots \oplus a_n x_n{}^I = 0^I \tag{2}$$

Definition 2.8: we call sub-matrices $S_X = \{S^i, i = 1, \dots, k\}$ of an $(m \times n)$ interval matrix X^I as *square set* and S^i as *sub-square matrices* and $s_X = \{s^i, i = 1, \dots, k\}$ where $k = \binom{m}{n}$ is called *index set* and s^i is called *index.*

Definition 2.9: The rank of an $(m \times n)$ interval matrix X^I is the maximum rank of its sub-matrices S^i, that is, $rank(M) = max\{rank(S^i), i = 1, \dots, k\}$.

Definition 2.10: For a square set S_X and its corresponding index set s_X of an $(m \times n)$ interval matrix X^I, we define the center and radius square matrices S_{X_c} and ΔS_X respectively as follows:

$S_{X_c} := \left\{S_0^i = \frac{\underline{s^i} + \overline{s^i}}{2}, i = 1, \dots, k\right\}$ and $\Delta S_X := \left\{\Delta S^i = \frac{\overline{s^i} - \underline{s^i}}{2}, i = 1, \dots, k\right\}$

III. System Formulation And The Nondelayed Transformation Of The System

Consider the linear interval system with delayed measurements described by:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t - \tau) \end{cases} \tag{3}$$

where $x \in IR^n$, $u \in IR^r$, $y \in IR^p$ are the state vector, input vector and measurement vector respectively . A, B and C are interval matrices given as $A \in A^I = [\underline{A} \quad \overline{A}] \in IR^{n \times n}$, $B \in B^I = [\underline{B} \quad \overline{B}] \in IR^{n \times r}$and $C \in C^I = [\underline{C} \quad \overline{C}] \in IR^{p \times n}$ respectively; with $rank(B) = r$ and $rank(C) = P$. $\tau > 0$ is a measurement delay assumed to be known .

To check the controllability and the observability test for system (3), we transform the delayed system into a non-delayed system by following the steps below:

Step1: let us first, consider system (3) without parametrical uncertainties i.e.

$$\begin{cases} \dot{x}(t) = A_0 x(t) + B_0 u(t) \\ y(t) = C_0\,x(t - \tau) \end{cases} \tag{4}$$

where A_0, B_0 and C_0 are constant matrices and $\tau > 0$ known measurement delay.

Step 2: Let us introduce the following functional transformation [9]

$$y(t) = \bar{y}(t) - C_0\,e^{-A_0 \tau} \int_{t-\tau}^{t} e^{A_0 (t - r)}\,B_0\,u(r) dr \tag{5}$$

System (4) by using equation (5) is transformed into the equivalent system without time-delay as follows:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + B_0 u(t), t > 0 \\ \bar{y}(t) = \bar{C}_0\,x(t) \end{cases} \tag{6}$$

where $\bar{C}_0 = C_0\,e^{-A_0 \tau}$ and $\bar{y} \in IR^p$ is called the equivalent output vector of non-delayed system.

IV. Linear independency of interval vectors

A. Linear independency of two interval vectors

To clarify the concept of linear independency of interval vectors, let us introduce the following theorem:

Theorem 4.1[5]: Two m dimensional LTI interval vectors x^I and y^I with $0 \notin x_1{}^I \cap x_2{}^I \dots \cap x_n{}^I$, $0 \notin y_1{}^I \cap y_2{}^I \dots \cap y_n{}^I$, are linearly independent iff, from the ratio μ_{xy} of x^I and y^I the following inequality holds:

$$(\mu_{xy})_1 \cap (\mu_{xy})_2 \cap \dots \cap (\mu_{xy})_n = \emptyset \tag{7}$$

Proof: Sufficiency: from $a_1 x^I \oplus a_2 y^I = 0^I$, we have

$$a_1 [x_1{}^I, x_2{}^I \dots, x_n{}^I]^T \oplus a_2 [y_1{}^I, y_2{}^I \dots, y_n{}^I]^T = 0^I$$

that we can simplify as

$$\begin{aligned} a_1 x_i{}^I \oplus a_2 y_i{}^I &= 0^I, i = \overline{k,n} \\ a_1 x_i{}^I &= \ominus\, a_2 y_i{}^I , i = \overline{k,n} \end{aligned} \tag{2}$$

let us define the ratio $(\mu_{xy})_i$ for each element:

$$(\mu_{xy})_i = x_i{}^I \oslash y_i{}^I$$

$$(\mu_{xy})_i = x_i{}^I \otimes \frac{1}{y_i{}^I} \tag{3}$$

by multiplying each side of (3) by ${y_i}^I$, we obtain:

$$\left(\mu_{xy}\right)_i \otimes {y_i}^I = {x_i}^I \quad (4)$$

let us replace (4) in (2):

$$a_1\left(\mu_{xy}\right)_i \otimes {y_i}^I = \ominus\, a_2 {y_i}^I,\ i = \overline{k,n}$$

$$\left(\mu_{xy}\right)_i = \ominus \frac{a_2}{a_1},\ i = \overline{k,n}$$

therefore, we can write

$$\left[\left(\mu_{xy}\right)_1, \left(\mu_{xy}\right)_2, \dots, \left(\mu_{xy}\right)_n\right]^T = \ominus \frac{a_2}{a_1} \quad (5)$$

from (5), we have:

$$\left(\mu_{xy}\right)_1 \cap \left(\mu_{xy}\right)_2 \cap \dots \cap \left(\mu_{xy}\right)_n = \ominus \frac{a_2}{a_1}$$

So, if $\left(\mu_{xy}\right)_1 \cap \left(\mu_{xy}\right)_2 \cap \dots \cap \left(\mu_{xy}\right)_n = \emptyset$ than $\frac{a_2}{a_1} = \emptyset$ and this is verified only for $a_1 = 0$ and from equation (2), $a_2 = 0$ as long as $0 \notin {y_1}^I \cap {y_2}^I \dots \cap {y_n}^I$.
Thus, from the solution $a_1 = a_2 = 0$, the linear independency condition of *definition 2.6* for the m interval vectors x^I and y^I is verified.
Necessity: let us suppose that

$$\left(\mu_{xy}\right)_1 \cap \left(\mu_{xy}\right)_2 \cap \dots \cap \left(\mu_{xy}\right)_n = \ominus \frac{a_2}{a_1} \neq \emptyset$$

then we can have $a_2 = 0$ and $a_1 \neq 0$ or $a_2 \neq 0$ and $a_1 \neq 0$. Thus, by *definition 2. 6* x^I and y^I are not linearly independent. The proof is completed

B. Linear independency of n interval vectors

Supposing that we have m dimensional LTI interval vectors given by : ${x_1}^I, {x_2}^I, \dots, {x_n}^I$. The interval vectors are said linearly independent iff they satisfy *definition 2.7*. Apparently, it is hard to solve equation (2) but it is possible to use anther writing for the interval vectors using an interval matrix form $X^I = ({x_1}^I, {x_2}^I, \dots, {x_n}^I)$ which is an $(m \times n)$ interval matrix. To check the linear independency of the m dimensional LTI interval vectors of the interval matrix X^I, we address the following lemma from the results of [12]:

Lemma 4.1: For interval sub-square matrix S^i of X^I, let its center matrix S_0 be nonsingular (invertible) and the spectral radius $\rho(|(S_0)^{-1}\Delta S|) < 1$, then S^i is nonsingular[1].

Remark 4.1: It is to notice that the spectral radius $\rho(|(S_0)^{-1}\Delta S|)$ is calculated using *definition 2.1.*

By using *Lemma 4.1* and *definition 2.9*, we give the theorem below, to check the linear independency of the m dimensional interval vectors.

Theorem 4.2: For $S^i \in S_X$, if there exists at least one corresponding center matrix $S_0 \in S_{X_c}$ and a radius matrix $\Delta S \in \Delta S_X$ such that S_0 is nonsingular (invertible) and $\rho(|(S_0)^{-1}\Delta S|) < 1$, then the interval vectors ${x_1}^I, {x_2}^I, \dots, {x_n}^I$ are linearly independent.

Proof: Let us consider an interval matrix form of the m dimensional interval vectors ${x_1}^I, {x_2}^I, \dots, {x_n}^I$ given as $X^I = (x_1^I, x_2^I, \dots \dots, x_n^I)$ which is an $(m \times n)$ interval matrix and let us suppose that $m > n$ (the demonstration for the case where $m < n$ is equivalent). The column interval vectors of X^I are said linearly independent if X^I has full rank, that is, $rank(X^I) = n$ and relying on the fact that the *row rank* is equal to *column rank* and the *full rank condition* is equivalent to *linear independency condition*. Therefore, by using *definition 2.9*, if any sub-square matrix $S^i \in S_X$ has row rank n, then X^I has n column rank and by using lemma 4.1, for S_0 and ΔS corresponding to the sub-matrix S^i, if S_0 is nonsingular (invertible) and $\rho(|(S_0)^{-1}\Delta S|) < 1$, then X^I has full rank, that is, the m dimensional interval vectors ${x_1}^I, {x_2}^I, \dots, {x_n}^I$ are linearly independent, the proof is completed.

[1] Nonsingular in this case means that S^i has full column rank

V. Linear Interval Systems With Both Delay Measurements And Parametrical Uncertainties

A. Robust controllability test

A.1 Case without interval parameters

Let us first study the case without interval uncertainties. The transformed system (6) is said completely controllable if the controllability matrix given by:

$$\mathcal{C}_0 = [B_0 \quad A_0B_0 \quad {A_0}^2B_0 \ \dots \ {A_0}^{n-r}B_0]$$

has always full rank i.e. $rank(\mathcal{C}_0) = n$
Let us now, consider a sub-matrix $(\mathcal{C}_0)'$ constructed from $\mathcal{C}_0$ given as:

$(\mathcal{C}_0)' = [B_0, A_0B_0, {A_0}^2B_0, \dots \dots, {A_0}^{n-r-q}B_0]$ where $q \geq 1$ and let us assume that it has full rank i.e. $rank((\mathcal{C}_0)') = rank(\mathcal{C}_0) = n$

A.2 Case with interval parameters

In this case, we introduce parametric uncertainties into system (6). The latter one is rewritten and given by the dynamic interval system below:

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t), & t > 0 \\ \bar{y}(t) = \bar{C}x(t) \end{cases} \quad (8)$$

where A, B are interval matrices given by $A \in A^I = [\underline{A} \quad \overline{A}] \in IR^{n\times n}$, $B \in B^I = [\underline{B} \quad \overline{B}] \in IR^{n\times r}$ with $rank(B) = r$ and $\bar{C} \in \bar{C}^I = [\underline{\bar{C}} \quad \overline{\bar{C}}] \in IR^{p\times n}$ is an interval matrix given by $\bar{C} = C \otimes e^{-A\tau}$ for $\forall A \in A^I = [\underline{A} \quad \overline{A}] \in IR^{n\times n}$ and $\forall C \in C^I = [\underline{C} \quad \overline{C}] \in IR^{p\times n}$ with $rank(\bar{C}) = rank(C) = p$
The controllability interval matrix in this case is given as:

$$\mathcal{C}^I = \left[B^I, A^I \otimes B^I, A^I \otimes A^I \otimes B^I, \dots, \underbrace{A^I \otimes A^I \dots \otimes A^I}_{n-r} \otimes B^I\right] \quad (10)$$

which is of dimension $(n \times m) = n \times \underbrace{(n - r + 1)\cdot r}_{m}$. The non-delayed interval system (8) is controllable if $rank(\mathcal{C}) = n$ $\forall \mathcal{C} \in \mathcal{C}^I$. But, since $\mathcal{C}^I$ is of dimension $(n \times m)$, it is not easy to find the rank of $\mathcal{C}^I$. However, we can check the rank of $\mathcal{C}^I$ using its sub-matrices $(\mathcal{C}^I)'$.
As in the case without interval parameters, let us consider a sub- matrix $(\mathcal{C}^I)'$ from the interval matrix $\mathcal{C}^I$, given as:

$$(\mathcal{C}^I)' = [B^I, A^I \otimes B^I, A^I \otimes A^I \otimes B^I, \dots, \underbrace{A^I \otimes \dots \otimes A^I}_{n-r-q} \otimes B^I]$$

Corollary 5.1: If sub-matrices $(\mathcal{C}^I)'$ constructed from $\mathcal{C}^I$ satisfy the linear independency condition of *theorem 4.2* then, the LTI interval system (8) is controllable.

Proof: The proof here is immediate, from the fact that the interval system is controllable if its controllability matrix has rank n and the full rank condition is equivalent to the linear independency condition.

B. Robust observability test

B.1 Case without interval parameters

The observability test for the transformed system (6) is formulated as follows:

Corollary 5.2: If the pair (A_0, C_0) is completely observable i.e.the rank of the observability matrix given by

$$\mathcal{O}_0 = [C_0 \quad C_0A_0 \quad C_0{A_0}^2 \ \dots \ C_0{A_0}^{n-p}]^T$$

has full rank, then the pair $(A_0, \bar{C}_0)$ of system (6) has the same observability without need to calculate $\bar{C}_0$ $(\bar{C}_0 = C_0\, e^{-A_0\tau})$

Proof: $e^{-A_0\tau}$ is a nonsingular matrix and noting that $e^{-A_0\tau}A_0 = A_0\ e^{-A_0\tau}$, so

$$rank\begin{bmatrix} \bar{C}_0 \\ \bar{C}_0A_0 \\ \bar{C}_0{A_0}^2 \\ \vdots \\ \bar{C}_0{A_0}^{n-p} \end{bmatrix} = rank\begin{bmatrix} C_0 \\ C_0A_0 \\ C_0{A_0}^2 \\ \vdots \\ C_oA_0{}^{n-p} \end{bmatrix} e^{-A_o\tau} = rank\begin{bmatrix} C_0 \\ C_0A_0 \\ C_0{A_0}^2 \\ \vdots \\ C_0{A_0}^{n-p} \end{bmatrix}$$

and as in the controllability test, let us consider a sub-matrix $(\mathcal{O}_0)'$ given as:

$(\mathcal{O}_0)' = [C_0 \quad C_0A_0 \quad C_0{A_0}^2 \ \dots \ C_0(A_0)^{n-p-q}]^T$, where $q \geq 1$ and let us assume that it has full rank i.e. $rank((\mathcal{O}_0)') = rank(\mathcal{O}_0) = n$

B.2 Case with interval parameters

In this case, the dynamical interval system (8) is considered. The observability interval matrix is calculated as follows:

$$\mathcal{O}^I = \begin{bmatrix} \bar{C} \\ \bar{C}\otimes A \\ \bar{C}\otimes A\otimes A \\ \vdots \\ \bar{C}\otimes \underbrace{A\otimes A \dots \otimes A}_{n-p} \end{bmatrix}, \text{with } \bar{C} = C \otimes e^{-A}$$

which is an $(m \times n)$interval matrix with $m = (n - p + 1).p$

Corollary 5.3: The observability test of the transformed interval dynamic system (8) depends on the observability test of the pair (A, C) for $\forall A \in A^I$ and $\forall C \in C^I$.

Proof: it is obvious that the observability test of the pair $(A, \bar{C})$ for $\forall A \in A^I$ and $\forall\, \bar{C} \in \bar{C}^I$ has the same observability test of the pair (A, C) for $\forall A \in A^I$ and $\forall C \in C^I$, without need to calculate the exponential interval matrix of A ($\bar{C} \in \bar{C}^I = [C \otimes e^{-A}$]) because the interval matrices C and $\bar{C}$ have the same rank.

Henceforth, we check the observability test for the pair (A, C) instead of the pair $(A, \bar{C})$. The interval observability matrix is then given as follows:

$$\mathcal{O}^I = \begin{bmatrix} C \\ C\otimes A \\ C\otimes A\otimes A \\ \vdots \\ C\otimes \underbrace{A\otimes A \dots \otimes A}_{n-p} \end{bmatrix} \tag{11}$$

Which is also an $(m \times n)$interval matrix with $m = (n - p + 1).p$. Therefore, the non-delayed interval system (8) is observable if $rank(\mathcal{O}) = n, \forall \mathcal{O} \in \mathcal{O}^I$.

The demonstration adopted here is the same as the one adopted in the controllability test. Since $\mathcal{O}^I$ is of dimension $(m \times n)$, it is not easy to find the rank of $\mathcal{O}^I$. For this reason, we check the rank of $\mathcal{O}^I$ using its sub-matrices $(\mathcal{O}^I)'$.

As in the case without interval parameters, let us consider a sub- matrix $(\mathcal{O}^I)'$ from the interval matrix $\mathcal{O}^I$, given as:

$$(\mathcal{O}^I)' = \begin{bmatrix} C \\ C\otimes A \\ C\otimes A\otimes A \\ \vdots \\ C\otimes \underbrace{A\otimes A \dots \otimes A}_{n-p-q} \end{bmatrix}$$

Corollary 5.4: If sub-matrices $(\mathcal{O}^I)'$ constructed from $\mathcal{O}^I$satisfy the linear independency condition of *theorem 4.2* then, the interval LTI system (8) is observable.

The proof is the same as the one adopted in the controllability test (section V.A.2)

Example:

Consider the transformed LTI interval system with free delay described by

$$\begin{cases} \dot{x}(t) = A\, x(t) + B\, u(t) \\ \quad \bar{y}(t) = \bar{C}x(t) \end{cases}$$

where $x \in IR^n, u \in IR^r, y \in IR^p (n = 3, r = 2, p = 2)$. A, B and $\bar{C}$ are interval matrices given as follows:

$$A \in A^I = \begin{pmatrix} [\,0.9, 1.1001] & [\,1.8, 2.2001] & [-1.1001, -0.9] \\ [\,-2.2001, -1.8] & [\,0.9, 1.1001] & [0.9, 1.1001] \\ [\,0.4498, 0.5502] & [\,-2.2001, -1.8] & [3.6, 4.4001] \end{pmatrix} \in IR^{n\times n}$$

$$B \in B^I = \begin{pmatrix} [0.9, 1.1001] & [-0.09, -0.0898] \\ [0.0898, 0.1102] & [\ \ 0.0898,\ \ 0.1102] \\ [-0.1102, -0.0898] & [\,0.9, 1.1002] \end{pmatrix} \in IR^{n\times r}$$

$\bar{C} \in \bar{C}^I = [\underline{\bar{C}}\ \ \overline{\bar{C}}] \in IR^{p\times n}$, $\bar{C} = C \otimes e^{-A\tau}$ $\forall C \in C^I$ with the interval matrix C is given as $C \in C^I = \begin{pmatrix} [0.98, 1.02] & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in IR^{p\times n}$. $rank(B) = r = 2$ and $rank(\bar{C}) = rank(C) = p = 2$. $\tau = 0.6$ is the measurement delay.

Robust Controllability test:

The controllability matrix calculated using (10) is as follows:

$$\mathcal{C} \in \mathcal{C}^I$$

$$\begin{pmatrix} [0.9000, 1.1001] & [-0.0900, -0.0898] & [1.0266, 1.5734] & [-1.1514, -0.6286] \\ [0.0898, 0.1102] & [0.0898,\ \ 0.1102] & [-2.4621, -1.5379] & [\,1.0309, 1.5291] \\ [-0.1101, -0.0899] & [\,0.9000,\ \ 1.1001] & [\,-0.3311, 0.1311] & [\,2.8682, 4.6423] \end{pmatrix} \in IR^{n\times m} = IR^{3\times 4}$$

which is an $(n \times m)$ interval matrix with $m = (n - r + 1) \cdot r$ is equal to 4. Here, we have the number of rows n is less than the number of columns m $(n < m)$. Therefore, the number of square sub- matrices S^i that can be formed from $\mathcal{C}^I$ is given by $k = \frac{m(m-1)(m-2)\dots\dots(m-n+1)}{n!}$, k is then equal to 4. The *column index set* is given as $s_c = \{s^i, i = \overline{1,k}\} = \{s^1, s^2, s^3, s^4\}$ and the *square set* as $S_c = \{S^i, i = \overline{1,k}\} = \{S^1, S^2, S^3, S^4\}$. The square sub-matrices, their center and their radius calculated are given below:

For the index $s^1 = \{1\ ,\ 2\ ,\ 3\}$, corresponding to the 1st, 2nd and 3rd column, we have sub-matrix S^1

$$S^1 = \begin{pmatrix} [0.9000, 1.1001] & [-0.0900, -0.0898] & [1.0266, 1.5734] \\ [0.0898, 0.1102] & [0.0898, 0.1102] & [-2.4623, -1.5377] \\ [-0.1102, -0.0898] & [0.9000, 1.1002] & [-0.3318, 0.1318] \end{pmatrix}$$

The center matrix of S^1 and its radius are as follows:

$$S_0^1 = \begin{pmatrix} 1 & -0.0899 & 1.3 \\ 0.1 & 0.1 & -2 \\ -0.1 & 1.0001 & -0.1 \end{pmatrix},$$

$$\Delta S^1 = \begin{pmatrix} 0.1 & 0.0001 & 0.2733 \\ 0.0101 & 0.0101 & 0.4622 \\ 0.0101 & 0.1001 & 0.2318 \end{pmatrix}$$

For the index $s^2 = \{1\ ,\ 2\ ,\ 4\}$, corresponding to the 1st, 2nd and 4th column, we have sub-matrix S^2 :

$$S^2 = \begin{pmatrix} [0.9000, 1.1001] & [-0.0900, -0.0898] & [-1.1514, -0.6286] \\ [0.0898, 0.1102] & [0.0898, 0.1102] & [1.0306, 1.5293] \\ [-0.1102, -0.0898] & [0.9000, 1.1002] & [2.8682, 4.6423] \end{pmatrix}$$

The center matrix of S^2 and its radius are as follows:

$$S_0^2 = \begin{pmatrix} 1 & -0.0899 & -0.89 \\ 0.1 & 0.1 & 1.28 \\ -0.1 & 1.0001 & 3.7552 \end{pmatrix},$$

$$\Delta S^2 = \begin{pmatrix} 0.1 & 0.0001 & 0.2613 \\ 0.0101 & 0.0101 & 0.2493 \\ 0.0101 & 0.1001 & 0.8870 \end{pmatrix}$$

For the index $s^3 = \{1\ ,\ 3\ ,4\}$ corresponding to the 1st, 3nd and 4th column, we have sub-matrix S^3 :

$$S^3 = \begin{pmatrix} [0.9000,\ 1.1001] & [1.0266, 1.5734] & [-1.1514, -0.6286] \\ [0.0898,\ 0.1102] & [-2.4623, -1.5377] & [1.0306, 1.5293] \\ [-0.1102, -0.0898] & [-0.3318, 0.1318] & [2.8682, 4.6423] \end{pmatrix}$$

The center matrix of S^3 and its radius are as follows:

$$S_0^3 = \begin{pmatrix} 1.0000 & 1.3000 & -0.8900 \\ 0.1000 & -2.0000 & 1.2800 \\ -0.1000 & -0.1000 & 3.7552 \end{pmatrix},$$

$$\Delta S^3 = \begin{pmatrix} 0.1000 & 0.2733 & 0.2613 \\ 0.0101 & 0.4622 & 0.2493 \\ 0.0101 & 0.2318 & 0.8870 \end{pmatrix}$$

For the index $s^4 = \{\ 2\ ,\ 3\ \ ,4\}$ corresponding to the 2nd, 3rd and 4th column, we have sub-matrix S^4 :

$$S^4 = \begin{pmatrix} [-0.0900, -0.0898] & [1.0266, 1.5734] & [-1.1514, -0.6286] \\ [0.0898, 0.1102] & [-2.4623, -1.5377] & [\ 1.0306, 1.5293] \\ [0.9000, 1.1002] & [-0.3318, 0.1318] & [\ 2.8682, 4.6423 \end{pmatrix}$$

The center matrix of S^4 and its radius are as follows:

$$S_0^4 = \begin{pmatrix} -0.0899 & 1.3 & -0.89 \\ 0.1 & -2 & 1.28 \\ 1.0001 & -0.1 & 3.7552 \end{pmatrix},$$

$$\Delta S^4 = \begin{pmatrix} 0.0001 & 0.2733 & 0.2613 \\ 0.0101 & 0.4622 & 0.2493 \\ 0.1001 & 0.2318 & 0.8870 \end{pmatrix}$$

- The center matrices S_0^1, S_0^2 , S_0^3 and S_0^4 are all nonsingular (invertible) matrices.
- For the square sub-matrices S^1 , S^2 and S^3, the spectral radius $\rho\left(\left|\left(S_0^i\right)^{-1}\right|\ \Delta S^{\mathrm{i}}\right)$ calculated is 0.2903, 0.5558 and 0.4325 respectively which is less than 1.
- The spectral radius of the square sub-matrix S^4 calculated is $\rho(|(S_0^4)^{-1}|\ \Delta S^4) = 21.1779$ which is greater than 1.

From the result obtained above, we conclude that the linear independency condition of the interval vectors of the interval controllability matrix $\mathcal{C}^I$ is satisfied (at least one center matrix is nonsingular and the spectral radius $\rho\left(\left|\left(S_0^i\right)^{-1}\right|\ \Delta S^{\mathrm{i}}\right) < 1$) this means that the interval system is completely controllable.

Observability test:

From *corollary 5.3* and *corollary 5.4*, we have proved that the observability of the pair $(A, \bar{C})$ has the same observability of the pair (A, C) and the latter depends on the linear independency condition of the interval vectors forming the observability matrix $\mathcal{O}^I$(*theorem 4.2*). The observability matrix calculated using (11) is as follows:

$$\mathcal{O} \in \mathcal{O}^I = \begin{pmatrix} [0.9799, 1.0201] & [0.00, 0.00] & [0.00, 0.00] \\ [0.00, 0.00] & [0.00, 0.00] & [1.00, 1.00] \\ [0.8779, 1.1221] & [1.7559, 2.2441] & [-1.1221, -0.8779] \\ [0.4498, 0.5502] & [-2.2001, -1.7999] & [3.6000, 4.4001] \end{pmatrix} \in IR^{l\times n} = IR^{4\times 3}$$

which is an $(l \times n)$ interval matrix with $l = (n - p + 1)\cdot p = 4$. Here, we have the number of rows l is greater than the number of columns n $(l > n)$. Therefore, the number of square sub- matrices S^i that can be formed from $\mathcal{O}^I$ is given by $k = \frac{l(l-1)(l-2)\ldots\ldots(l-n+1)}{n!}$, k is then equal to 4. The *row index set* is given as $s_O = \{s^i, i = \overline{1,k}\} = \{s^1, s^2, s^3, s^4\}$ and the *square set* as $S_O = \{S^i, i = \overline{1,k}\} = \{S^1, S^2, S^3, S^4\}$. The square sub-matrices, their center and their radius calculated are given below:

For the index $s^1 = \{1\ ,2\ \ ,3\}$ corresponding to the 1st, 2nd and 3rd row, we have sub-matrix S^1:

$$S^1 = \begin{pmatrix} [0.9799, 1.0201] & [0.00, 0.00] & [0.00, 0.00] \\ [0.00, 0.00] & [0.00, 0.00] & [1.00, 1.00] \\ [0.8779, 1.1221] & [1.7559, 2.2441] & [-1.1221, -0.8779] \end{pmatrix}$$

The center matrix of S^1 and its radius are as follows:

$$S_0^1 = \begin{pmatrix} 1.00 & 0 & 0 \\ 0 & 0 & 1.00 \\ 1.00 & 2.00 & -1.00 \end{pmatrix}, \Delta S^1 = \begin{pmatrix} 0.02 & 0 & 0 \\ 0 & 0 & 0 \\ 0.122 & 0.244 & 0.122 \end{pmatrix}$$

For the index $s^2 = \{1\ ,\ 2\ \ ,4\}$ corresponding to the 1st, 2nd and 4th row, we have sub-matrix S^2 :

$$S^2 = \begin{pmatrix} [0.9799,\ 1.0201] & [0.00,\ 0.00] & [0.00, 0.00] \\ [0.00,\ 0.00] & [0.00,\ 0.00] & [1.00, 1.00] \\ [0.4498,\ 0.5502] & [-2.2001, -1.7999] & [3.6000, 4.4001] \end{pmatrix}$$

The center matrix of S^2 and its radius are as follows:

$$S_0^2 = \begin{pmatrix} 1.00 & 0 & 0 \\ 0 & 0 & 1.00 \\ 0.50 & -2.00 & 4.00 \end{pmatrix}, \Delta S^2 = \begin{pmatrix} 0.02 & 0 & 0 \\ 0 & 0 & 0 \\ 0.0501 & 0.20 & 0.40 \end{pmatrix}$$

For the index $s^3 = \{1 \;\;, 3 \;, \;4\}$ corresponding to the 1st, 3rd and 4th row, we have sub-matrix S^3:

$$S^3 = \begin{pmatrix} [0.9799, 1.0201] & [0.00, 0.00] & [0.00, 0.00] \\ [0.8779, 1.1221] & [1.7559, 2.2441] & [-1.1221, -0.8779] \\ [0.4498, 0.5502] & [-2.2001, -1.7999] & [3.6000, 4.4001] \end{pmatrix}$$

The center matrix of S^3 and its radius are as follows:

$$S_0^3 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 0.5 & -2 & 4 \end{pmatrix}, \;\Delta S^3 = \begin{pmatrix} 0.02 & 0 & 0 \\ 0.122 & 0.244 & 0.122 \\ 0.0501 & 0.20 & 0.40 \end{pmatrix}$$

For the index $s^4 = \{\,2 \;, \;\; 3 \;\;\; , 4\}$ corresponding to the 2nd, 3rd and 4th row, we have sub-matrix S^4:

$$S^4 = \begin{pmatrix} [\,0.00, 0.00] & [0.00, 0.00] & [\,1.00, 1.00] \\ [0.8779, 1.1221] & [1.7559, 2.2441] & [-1.1221, -0.8779] \\ [0.4498, 0.5502] & [-2.2001, -1.7999] & [3.6000, 4.4001] \end{pmatrix}$$

The center matrix of S^4 and its radius are as follows:

$$S_0^4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & -1 \\ 0.5 & -2 & 4 \end{pmatrix} \Delta S^4 = \begin{pmatrix} 0 & 0 & 0 \\ 0.122 & 0.244 & 0.122 \\ 0.0501 & 0.200 & 0.40 \end{pmatrix}$$

- The center matrices S_0^1, S_0^2 ,S_0^3 and S_0^4 are all nonsingular (invertible) matrices.
- For the square sub-matrices S^1 ,S^2, $S^3$and S^4 the spectral radius $\rho\left(\left|\left(S_0^i\right)^{-1}\right| \;\Delta S^{\mathrm{i}}\right)$ calculated is $0.1220, 0.1000, 0.3334$ and 0.2158 respectively which is less than 1.

From the result obtained above, we conclude that the linear independency condition of the interval vectors of the observability matrix O^I is satisfied (at least one center matrix is nonsingular and the spectral radius $\rho\left(\left|\left(S_0^i\right)^{-1}\right| \;\Delta S^{\mathrm{i}}\right) < 1$)

Remark 5.1: Here we checked the linear independency condition of the *theorem 4.2* using row interval vectors. Since *row rank* is equal to *column rank* and the *full rank condition* is equivalent to *linear independency condition* then we conclude that the pair (A, C) is observable. Therefore, the pair $(A, \bar{C})$ is also observable by using *corollary 5.3* and *corollary 5.4*.

VI. Conclusion

In this paper, we introduced interval arithmetic to redefine the structural properties of controllability and observability tests of an LTI system with parametric uncertainties and delayed measurements. The robust controllability test was based on the linear independency condition of the interval vectors of the controllability interval matrix. For the robust observability test, a functional-based transformation was first performed to obtain free- delay interval system, and then the same linear independency condition was applied to the observability interval matrix of the transformed interval system. The developed method is characterized by its robustness respect to parametrical uncertainties and its simplicity by using an adequate transformation to overcome the delay in the output measurements. The case where the delay affect the control input or the state vector of an LTI system with parametric uncertainties can be investigated in future works.

VII. References

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