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Existence of solution for a class of heat equation involving the 1-Laplacian operator

Claudianor O. Alves[a,1], Tahir Boudjeriou[b,*]

[a] *Unidade Acadêmica de Matemática, Universidade Federal de Campina Grande, 58429-970, Campina Grande - PB, Brazil*
[b] *Department of Basic Teaching, Institute of Electrical & Electronic Engineering, University of Boumerdes, Boumerdes, 35000, Algeria*

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ABSTRACT

This paper concerns the existence of global solutions for the following class of heat equations involving the 1-Laplacian operator for the Dirichlet problem

$$\begin{cases} u_t - \Delta_1 u = f(u) & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{in} \quad \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying some technical conditions and $\Delta_1 u = \operatorname{div}\left(\frac{Du}{|Du|}\right)$ denotes the 1-Laplacian operator. The existence of global solutions is done by using an approximation technique that consists in working with a class of p-Laplacian problems associated with (P) and then taking the limit when $p \to 1^+$ to get our results.

1. Introduction and the main results

In this paper, we are concerned with the existence of global solutions for the following class of heat equations involving the 1-Laplacian operator for the Dirichlet problem

$$\begin{cases} u_t - \Delta_1 u = f(u) & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{in} \quad \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: coalves@mat.ufcg.edu.br (C.O. Alves), re.tahar@yahoo.com (T. Boudjeriou).
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where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\Delta_1 u = \operatorname{div}\left(\frac{Du}{|Du|}\right)$ denotes the 1-Laplacian operator.

In recent decades, some problems involving the operator Δ_1 has received special attention after the pioneering works involving this operator that were written by Andreu, Ballester, Caselles, and Mazón (among them [4–6]). For example, in [4], Andreu, Ballester, Caselles, and Mazón considered the following class of evolution equations

$$\begin{cases} u_t - \operatorname{div}\left(\frac{Du}{|Du|}\right) = 0 & \text{in} \quad \Omega \times (0, +\infty), \\ u = \varphi & \text{in} \quad \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.2}$$

where $u_0 \in L^1(\Omega)$ and $\varphi \in L^\infty(\Omega)$. By using the techniques of completely accretive operators and the Crandall-Liggett semigroup generation theorem [12], they were able to prove the existence and uniqueness of entropy solution. The same problem, but involving the Neumann boundary condition, that is $\frac{\partial u}{\partial \nu} = 0$ in $\partial\Omega \times (0, +\infty)$, was considered by Andreu, Ballester, Caselles, and Mazón in [5]. Still related to the existence of solutions to problem (1.2), we would like to cite a paper by Hardt and Zhou [21], where the authors used an approximation technique that consists in working with a class of nondegenerate parabolic approximation problems, and after some estimates, they were able to prove the existence of a solution for the original problem. In [7], Andreu, Caselles, Díaz, and Mazón studied the asymptotic profile of solutions to (1.2) near the extinction time for Dirichlet and Neumann boundary conditions.

Related to the stationary case, that is, for the following class of problems

$$\begin{cases} -\operatorname{div}\left(\frac{Du}{|Du|}\right) = f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega, \end{cases} \tag{1.3}$$

Degiovanni and Magrone [13] studied a version of the Brézis-Nirenberg problem for (1.3) by applying the Linking theorem. The functional energy has been extended to Lebesgue space $L^{N/N-1}(\Omega)$, in order to recover the compact embedding. Concerning the spectrum of the 1-Laplacian operator, by using the same approach, Chang [11] proved the existence of a sequence of eigenvalues. Different approaches have been taken to attack (1.3) under various hypotheses on the nonlinearity f. In [19], Figueiredo and Pimenta studied a problem related to (1.3), where the nonlinearity has a subcritical growth and their main results establish the existence of a nontrivial ground-state solution.

As regards quasilinear problems, depending on some features of the differential operator to be considered, it is worthwhile to work with it in a suitable space, like the space of functions of bounded variation, hereafter denoted by $BV(\Omega)$. We may address the question of finding critical points for a functional in $BV(\Omega)$, where the coerciveness and smoothness are lost. In other words, the main difficulties arise mainly due to the lack of smoothness on the energy functional associated with (1.3) and the lack of reflexiveness of $BV(\Omega)$. Indeed, the energy functional is not C^1 and we find some hindrances to show that functionals defined in this space satisfy compactness conditions like the Palais-Smale. Meanwhile, a lot of attention has been paid recently to that space, for example, see [3,9–11,13,14,17,23,30,31] and references therein, since it is the natural environment in which minimizers of many problems can be found, especially in problems involving interesting physical situations, in capillarity theory and existence of minimal surfaces, and as an application of the variational approach to image restoration.

In conclusion, it is important to point out that the literature concerning the inhomogeneous case of problem (1.2) is poor and, to the best of our knowledge, there are a few papers in which the authors studied the existence and uniqueness of solutions. In that direction, we mention a result by Segura de León and Webler [25] involving the following problem

$$\begin{cases} u_t - \operatorname{div}\left(\frac{Du}{|Du|}\right) = f(x,t) & \text{in} \quad \Omega \times (0,+\infty), \\ u = 0 & \text{in} \quad \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.4}$$

where $u_0 \in L^2(\Omega)$ and $f \in L^1_{loc}(0,+\infty; L^2(\Omega))$. In that work the authors proved global existence and uniqueness of solutions for (1.4) via a parabolic p-Laplacian problem and then taking the limit as $p \to 1^+$. In [22], by means of nonlinear semigroup, Hauer and Mazón studied the existence of strong solutions for problem (1.4) with a global Lipschitz continuous function $f(x,u)$ in the second variable instead of $f(x,t)$. Finally, we refer the reader to [22] and [27] for some recent results on parabolic equations involving the fractional 1-Laplacian operator.

Motivated by the studies made in [4,5,7,21,22,25], we intend to prove two existence results of solutions for (1.1) by supposing different conditions on f and Ω. Our first result is devoted to the radial case, where we assume that Ω is an annulus region of the form

$$\Omega = \{x \in \mathbb{R}^N \ : \ a < |x| < b\}, \tag{1.5}$$

where $0 < a < b < +\infty$, while $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following conditions: There is $p_0 \in (1,2)$ such that

(f_1) $$\lim_{s \to 0} \frac{f(s)}{|s|^{p_0-1}} = 0.$$

There is $\theta > 1$ such that

(f_2) $$0 < \theta F(s) \le f(s)s, \quad s \in \mathbb{R} \setminus \{0\},$$

where $F(s) = \int_0^s f(r)\, dr$.

In order to state our condition on the initial datum u_0, we need to fix some notations. Hereafter, we denote by $W^{1,p}_{0,rad}(\Omega)$ the subspace of $W^{1,p}_0(\Omega)$ that is formed by radial functions. It is well known that the embedding below

$$W^{1,p}_{0,rad}(\Omega) \hookrightarrow C(\overline{\Omega}) \tag{1.6}$$

is compact, whose proof is an immediate consequence of [24, Chapitre 6, Lemme 1.1]. In particular, we have the compact embedding

$$W^{1,p}_{0,rad}(\Omega) \hookrightarrow L^q(\Omega), \quad \forall q \in [1,\infty].$$

The embedding (1.6) permits to consider the functional $E : W^{1,1}_{0,rad}(\Omega) \to \mathbb{R}$ given by

$$E(u) = \int_\Omega |\nabla u|\, dx - \int_\Omega F(u)\, dx.$$

Associated with E we have the Nehari set defined by

$$\mathcal{N}_{rad} = \left\{ u \in W^{1,1}_{0,rad}(\Omega) \setminus \{0\} \ : \ \int_\Omega |\nabla u|\, dx = \int_\Omega f(u)u\, dx \right\},$$

and the real number

$$d = \inf_{u \in \mathcal{N}_{rad}} E(u).$$

The potential well associated with problem (1.1) is the set

$$W_{rad} = \left\{ u \in W^{1,1}_{0,rad}(\Omega) \, : \, E(u) < d \text{ and } I(u) > 0 \right\} \cup \{0\}, \tag{1.7}$$

where $I(u) = \int_\Omega |\nabla u| \, dx - \int_\Omega f(u) u \, dx$ for all $u \in W^{1,1}_{0,rad}(\Omega)$.

For each $r \in [1, \infty]$, we define the function space $X_r(\Omega)$ by

$$X_r(\Omega) := \left\{ z \in L^\infty(\Omega; \mathbb{R}^N) : \, \text{div} z \in L^r(\Omega) \right\}. \tag{1.8}$$

In the sequel, we give the following two definitions.

Definition 1.1. (radial solution) A function $u \in L^\infty(0, +\infty; BV_{rad}(\Omega))$ with $u_t \in L^2(0, +\infty; L^2(\Omega))$ will be called a radial weak solution of (1.1), if $u(0) = u_0$ and there exists a vector field $z(t) \in X_2(\Omega)$ with $\|z(t)\|_\infty \leq 1$ a.e. $t \in (0, +\infty)$ such that

(1) $u_t(t) - \text{div}(z(t)) = f(u(t))$, in $\mathcal{D}'(\Omega)$ a.e. $t \in (0, +\infty)$,
(2) $\int_\Omega (z(t), Du(t)) = \int_\Omega |Du(t)|$,
(3) $[z(t), \nu] \in \text{sign}(-u(t))$ $\mathcal{H}^{N-1}-$a.e. on $\partial\Omega$.

Definition 1.2. (Maximal existence time) Let $u(t)$ be a solution of problem (1.1). We define the maximal existence time $T_{\max}$ of u as follows:

$$T_{\max} = \sup\{t > 0 \, : u = u(t) \text{ exists on } [0, T]\}.$$

(1) If $T_{\max} < \infty$ we say that the solution of (1.1) blows up and $T_{\max}$ is the blow up time.
(2) If $T_{\max} = \infty$, we say that the solution is global.

Now we are in a position to state the first result.

Theorem 1.1. *Assume (1.5), $(f_1) - (f_2)$ and $u_0 \in W^{1,p_0}_{0,rad}(\Omega)$, where p_0 was given in (f_1). Moreover, suppose that*

$$E(u_0) < d \quad \textit{and} \quad I(u_0) > 0. \tag{1.9}$$

Then there exists a global weak radial solution to problem (1.1). Furthermore, there holds

$$\int_0^t \|u_s(s)\|_2^2 \, ds + \int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)| \, d\mathcal{H}^{N-1} - \int_\Omega F(u(t)) \, dx \leq E(u_0) \;\; \textit{a.e. } t \in [0, +\infty). \tag{1.10}$$

Remark 1. From here on out, $BV_{rad}(\Omega)$ denotes the subspace of $BV(\Omega)$ that is formed by radial functions. It was shown in Lemma 2.1, see Section 3, that the following continuous embedding holds

$$BV_{rad}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{for} \quad p \in [1, \infty),$$

where Ω is of the form (1.5). Hence, by [34, Corollary 4, p. 85], $u \in C([0, T]; L^p(\Omega))$ for all $p \in [1, \infty)$ and for any $T > 0$. Then, the initial condition $u(0) = u_0$ in (1.1) exists and makes sense.

Concerning the non-radial case, that is, the case where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded set with Lipschitz boundary, beside the conditions $(f_1) - (f_2)$, we assume the following condition on the function f:
(f_3) There exist $q \in (1, 1^*)$ and $C > 0$

$$|f(s)| \leq C(1 + |s|^{q-1}), \ \ \forall s \in \mathbb{R},$$

where $1^* = \frac{N}{N-1}$ if $N \geq 2$ and $1^* = +\infty$ when $N = 1$.

Here, the Nehari set associated with E is given by

$$\mathcal{N} = \left\{ u \in W_0^{1,1}(\Omega) \setminus \{0\} \, : \, \int_\Omega |\nabla u| \, dx = \int_\Omega f(u)u \, dx \right\},$$

and the potential well associated with problem (1.1) is the set

$$W = \left\{ u \in W_0^{1,1}(\Omega) \, : \, E(u) < d \text{ and } I(u) > 0 \right\} \cup \{0\}, \tag{1.11}$$

where $I(u) = \int_\Omega |\nabla u| \, dx - \int_\Omega f(u)u \, dx$.

The second result reads as follows:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a bounded set with Lipschitz boundary. Assume that the assumptions $(f_1) - (f_3)$ hold and $u_0 \in W_0^{1,p_0}(\Omega)$ where p_0 was given in (f_1). In addition suppose the following condition holds*

$$E(u_0) < d \quad \text{and} \quad I(u_0) > 0. \tag{1.12}$$

Then, there exists a global weak solution to problem (1.1). Moreover, there holds

$$\int_0^t \|u_s(s)\|_2^2 \, ds + \int_\Omega |Du(t)| + \int_{\partial\Omega} |u(t)| \, d\mathcal{H}^{N-1} - \int_\Omega F(u(t)) \, dx \leq E(u_0) \ \ a.e. \ t \in [0, +\infty). \tag{1.13}$$

Related to Theorem 1.2, we are using the following definition of solution:

Definition 1.3. A function $u \in L^\infty(0, +\infty; BV(\Omega) \cap L^2(\Omega))$ with $u_t \in L^2(0, +\infty; L^2(\Omega))$ will be called a weak solution of (1.1) if $u(0) = u_0$, and there exists a vector field $z(t) \in X_2(\Omega)$ with $\|z(t)\|_\infty \leq 1$, $\text{div}(z(t)) \in L^2(\Omega)$ a.e. $t \in (0, +\infty)$ such that

(1) $u_t(t) - \text{div}(z(t)) = f(u(t))$, in $\mathcal{D}'(\Omega)$ a.e. $t \in (0, +\infty)$,
(2) $\int_\Omega (z(t), Du(t)) = \int_\Omega |Du(t)|$,
(3) $[z(t), \nu] \in \text{sign}(-u(t))$ $\mathcal{H}^{N-1}$ – a.e. on $\partial\Omega$.

Remark 2. In view of [26, Lemma 1.2] and the regularity of the solution u stated in Definition 1.2, we have that $u \in C([0, T]; L^q(\Omega))$ for each $q \in [1, 2]$ and for any $T > 0$. Thereby, the initial condition $u(0) = u_0$ exists and makes sense.

1.1. Our approach

In the proof of Theorems 1.1 and 1.2 was used an approximation technique that consists in working with a class of p-Laplacian problems associated with (1.1) and then taking the limit as $p \to 1^+$ to get our results. More precisely, employing the potential well theory combined with Galerkin methods we prove the existence of a global solution for the following class of quasilinear heat equations

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(u) & \text{in} \quad \Omega \times (0,+\infty), \\ u = 0 & \text{in} \quad \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

for all $p > 1$, which is denoted by u_p. After that, we consider a sequence $p_m \to 1^+$ and show that the sequence (u_m) converges to a solution of (1.1) in the sense of Definitions 1.2 and 1.3 respectively. In Theorem 1.1, the reader is invited to see that we do not assume on the function f any growth condition from above at infinite, because in this case the domain Ω is an annulus and the properties of the spaces $W^{1,p}_{0,rad}(\Omega)$ and $BV_{rad}(\Omega)$ play an important role in our approach. However, in the proof of Theorem 1.2 we assumed that f has a subcritical growth because Ω is a general bounded domain. Finally, we would like to point out that we will work only with the case $N \geq 2$, because the case $N = 1$ follows with few modifications.

The approximation technique by using p-Laplacian problems is well known to get a solution for problems involving the 1-Laplacian operator for stationary case, see for example Alves [2], Demengen [14,15], Figueiredo and Pimenta [18], Mercaldo, Rossi, Segura de León and Trombetti [28], Mercaldo, Segura de León and Trombetti [29], Salas and Segura de León [30]. Related to the evolution case we only know a paper by Segura de León and Webler [25]. However, up to our knowledge, this is the first time that this approach is used to prove the existence of a global solution for a heat equation involving the 1-Laplacian operator when the nonlinearity f is of the form $f(u)$, that is when f can be a nonlinear function in the variable u. For example, Theorem 1.1 can be used for the nonlinearity $f(u) = |u|^{q-2}ue^{\alpha|u|^2}$ for $q > 1$ and $\alpha > 0$, whereas in Theorem 1.2 we can work with $f(u) = |u|^{q-2}u + |u|^{s-2}u$ with $q, s \in (1, 1^*)$.

1.2. Organization of the article

This article is organized as follows: In Section 2, we recall some notations and results involving the $BV(\Omega)$ space. In Sections 3 and 4, we prove Theorems 1.1 and 1.2 respectively.

1.3. Notations

Throughout this paper, the letters c, c_i, C, C_i, $i = 1, 2, \ldots$, denote positive constants which vary from line to line, but are independent of terms that take part in any limit process. Furthermore, we denote the norm of $L^p(\Omega)$ for any $p \geq 1$ by $\| \, . \, \|_p$. In some places we will use "$\to$", "$\rightharpoonup$" and "$\overset{*}{\rightharpoonup}$" to denote the strong convergence, weak convergence, and weak star convergence respectively.

2. Notation and preliminaries involving the space $BV(\Omega)$

In this section, we recall several facts on functions of bounded variation that we shall use.

Throughout the paper, without further mentioning, given an open bounded set Ω in $\mathbb{R}^N$ with Lipschitz boundary, we denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure and $|\Omega|$ stands for the N-dimensional Lebesgue measure. Moreover, we shall denote by $\mathcal{D}(\Omega)$ or $C_0^\infty(\Omega)$, the space of infinitely differentiable functions with compact support in Ω and $\nu(x)$ is the outer vector normal defined for $\mathcal{H}^{N-1}$-almost everywhere $x \in \partial\Omega$.

We will denote by $BV(\Omega)$ the space of functions of bounded variation, that is,

$$BV(\Omega) = \left\{ u \in L^1(\Omega) : \, Du \text{ is a bounded Radon measure} \right\},$$

where $Du : \Omega \to \mathbb{R}^N$ denotes the distributional gradient of u. It can be proved that $u \in BV(\Omega)$ is equivalent to $u \in L^1(\Omega)$ and

$$\int_\Omega |Du| := \sup \left\{ \int_\Omega u \, \mathrm{div}\varphi \, dx : \, \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), \, |\varphi(x)| \leq 1 \; \forall x \in \Omega \right\} < +\infty,$$

where $|Du|$ is the total variation of the vectorial Radon measure. It is well known that the space $BV(\Omega)$ endowed with the norm

$$\|u\|_{BV(\Omega)} := \int_\Omega |Du| + \|u\|_{L^1(\Omega)}$$

is a Banach space that is non reflexive and non separable. For more information on functions of bounded variation we refer the reader to [9,16,36].

From [9, Theorem 3.87], the notion of a trace on the boundary can be extended to functions $u \in BV(\Omega)$, through a bounded operator $BV(\Omega) \hookrightarrow L^1(\partial\Omega)$, which is also onto. As a consequence, an equivalent norm on $BV(\Omega)$ can be defined by

$$\|u\| := \int_\Omega |Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}. \tag{2.1}$$

In addition, by [9, Corollary 3.49] the following continuous embeddings hold

$$BV(\Omega) \hookrightarrow L^m(\Omega) \;\text{ for every }\; 1 \leq m \leq 1^* = \frac{N}{N-1}, \tag{2.2}$$

which are compact for $1 \leq m < 1^*$.

In what follows, let us recall several important results from [8] that will be used throughout the paper. Following [8], for each $z \in X_r(\Omega)$ and $w \in BV(\Omega) \cap L^{r'}(\Omega)$ where r' is the conjugate of r, we define the functional $(z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by formula

$$\langle (z, Dw), \varphi \rangle = -\int_\Omega w\varphi \mathrm{div}(z) \, dx - \int_\Omega wz.\nabla\varphi \, dx, \;\; \forall \varphi \in C_0^\infty(\Omega). \tag{2.3}$$

Then by [8, Theorem 1.5], (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z.\nabla w \, dx, \quad \forall w \in W^{1,1}(\Omega)$$

and

$$\left| \int_\Omega (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B |Dw|, \tag{2.4}$$

for every Borel set B with $B \subseteq \Omega$. Moreover, besides the BV-norm, for any nonnegative smooth function φ the functional given by

$$w \mapsto \int_{\Omega} \varphi |Dw|$$

is lower semicontinuous with respect to the L^1-convergence, for details see [1].

In [8], a weak trace on $\partial\Omega$ of normal component of $z \in X_r(\Omega)$ is defined as the application $[z,\nu] : \partial\Omega \to \mathbb{R}$, such that $[z,\nu] \in L^\infty(\partial\Omega)$ and $\|[z,\nu]\|_\infty \leq \|z\|_\infty$. In addition, this definition coincides with the classical one, that is

$$[z,\nu] = z.\nu, \text{ for } z \in C^1(\overline{\Omega_\delta}, \mathbb{R}^N), \tag{2.5}$$

where $\Omega_\delta = \{x \in \Omega : d(x,\Omega) < \delta\}$, for some $\delta > 0$ sufficiently small. We recall the Green formula involving the measure (z, Dw) and the weak trace $[z,\nu]$ that was given in [8, Theorem 1.9], namely:

$$\int_{\Omega} (z, Dw) + \int_{\Omega} w \operatorname{div} z \, dx = \int_{\partial\Omega} w[z,\nu] \, d\mathcal{H}^{N-1}, \tag{2.6}$$

for $z \in X_r(\Omega)$ and $w \in BV(\Omega) \cap L^{r'}(\Omega)$.

Next, we prove the following lemma that will be crucial in the proof of Theorem 1.1.

Lemma 2.1. *Assume (1.5) and let $BV_{rad}(\Omega) = \{u \in BV(\Omega) : u(x) = u(|x|)\}$. Then, there exists $C > 0$ such that*

$$\sup_{x \in \overline{\Omega}} |u(x)| \leq C a^{1-N} \|u\|, \quad \forall u \in BV_{rad}(\Omega). \tag{2.7}$$

Hence, the embedding $BV_{rad}(\Omega) \hookrightarrow L^\infty(\Omega)$ is continuous and $BV_{rad}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $p \in [1,\infty)$.

Proof. From [20, Lemma 4.1], if $u \in BV_{rad}(\mathbb{R}^N)$ we have that

$$|u(x)| \leq \frac{1}{|x|^{N-1}} \|u\|, \quad \text{a.e. in} \quad \mathbb{R}^N. \tag{2.8}$$

Setting

$$\widetilde{u} = \begin{cases} u \text{ if } x \in \Omega, \\ 0 \text{ if } x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{2.9}$$

in view of [16, Theorem 5.8], we have $\widetilde{u} \in BV(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |D\widetilde{u}| = \int_{\Omega} |Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}.$$

This combined with (2.8) shows (2.7) and the continuous embedding $BV_{rad}(\Omega) \hookrightarrow L^\infty(\Omega)$. The compact embedding $BV_{rad}(\Omega) \hookrightarrow L^p(\Omega)$ for $p \in [1,\infty)$ follows combining the interpolation in the Lebesgue's space together with the compact embedding $BV_{rad}(\Omega) \hookrightarrow L^1(\Omega)$ and the continuous embedding $BV_{rad}(\Omega) \hookrightarrow L^\infty(\Omega)$. □

3. Proof of Theorem 1.1

This section is devoted to prove Theorem 1.1. From now on, p_0 is the constant fixed in (f_1). For each $p \in (1, p_0)$, let us consider the following problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(u) & \text{in} \quad \Omega \times (0, +\infty), \\ u = 0 & \text{in} \quad \partial\Omega \times (0, +\infty), \\ u(x, 0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{3.1}$$

where

$$\Omega = \{x \in \mathbb{R}^N \ : \ 0 < a < |x| < b\}.$$

In the sequel, we denote by $E_p : W^{1,p}_{0,rad}(\Omega) \to \mathbb{R}$ the energy functional associated with problem (3.1) given by

$$E_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \, dx - \int_\Omega F(u) \, dx, \tag{3.2}$$

and the Nehari set associated with E_p given by

$$\mathcal{N}_p = \left\{ u \in W^{1,p}_{0,rad}(\Omega) \setminus \{0\} \ : \ E_p'(u)u = 0 \right\}.$$

Hereafter, let us also denote by d_p the following real number

$$d_p = \inf_{u \in \mathcal{N}_p} E_p(u).$$

The potential well associated with problem (3.1) is the set

$$W_p = \left\{ u \in W^{1,p}_{0,rad}(\Omega) \ : \ E_p(u) < d_p \text{ and } I_p(u) > 0 \right\} \cup \{0\}, \tag{3.3}$$

where $I_p(u) = E_p'(u)u$ for all $u \in W^{1,p}_{0,rad}(\Omega)$.

Our first lemma establishes an estimate from above for d_p that will be used later on.

Lemma 3.1. *There are $p_1 \in (1, p_0)$ and $M > 0$ such that $d_p \le M$ for all $p \in (1, p_1]$.*

Proof. Let $\varphi \in C^\infty_{0,rad}(\Omega) \setminus \{0\}$. By (f_1) and (f_2), for each $p \in (1, p_0)$ there is $s_p > 0$ such that $s_p\varphi \in \mathcal{N}_p$, that is,

$$s_p^{p-1}\int_\Omega |\nabla \varphi|^p \, dx = \int_\Omega f(s_p\varphi)\varphi \, dx.$$

Since $\lim_{p \to 1^+}\int_\Omega |\nabla \varphi|^p \, dx = \int_\Omega |\nabla \varphi| \, dx$, the condition (f_2) ensures that (s_p) is bounded for p close to 1. Now, using the inequality below

$$d_p \le E_p(s_p\varphi) \le \frac{s_p^p}{p}\int_\Omega |\nabla \varphi|^p \, dx,$$

we deduce that there are $p_1 \in (1, p_0)$ and $M > 0$ such that $d_p \leq M$ for all $p \in (1, p_1)$. This proves the desired result. $\square$

As a byproduct, decreasing $p_1 \in (1, p_0)$ if necessary, we may assume from (1.9) that

$$E_p(u_0) < d_p \quad \text{and} \quad I_p(u_0) > 0, \quad \forall p \in (1, p_1).$$

In the sequel, we are supposing that $p \in (1, p_1)$ and $p_1 < \theta$.

For the reader's convenience, we state the definition of weak solution to (3.1).

Definition 3.1. (global weak solution) We say that $u \in L^\infty(0, +\infty; W^{1,p}_{0,rad}(\Omega))$ is a global weak solution of problem (3.1) if $u_t \in L^2(0, +\infty; L^2(\Omega))$ and the following equalities hold:

(1) $\int_\Omega u_t(t)v\,dx + \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v\,dx = \int_\Omega f(u(t))v\,dx$
for each $v \in W^{1,p}_0(\Omega) \cap L^2(\Omega)$ and a.e. time $0 < t < +\infty$, and
(2) $u(0) = u_0$.

In order to show the global existence of solutions to (3.1) we will apply the Galerkin method. The proof will be divided into three steps:

Step 1: For $N \geq 2$, we have the Gelfand triple

$$W^{1,p}_{0,rad}(\Omega) \hookrightarrow^{c,d} L^2(\Omega) \hookrightarrow^{c,d} \left(W^{1,p}_{0,rad}(\Omega)\right)',$$

which is well defined because of (1.6). Here, $\hookrightarrow^{c,d}$ denotes a dense and continuous embedding. Let $\{V_m\}_{m\in\mathbb{N}}$ be a Galerkin scheme of the separable Banach space $W^{1,p}_{0,rad}(\Omega)$, i.e.,

$$V_m = span\{w_1, w_2, \ldots, w_m\}, \quad \overline{\bigcup_{m\in\mathbb{N}} V_m}^{\|\cdot\|_{W^{1,p}_{0,rad}(\Omega)}} = W^{1,p}_{0,rad}(\Omega), \tag{3.4}$$

where $\{w_j\}_{j=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$. Since $u_0 \in W^{1,p}_{0,rad}(\Omega)$, there exists $u_{0m} \in V_m$ such that

$$u_m(0) = u_{0m} = \sum_{j=1}^m a_{jm}w_j \to u_0 \text{ strongly in } W^{1,p}_{0,rad}(\Omega) \text{ as } m \to \infty. \tag{3.5}$$

From now on, let us denote by $\|\cdot\|_{1,p}$ the usual norm in $W^{1,p}_{0,rad}(\Omega)$ given by

$$\|u\|_{1,p} = \|\nabla u\|_p, \quad \forall u \in W^{1,p}_{0,rad}(\Omega).$$

For each m, we look for the approximate solutions $u_m(x,t) = \sum_{j=1}^m g_{jm}(t)w_j(x)$ satisfying the following identities:

$$\int_\Omega u_{mt}(t)w_j\,dx + \int_\Omega |\nabla u_m(t)|^{p-2}\nabla u_m(t).\nabla w_j\,dx = \int_\Omega f(u_m(t))w_j\,dx, \quad j \in \{1, \ldots, m\}, \tag{3.6}$$

with the initial conditions

$$u_m(0) = u_{0m}. \tag{3.7}$$

Then (3.6) − (3.7) is equivalent to the following initial value problem for a system of nonlinear ordinary differential equations on g_{jm}:

$$\begin{cases} g'_{jm}(t) = H_j(g(t)),\ j = 1, 2, \dots, m,\ t \in [0, t_0], \\ g_{jm}(0) = a_{jm},\ j = 1, 2, \dots, m, \end{cases} \tag{3.8}$$

where $H_j(g(t)) = -\int_\Omega |\nabla u_m(t)|^{p-2}\nabla u_m(t).\nabla w_j\, dx + \int_\Omega f(u_m)w_j\, dx$. By the Picard iteration method, there is $t_{0,m} > 0$ depending on $|a_{jm}|$ such that problem (3.8) admits a unique local solution $g_{jm} \in C^1([0, t_{0,m}])$. Hereafter, we will assume that $[0, T_{0,m})$ is the maximal interval of existence of the solution $u_m(t)$.

Step 2: Multiplying the j^{th} equation in (3.6) by $g'_{jm}(t)$ and summing over j from 1 to m, we obtain

$$\|u_{mt}(t)\|_2^2 + \frac{d}{dt}E_p(u_m(t)) = 0, \quad t \in [0, T_{0,m}). \tag{3.9}$$

Integrating (3.9) over $(0, t)$ we arrive at

$$\int_0^t \|u_{ms}(s)\|_2^2\, ds + E_p(u_m(t)) = E_p(u_{0m}),\ t \in [0, T_{0,m}). \tag{3.10}$$

Since (u_{0m}) converges to u_0 strongly in $W^{1,p}_{0,rad}(\Omega)$, the continuity of E ensures that

$$E_p(u_{0m}) \to E_p(u_0),\ \text{as} \quad m \to +\infty.$$

From the assumption that $E_p(u_0) < d_p$, we have $E_p(u_{0m}) < d_p$ for sufficiently large m. This combined with (3.10) leads to

$$E_p(u_m(t)) < d_p, \quad t \in [0, T_{0,m}), \tag{3.11}$$

for sufficiently large m. Now, we are going to show that $T_{0,m} = +\infty$ and

$$u_m(t) \in W_p,\ \ \forall t \geq 0, \tag{3.12}$$

for sufficiently large m. Suppose by contradiction that $u_m(t_1) \notin W_p$ for some $t_1 \in [0, T_{0,m})$. Let $t_* \in [0, T_{0,m})$ be the smallest time for which $u_m(t_*) \notin W_p$. Then, by continuity of $u_m(t)$, we get $u_m(t_*) \in \partial W_p$. Hence, it turns out that

$$E_p(u_m(t_*)) = d_p, \tag{3.13}$$

or

$$u_m(t_*) \neq 0 \quad \text{and} \quad I_p(u_m(t_*)) = 0. \tag{3.14}$$

It is clear that (3.13) could not occur by (3.11). On the other hand, if (3.14) holds, then the definition of d_p implies in the inequality below

$$E_p(u_m(t_*)) \geq \inf_{u \in \mathcal{N}} E_p(u) = d_p,$$

which also contradicts (3.11). Consequently, (3.12) holds.
From (3.11) and (3.12),

$$\int_0^t \|u_{ms}(s)\|_2^2\,ds + \left(\frac{1}{p}-\frac{1}{\theta}\right)\int_\Omega |\nabla u_m(t)|^p\,dx < d_p,\ t\in[0,T_{0,m}). \tag{3.15}$$

Thus, it turns out that

$$\int_\Omega |\nabla u_m(t)|^p\,dx < \frac{\theta p d_p}{\theta - p} \ \text{ and } \ \int_0^t \|u_{ms}(s)\|_2^2\,ds < d_p,\ \ t\in[0,T_{0,m}), \tag{3.16}$$

for m large enough. Hence, the above estimates give $T_{0,m} = +\infty$. Here we are using the fact that if $T_{0,m} < +\infty$, then we must have $\lim\limits_{t\to T_{0,m}^-} \|u_m(t)\|_{1,p} = +\infty$, see [32, Lemma 2.4, p. 48].
An important inequality that we will be used later on is the following:

Claim 3.2. $\|u_m(t)\|_2 \le \|u_0\|_2$ *for all* $t \ge 0$ *and* m *large enough.*

Indeed, multiplying the j^{th} equation in (3.6) by $g_{jm}(t)$ and summing up over $j = 1,\dots m$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 = -I_p(u_m(t)) < 0, \quad \forall t\in[0,+\infty),$$

for m large. This yields that $t \mapsto \|u_m(t)\|_2^2$ is a decreasing function in $[0,+\infty)$. Thereby,

$$\|u_m(t)\|_2^2 \le \|u_0\|_2^2,\ \ \forall t\in[0,+\infty), \tag{3.17}$$

for m large enough, showing the claim.
Step 3: From (3.16) – (3.17), there is a function u and a subsequence of (u_m), still denoted by (u_m), such that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u & \text{in } \ L^\infty(0,+\infty;W^{1,p}_{0,rad}(\Omega)),\\ u_{mt} \rightharpoonup u_t & \text{in } \ L^2(0,+\infty;L^2(\Omega)),\\ u_m \overset{*}{\rightharpoonup} u & \text{in } \ L^\infty(0,+\infty;L^2(\Omega)),\\ -\text{div}\left(|\nabla u_m|^{p-2}\nabla u_m\right) \overset{*}{\rightharpoonup} \chi & \text{in } \ L^\infty\left(0,+\infty;\left(W^{1,p}_{0,rad}(\Omega)\right)'\right). \end{cases} \tag{3.18}$$

Moreover, in view of (1.6), (3.16) and [34, Corollary 4, p. 85], for any $T > 0$ we have

$$u_m \to u \ \text{ in } C([0,T];C(\overline{\Omega})). \tag{3.19}$$

In particular,

$$u_m \to u \ \text{ in } C([0,T];L^\kappa(\Omega)),\ \forall \kappa\in[1,\infty] \tag{3.20}$$

and

$$u_m(x,t) \to u(x,t) \ \text{ a.e. } (x,t)\in\Omega\times[0,T]. \tag{3.21}$$

Since f is a continuous function, the limit (3.19) ensures that

$$f(u_m) \to f(u) \ \text{ in } C([0,T];C(\overline{\Omega})). \tag{3.22}$$

Thereby, for any fixed j, letting $m \to \infty$ in (3.6), we obtain

$$\int_0^t \int_\Omega u_\tau(\tau) w_j \, dx d\tau + \int_0^t \langle \chi(\tau)), w_j \rangle \, d\tau = \int_0^t \int_\Omega f(u(\tau)) w_j \, dx d\tau, \quad \forall t \in [0,T]. \tag{3.23}$$

From the density of V_m in $W^{1,p}_{0,rad}(\Omega)$, for any $v \in W^{1,p}_{0,rad}(\Omega)$ it follows that

$$\int_\Omega u_t(t) v \, dx + \langle \chi(t), v \rangle = \int_\Omega f(u(t)) v \, dx, \ \text{a.e. } t \in (0,T). \tag{3.24}$$

By (3.18) and [35, see, Lemma 3.1.7],

$$u_m(0) \to u(0) \text{ weakly in } L^2(\Omega).$$

However, by (3.5) we know that $u_m(0) \to u_0$ in $W^{1,p}_{0,rad}(\Omega)$, in particular $u_m(0) \to u_0$ in $L^2(\Omega)$, and so, $u(0) = u_0$. This shows that u satisfies the initial condition.

Our next step is to prove that

$$-\text{div}\left(|\nabla u|^{p-2} \nabla u\right) = \chi. \tag{3.25}$$

In doing so, multiplying (3.6) by $g_{jm}(t)$ and summing up from 1 to m, afterward integrating over $(0,T)$ yields

$$\int_0^T \langle -\text{div}\left(|\nabla u_m(t)|^{p-2} \nabla u_m(t)\right), u_m(t) \rangle \, dt = -\frac{1}{2}\|u_m(T)\|_2^2 + \frac{1}{2}\|u_m(0)\|_2^2 + \int_0^T \int_\Omega f(u_m(t)) u_m(t) \, dx dt. \tag{3.26}$$

From (3.19) and (3.22),

$$\int_0^T \int_\Omega f(u_m(t)) u_m(t) \, dx dt \to \int_0^T \int_\Omega f(u(t)) u(t) \, dx dt. \tag{3.27}$$

Letting $m \to \infty$ in (3.26), we get

$$\begin{aligned}
\limsup_{m\to\infty} \int_0^T \langle -\text{div}\left(|\nabla u_m|^{p-2} \nabla u_m\right), u_m(t) \rangle \, dt &= -\frac{1}{2} \liminf_{m\to\infty} \|u_m(T)\|_2^2 + \frac{1}{2} \lim_{m\to\infty} \|u_{0m}\|_2^2 \\
&\quad + \lim_{m\to\infty} \int_0^T \int_\Omega f(u_m) u_m \, dx dt \\
&\le -\frac{1}{2}\|u(T)\|_2^2 + \frac{1}{2}\|u_0\|_2^2 + \int_0^T \int_\Omega f(u) u \, dx dt \\
&= \int_0^T \langle \chi(t), u(t) \rangle \, dt.
\end{aligned}$$

Hence, from this and the theory of monotone operators (see, e.g., [35, Remark 3.2.2]), we conclude

$$-\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)=\chi. \tag{3.28}$$

Replacing (3.28) in (3.24) we find

$$\int_\Omega u_t(t)v\,dx+\int_\Omega |\nabla u(t)|^{p-2}\nabla u(t).\nabla v\,dx=\int_\Omega f(u(t))v\,dx,\ \text{a.e. in } (0,T). \tag{3.29}$$

As $T>0$ is arbitrary, it follows that

$$\int_\Omega u_t(t)v\,dx+\int_\Omega |\nabla u(t)|^{p-2}\nabla u(t).\nabla v\,dx=\int_\Omega f(u(t))v\,dx,\ \text{a.e. in } (0,+\infty), \tag{3.30}$$

for any $v\in W^{1,p}_{0,rad}(\Omega)$.

Claim 3.3.

$$\int_\Omega u_t(t)v\,dx+\int_\Omega |\nabla u(t)|^{p-2}\nabla u(t).\nabla v\,dx=\int_\Omega f(u(t))v\,dx,\ \textit{a.e. in } (0,+\infty), \tag{3.31}$$

for all $v\in W_0^{1,p}(\Omega)\cap L^2(\Omega)$.

In order to prove Claim 3.3, we will use the Palais' principle due to Squassina [33, Theorem 4]. However, since the energy functional E_p given in (3.2) is not well defined in whole $v\in W_0^{1,p}(\Omega)$, we cannot use this principle directly in our problem. Here, we need to do the following trick: First of all, we fix $t>0$ such that equality in (3.31) is true. Setting $M=\|u(t)\|_\infty+1$, $g(x)=-u_t(t)(x)$ for all $x\in\Omega$ and $h:\mathbb{R}\to\mathbb{R}$ by

$$h(t)=\begin{cases} f(M), & \text{if } t\geq M,\\ f(t), & \text{if } |t|\leq M,\\ f(-M), & \text{if } t\leq -M,\end{cases}$$

it follows that

$$\int_\Omega |\nabla u(t)|^{p-2}\nabla u(t).\nabla v\,dx=\int_\Omega h(u(t))v\,dx+\int_\Omega g(x)v\,dx,\ \forall v\in W^{1,p}_{0,rad}(\Omega). \tag{3.32}$$

Considering the functional $J:W_0^{1,p}(\Omega)\cap L^2(\Omega)\to\mathbb{R}$ given by

$$J(w)=\frac{1}{p}\int_\Omega |\nabla w|^p\,dx-\int_\Omega H(w)\,dx-\int_\Omega g(x)w(x)\,dx,$$

where $H(t)=\int_0^t h(s)\,ds$ and the space $W_0^{1,p}(\Omega)\cap L^2(\Omega)$ endowed with its usual norm, that is,

$$\|u\|_{1,p,2}=\|\nabla u\|_p+\|u\|_2,\quad \forall u\in W_0^{1,p}(\Omega)\cap L^2(\Omega),$$

which is a Banach space. A simple computation gives $J\in C^1(W_0^{1,p}(\Omega)\cap L^2(\Omega),\mathbb{R})$ and that $u(t)$ is a critical point of J restricts to $W^{1,p}_{0,rad}(\Omega)$. Therefore, by Palais' principle due to Squassina [33], we deduce that $u(t)$ is a critical point of J in whole $W_0^{1,p}(\Omega)\cap L^2(\Omega)$, that is,

$$\int_\Omega |\nabla u(t)|^{p-2}\nabla u(t).\nabla v\,dx = \int_\Omega h(u(t))v\,dx + \int_\Omega g(x)v\,dx,\ \forall v \in W_0^{1,p}(\Omega)\cap L^2(\Omega), \tag{3.33}$$

or equivalently

$$\int_\Omega u_t(t)v\,dx + \int_\Omega |\nabla u(t)|^{p-2}\nabla u(t).\nabla v\,dx = \int_\Omega f(u(t))v\,dx,\ \forall v \in W_0^{1,p}(\Omega)\cap L^2(\Omega), \tag{3.34}$$

showing the Claim 3.3.

Next, we show that the solution u satisfies the following energy inequality

$$\int_0^t \|u_\tau(\tau)\|_2^2\,d\tau + E_p(u(t)) \le E_p(u_0),\ \text{a.e. } t\in[0,+\infty). \tag{3.35}$$

To this end, let ψ be a nonnegative function that belongs to $C_0([0,+\infty))$. From (3.10),

$$\int_0^T \psi(t)\,dt \int_0^T \|u_{m\tau}(\tau)\|_2^2\,d\tau + \int_0^T E_p(u_m(t))\psi(t)\,dt = \int_0^T E(u_m(0))\psi(t)\,dt. \tag{3.36}$$

The right-hand side of (3.36) converges to

$$\int_0^T E_p(u_0)\psi(t)\,dt,$$

as $m\to\infty$. The second term in the left-hand side $\int_0^T E_p(u_m(t))\psi(t)\,dt$ is lower semicontinuous with respect to the weak topology of $W_0^{1,p}(\Omega)$. Hence

$$\int_0^T E_p(u(t))\psi(t)\,dt \le \liminf_{m\to\infty}\int_0^T E_p(u_m(t))\psi(t)\,dt. \tag{3.37}$$

Thus, the proof is now complete.

3.1. Existence of solution for (1.1)

In what follows, we set $p_m\to 1^+$ and $u_m = u_{p_m}$ the solution obtained in the last subsection, that is,

$$u_m \in L^\infty(0,+\infty; W_{0,rad}^{1,p_m}(\Omega)),$$
$$u_{mt} \in L^2(0,+\infty; L^2(\Omega))$$

and

$$\int_\Omega u_{mt}(t)v\,dx + \int_\Omega |\nabla u_m(t)|^{p_m-2}\nabla u_m(t).\nabla v\,dx = \int_\Omega f(u_m(t))v\,dx,\ \text{a.e. in } (0,+\infty), \tag{3.38}$$

for all $v\in W_0^{1,p_m}(\Omega)\cap L^2(\Omega)$ and $m\in\mathbb{N}$. Moreover, from above we have

$$\int_\Omega |\nabla u_m(t)|^{p_m}\,dx \le \frac{\theta p_m d_{p_m}}{\theta - p_m}, \qquad \int_0^t \|u_{ms}(s)\|_2\,ds \le d_{p_m} \text{ and } \|u_m(t)\|_2 \le \|u_0\|_2, \text{ for } t \in [0,+\infty). \tag{3.39}$$

Since $\theta > 1$, $p_m \to 1^+$ and (d_{p_m}) is bounded by Lemma 3.1, there exists $C_1 > 0$ such that

$$\int_\Omega |\nabla u_m(t)|^{p_m}\,dx < C_1, \quad \int_0^t \|u_{ms}(s)\|_2^2\,ds < C_1, \quad \|u_m(t)\|_2 \le \|u_0\|_2, \ \forall\, t \in [0,+\infty) \text{ and } m \in \mathbb{N}. \tag{3.40}$$

By Young's inequality,

$$\int_\Omega |\nabla u_m(t)|\,dx \le \frac{1}{p_m}\int_\Omega |\nabla u_m(t)|^{p_m}\,dx + \frac{p_m - 1}{p_m}|\Omega|, \ \forall t \in [0,+\infty) \text{ and } m \in \mathbb{N}. \tag{3.41}$$

Hence, there exists $C_2 > 0$ such that

$$\int_\Omega |\nabla u_m(t)|\,dx \le C_2, \ \forall t \in [0,+\infty) \text{ and } m \in \mathbb{N}. \tag{3.42}$$

Using Hölder's inequality and (3.40),

$$\|u_m(t)\|_1 \le |\Omega|^{1/2}\|u_m\|_2 \le |\Omega|^{1/2}\|u_0\|_2, \text{ for any } \ t \in [0,+\infty) \text{ and } m \in \mathbb{N}, \tag{3.43}$$

showing that (u_m) is a bounded sequence in $L^\infty(0,+\infty; L^1(\Omega))$. Recalling that the usual norm in $BV(\Omega)$ is

$$\|u\|_{BV(\Omega)} = \int_\Omega |Du| + \|u\|_1, \quad \forall u \in BV(\Omega),$$

it follows from (3.42) and (3.43) that there is $C_4 > 0$ such that

$$\int_\Omega |\nabla u_m|\,dx + \int_\Omega |u_m|\,dx \le C_4, \quad \forall t \in (0,+\infty),$$

showing that (u_m) is bounded in $L^\infty(0,+\infty; BV_{rad}(\Omega))$. Then, by Lemma 2.1, we derive that (u_m) is also bounded in $L^\infty(0,+\infty; L^\infty(\Omega))$. Moreover, this implies that $(f(u_m))$ is a bounded sequence in $L^\infty(0,+\infty; L^\infty(\Omega))$.

As an immediate consequence of the above analysis, we deduce that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u & \text{in} \quad L^\infty(0,+\infty; BV_{rad}(\Omega)), \\ u_{mt} \rightharpoonup u_t & \text{in} \quad L^2(0,+\infty; L^2(\Omega)). \end{cases} \tag{3.44}$$

By Lemma 2.1 and [34, Corollary 4, p. 85], for any $T > 0$ we have

$$u_m \to u \ \text{ in } C([0,T]; L^\kappa(\Omega)), \ \forall \kappa \in [1,\infty] \tag{3.45}$$

and

$$u_m(x,t) \to u(x,t) \ \text{ a.e. } (x,t) \in \Omega \times [0,T]. \tag{3.46}$$

In view of (3.45), $u_m(t) \to u(t)$ in $L^1(\Omega)$ for any $t \in [0,+\infty)$. Therefore, $u(t) \in BV(\Omega)$ for any $t \in [0,+\infty)$ and

$$\liminf_{m\to+\infty} \int_\Omega \varphi|\nabla u_m(t)|\,dx \geq \int_\Omega \varphi|Du(t)|, \quad \forall 0 \leq \varphi \in C_0^1(\Omega) \text{ and } \forall t \in [0,+\infty) \text{ (see [1]).}$$

Moreover, from (2.1) and (3.41),

$$\liminf_{m\to+\infty} \int_\Omega |\nabla u_m(t)|^{p_m}\,dx \geq \liminf_{m\to+\infty} \left(\int_\Omega |\nabla u_m(t)|\,dx + \int_{\partial\Omega} |u_m|\,d\mathcal{H}^{N-1} \right) \geq \|u(t)\|, \quad \forall t \in [0,+\infty) \tag{3.47}$$

and by (3.46),

$$f(u_m) \overset{*}{\rightharpoonup} f(u) \text{ in } L^\infty(0,T;L^s(\Omega)), \quad \forall s \in (1,\infty),\ \forall T > 0. \tag{3.48}$$

Claim 3.4. *There exists a vector field $z \in L^\infty(0,+\infty;L^\infty(\Omega))$ with $div\, z(t) \in L^2(\Omega)$ such that, up to subsequence,*

(i) $$|\nabla u_m|^{p_n-2}\nabla u_n \rightharpoonup z \quad in \quad L^s(0,T;L^s(\Omega)), \quad \forall s > 1 \quad and \quad \forall T > 0,$$

(ii) $$|z(t)|_\infty \leq 1, \quad \forall t > 0,$$

(iii) $$(z(t), Du(t)) = |Du(t)|, \quad as\ measures\ on \quad \Omega,\ a.e.\ in\ (0,+\infty),$$

(iv) $$[z(t),\nu] \in sign(-u(t))\ \ \mathcal{H}^{N-1} - a.e\ \ on\ \partial\Omega,$$

and

(v) $$u_t(t) - div\, z(t) = f(u(t)), \quad in \quad \mathcal{D}'(\Omega),\ a.e.\ in\ (0,+\infty).$$

Let us prove this claim. For each $s > 1$, there exists $m_0 = m_0(s) \in \mathbb{N}$ such that $s(p_m - 1) < p_m$ for all $m \geq m_0$. Thus, for each $T > 0$, $|\nabla u_m|^{p_m-2}\nabla u_m \in L^s(0,T;L^s(\Omega))$ for all $m \geq m_0$ and

$$\left(\int_0^T \||\nabla u_m(t)|^{p_m-2}\nabla u_m(t)\|_{L^s(\Omega)}^s\,dt \right)^{\frac{1}{s}} \leq |\Omega|^{\frac{1}{s}-\frac{p_m-1}{p_m}} \left(\int_0^T \|\nabla u_m\|_{p_m}^{(p_m-1)s}\,dt \right)^{\frac{1}{s}}, \quad \forall m \geq m_0.$$

Thus, by (3.40),

$$\left(\int_0^T \||\nabla u_m(t)|^{p_m-2}\nabla u_m(t)\|_{L^s(\Omega)}^s\,dt \right)^{\frac{1}{s}} \leq M^{\frac{(p_m-1)}{p_m}} |\Omega|^{\frac{1}{s}-\frac{p_m-1}{p_m}} T^{\frac{1}{s}}, \quad \forall m \geq n_0, \tag{3.49}$$

where $M = \sup_{m\in\mathbb{N}} \int_\Omega |\nabla u_m|^{p_m}\,dx$. Since $L^s(0,T;L^s(\Omega))$ is reflexive, there is $z \in L^s_{loc}(0,+\infty;L^s(\Omega))$, such that for all $s > 1$,

$$|\nabla u_m|^{p_m-2}\nabla u_m \rightharpoonup z \quad \text{in} \quad L^s(0,T;L^s(\Omega)), \quad \forall T > 0. \tag{3.50}$$

The last limit combined with (3.49) gives

$$|z|_{L^s(0,T;L^s(\Omega))} \leq (|\Omega|T)^{\frac{1}{s}}, \quad \forall s > 1,$$

from where it follows that $z \in L^\infty(0,T;L^\infty(\Omega))$ with

$$|z|_{L^\infty(0,T;L^\infty(\Omega))} \leq 1, \quad \forall T > 0.$$

Hence, $z \in L^\infty(0,+\infty;L^\infty(\Omega)))$ with $|z(t)|_\infty \leq 1$ for all $t > 0$. Finally the equality below

$$\int_\Omega u_{mt}(t)\varphi\,dx + \int_\Omega |\nabla u_m(t)|^{p_m-2}\nabla u_m.\nabla\varphi\,dx = \int_\Omega f(u_m)\varphi\,dx, \quad \forall \varphi \in C_0^1(\Omega)$$

leads to

$$\int_0^T\int_\Omega u_{mt}(t)\varphi\,dxdt + \int_0^T\int_\Omega |\nabla u_m(t)|^{p_m-2}\nabla u_m.\nabla\varphi\,dxdt = \int_0^T\int_\Omega f(u_m)\varphi\,dxdt,$$

for all $\varphi \in C_0^1(\Omega)$ and $T > 0$. This together with the limits (3.50) and (3.48) gives

$$\int_0^T\int_\Omega u_t(t)\varphi\,dxdt + \int_0^T\int_\Omega z(t).\nabla\varphi\,dxdt = \int_0^T\int_\Omega f(u)\varphi\,dxdt, \quad \forall \varphi \in C_0^1(\Omega) \quad \text{and} \quad T > 0,$$

that is,

$$u_t(t) - div\,z(t) = f(u(t)), \quad \text{in} \quad \mathcal{D}'(\Omega) \text{ a.e. in } (0,+\infty),$$

showing (v), and that $\operatorname{div} z(t) \in L^2(\Omega)$. Finally, in order to prove $(iii)-(iv)$, we will adapt some arguments developed in [30, page 57]. As mentioned in [30], the item (iii) follows if we prove the below inequality

$$-\int_\Omega u\operatorname{div}z(t)\,dx - \int_\Omega uz(t).\nabla\varphi\,dx \geq \int_\Omega \varphi|Du(t)|, \quad \text{for all} \quad 0 \leq \varphi \in C_0^1(\Omega) \quad \text{a.e. in } (0,+\infty).$$

From definition of u_m, it follows that for any $0 \leq \varphi \in C_0^1(\Omega)$ and $T > 0$,

$$\int_0^T\int_\Omega u_{mt}(t)u_m(t)\varphi\,dxdt + \int_0^T\int_\Omega |\nabla u_m(t)|^{p_m-2}\nabla u_m.\nabla(u_m\varphi)\,dxdt = \int_0^T\int_\Omega f(u_m)u_m\varphi\,dxdt,$$

and so,

$$\int_0^T\int_\Omega u_{mt}(t)u_m(t)\varphi\,dxdt + \int_0^T\int_\Omega |\nabla u_m(t)|^{p_m}\varphi\,dxdt + \int_0^T\int_\Omega |\nabla u_m(t)|^{p_m-2}u_m(t)\nabla u_m(t).\nabla\varphi\,dxdt$$
$$= \int_0^T\int_\Omega f(u_m(t))u_m(t)\varphi\,dxdt.$$

A direct computation gives

$$\int_0^T\int_\Omega u_{mt}(t)u_m(t)\varphi\,dxdt \to \int_0^T\int_\Omega u_t(t)u(t)\varphi\,dxdt,$$

and

$$\int_0^T\int_\Omega f(u_m(t))u_m(t)\varphi\,dxdt \to \int_0^T\int_\Omega f(u(t))u(t)\varphi\,dxdt.$$

Finally, we would like to point out that by lower semicontinuity,

$$\liminf_{n\to+\infty}\int_\Omega |\nabla u_m(t)|^{p_m}\varphi\,dx \geq \liminf_{n\to+\infty}\int_\Omega \varphi|\nabla u_m(t)|\,dx \geq \int_\Omega \varphi|Du(t)| \quad \text{in } (0,+\infty).$$

The above analysis implies that

$$-\int_0^T\int_\Omega u\mathrm{div}z(t)\,dxdt - \int_0^T\int_\Omega uz(t).\nabla\varphi\,dxdt \geq \int_0^T\int_\Omega \varphi|Du|\,dt, \quad \forall T>0.$$

Therefore,

$$-\int_\Omega u\mathrm{div}z(t)\,dx - \int_\Omega uz(t).\nabla\varphi\,dx \geq \int_\Omega \varphi|Du|, \quad \text{for all} \quad 0\leq\varphi\in C_0^1(\Omega) \quad \text{a.e. in } (0,+\infty),$$

proving (iii).

Now, in order to prove (iv), as shown in [30], it is enough to prove

$$\int_{\partial\Omega} (|u(t)| + u[z(t),\nu])\ d\mathcal{H}^{N-1} \leq 0 \quad \text{a.e. in } (0,+\infty).$$

Using $(u_m-\varphi)$ as a test function, we get

$$\int_0^T\int_\Omega u_{mt}(t)(u_m(t)-\varphi)\,dxdt + \int_0^T\int_\Omega |\nabla u_m(t)|^{p_m}\,dxdt - \int_0^T\int_\Omega |\nabla u_m(t)|^{p_m-2}\nabla u_m(t).\nabla\varphi\,dxdt$$
$$= \int_0^T\int_\Omega f(u_m(t))(u_m(t)-\varphi)\,dxdt, \quad \forall T>0.$$

Letting $m\to+\infty$ and using (3.47), we find

$$\int_0^T\int_\Omega |Du(t)|\,dt + \int_0^T\int_{\partial\Omega} |u(t)|\,d\mathcal{H}^{N-1}dt \leq -\int_0^T\int_\Omega u_t(t)(u-\varphi)\,dxdt + \int_0^T\int_\Omega z(t).\nabla\varphi\,dxdt$$
$$-\int_0^T\int_\Omega f(u)\varphi\,dxdt + \int_0^T\int_\Omega f(u)u\,dxdt,$$

that is,

$$\int_0^T\int_\Omega |Du(t)|\,dt+\int_0^T\int_{\partial\Omega}|u(t)|\,d\mathcal{H}^{N-1}dt\le-\int_0^T\int_\Omega u_t(t)u(t)+\int_0^T\int_\Omega f(u)u\,dxdt$$
$$=-\int_0^T\int_\Omega \mathrm{div}z(t)u(t)\,dxdt.$$

Then by Green's formula (2.6),

$$\int_0^T\int_\Omega |Du(t)|\,dt+\int_0^T\int_{\partial\Omega}|u(t)|\,d\mathcal{H}^{N-1}dt\le-\int_0^T\int_{\partial\Omega} u(t)[z(t),\nu]\,d\mathcal{H}^{N-1}+\int_0^T\int_\Omega |Du(t)|\,dt$$

which leads to

$$\int_0^T\int_{\partial\Omega}(|u(t)|+u(t)[z(t),\nu])\;d\mathcal{H}^{N-1}dt\le 0,\quad \forall T>0.$$

Therefore,

$$\int_{\partial\Omega}(|u(t)|+u(t)[z(t),\nu])\,d\mathcal{H}^{N-1}\le 0,\quad \text{a.e. in}\quad (0,+\infty),$$

showing the desired result. In order to prove (1.10), we can use a similar argument as in the proof of (3.36) and the weak lower semicontinuity of the total variation combined with (3.44) and (3.48). Hence, the proof is now complete.

4. Proof of Theorem 1.2

In this section, we are concerned with the proof Theorem 1.2. Here we just sketch it since it follows similarly as above.

Step 1: For $N\ge 2$, we have the Gelfand triple

$$W_0^{1,p}(\Omega)\cap L^2(\Omega)\hookrightarrow^{c,d} L^2(\Omega)\hookrightarrow^{c,d}\left(W_0^{1,p}(\Omega)\cap L^2(\Omega)\right)'.$$

In this section, we denote by $\|\ \|_{1,p,2}$ the usual norm in $W_0^{1,p}(\Omega)\cap L^2(\Omega)$ given by

$$\|u\|_{1,p,2}=\|\nabla u\|_p+\|u\|_2,\quad \forall u\in W_0^{1,p}(\Omega)\cap L^2(\Omega).$$

Let $\{V_m\}_{m\in\mathbb{N}}$ be a Galerkin scheme of the separable Banach space $W_0^{1,p}(\Omega)\cap L^2(\Omega)$, i.e.,

$$V_m=span\{w_1,w_2,\dots,w_m\},\quad \overline{\bigcup_{m\in\mathbb{N}}V_m}^{\|\cdot\|_{1,p,2}}=W_0^{1,p}(\Omega)\cap L^2(\Omega),\tag{4.1}$$

where $\{w_j\}_{j=1}^\infty$ is an orthonormal basis in $L^2(\Omega)$. Since $u_0\in W_0^{1,p}(\Omega)\cap L^2(\Omega)$ then we can find $u_{0m}\in V_m$ such that

$$u_m(0)=u_{0m}\to u_0 \text{ strongly in } W_0^{1,p}(\Omega)\cap L^2(\Omega) \text{ as } m\to\infty.\tag{4.2}$$

For each m, we look for the approximate solutions $u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t)w_j(x)$ satisfying the following identities:

$$\int_\Omega u_{mt}(t)w_j\,dx + \int_\Omega |\nabla u_m(t)|^{p-2}\nabla u_m(t).\nabla w_j\,dx = \int_\Omega f(u_m(t))w_j\,dx,\ \ j \in \{1,\dots,m\}, \tag{4.3}$$

with the initial conditions

$$u_m(0) = u_{0m}. \tag{4.4}$$

As in the last section, (4.3) − (4.4) is equivalent to the following initial value problem for a system of nonlinear ordinary differential equations on g_{jm}:

$$\begin{cases} g'_{jm}(t) = H_j(g(t)),\ j = 1,2,\dots,m,\ t \in [0,t_0], \\ g_{jm}(0) = a_{jm},\ j = 1,2,\dots,m, \end{cases} \tag{4.5}$$

where $H_j(g(t)) = -\int_\Omega |\nabla u_m(t)|^{p-2}\nabla u_m(t).\nabla w_j\,dx + \int_\Omega f(u_m)w_j\,dx$. By the Picard iteration method, there is $t_{0,m} > 0$ depending on $|a_{jm}|$ such that problem (3.8) admits a unique local solution $g_{jm} \in C^1([0,t_{0,m}])$. Hereafter, we will assume that $[0,T_{0,m})$ is the maximal interval of existence of the solution $u_m(t)$.

Step 2: Multiplying the j^{th} equation in (4.3) by $g'_{jm}(t)$ and summing over j from 1 to m, we obtain

$$\|u_{mt}(t)\|_2^2 + \frac{d}{dt}E_p(u_m(t)) = 0,\ \ t \in [0,T_{0,m}). \tag{4.6}$$

Integrating (3.9) over $(0,t)$ we arrive at

$$\int_0^t \|u_{mt}(s)\|_2^2\,ds + E_p(u_m(t)) = E(u_{0m}),\ t \in [0,T_{0,m}). \tag{4.7}$$

Since u_{0m} converges to u_0 strongly in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$, the continuity of E ensures that

$$E_p(u_{0m}) \to E_p(u_0),\ \text{as}\quad m \to +\infty.$$

From the assumption that $E_p(u_0) < d_p$, we have $E_p(u_{0m}) < d_p$ for sufficiently large m. This combined with (3.10) yields

$$E_p(u_m(t)) < d_p,\quad t \in [0,T_{0,m}), \tag{4.8}$$

for sufficiently large m. In a similar fashion to above, we immediately obtain

$$u_m(t) \in W_p,\ \ \forall t \in [0,T_{0,m}). \tag{4.9}$$

Gathering (3.11) and (3.12), we deduce

$$\int_0^t \|u_{ms}(s)\|_2^2\,ds + \left(\frac{1}{p} - \frac{1}{\theta}\right)\int_\Omega |\nabla u_m(t)|^p\,dx < d_p,\ t \in [0,T_{0,m}). \tag{4.10}$$

Thus, it turns out that

$$\int_{\Omega} |\nabla u_m(t)|^p \, dx < \frac{\theta p d_p}{\theta - p}, \quad \int_0^t \|u_{ms}(s)\|_2^2 \, ds < d_p, \quad t \in [0, T_{0,m}). \tag{4.11}$$

Next, multiplying the j^{th} equation in (4.3) by $g_{jm}(t)$ and summing up over $j = 1, \dots m$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_2^2 = -I_p(u_m(t)) < 0 \quad \forall t \in [0, T_{0,m}).$$

Hence, the function $t \mapsto \|u_m(t)\|_2^2$ for $t \in [0, T_{0,m})$ is decreasing, and so,

$$\|u_m(t)\|_2^2 \le \|u_0\|_2^2, \quad t \in [0, T_{0,m}), \tag{4.12}$$

for m large enough. The above estimates ensure that $T_{0,m} = +\infty$. Here we are using the fact that if $T_{0,m} < +\infty$, then we must have $\lim_{t \to T_{0,m}^-} \|u_m(t)\|_{1,p,2} = +\infty$, see [32, Lemma 2.4, p. 48].

Step 3: From (4.11) − (4.12) we get the existence of a function u and a subsequence of (u_m) still denoted by (u_m) such that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u & \text{in} \quad L^\infty(0, +\infty; W_0^{1,p}(\Omega) \cap L^2(\Omega)), \\ u_{mt} \rightharpoonup u_t & \text{in} \quad L^2(0, +\infty; L^2(\Omega)), \\ -\text{div}\left(|\nabla u_m|^{p-2}\nabla u_m\right) \overset{*}{\rightharpoonup} \chi & \text{in} \quad L^\infty\left(0, +\infty; \left(W_0^{1,p}(\Omega)\right)'\right). \end{cases} \tag{4.13}$$

Moreover, from (4.11) and [34, Corollary 4, p. 85], for any $T > 0$ we have

$$u_m \to u \ \text{ in } C([0, T]; L^\kappa(\Omega)), \ \forall \kappa \in [1, p^*) \tag{4.14}$$

and

$$u_m(x, t) \to u(x, t) \ \text{ a.e. } (x, t) \in \Omega \times [0, T]. \tag{4.15}$$

Since f is a continuous function, the limit (4.15) yields that

$$f(u_m) \to f(u) \ \text{ a.e. } (x, t) \in \Omega \times [0, T]. \tag{4.16}$$

On the other hand, from (f_3), (4.12), and the Hölder inequality,

$$\int_{\Omega} |f(u_m)|^2 \, dx \le 2C\left(|\Omega| + |\Omega|^{\frac{4-2q}{2}}\|u_0\|_2^{2q-2}\right) \tag{4.17}$$

for m large enough. Therefore, from (4.15) and (4.17),

$$f(u_m) \overset{*}{\rightharpoonup} f(u) \quad \text{in } L^\infty(0, T; L^2(\Omega)), \quad \forall T > 0. \tag{4.18}$$

For any fixed j, letting $m \to \infty$ in (4.3), we obtain

$$\int_0^t \int_{\Omega} u_\tau(\tau) w_j \, dx d\tau + \int_0^t \langle \chi(\tau)), w_j \rangle \, d\tau = \int_0^t \int_{\Omega} f(u(\tau)) w_j \, dx d\tau, \quad \forall t \in [0, T]. \tag{4.19}$$

From the density of V_m in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$, for any $v \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ we have

$$\int_\Omega u_t(t)v\,dx + \langle \chi(t), v\rangle = \int_\Omega f(u(t))v\,dx, \text{ a.e. } t \in (0,T). \tag{4.20}$$

By using a similar argument as in the proof of Theorem 1.1, one can shows that u satisfies the first initial condition. Next step is to prove that

$$-\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \chi. \tag{4.21}$$

Since the proof of (4.21) will be similar to that of Theorem 1.1, we only need to prove the following claim

Claim 4.1. *$\int_0^T \int_\Omega f(u_m(t))u_m(t)\,dxdt \to \int_0^T \int_\Omega f(u(t))u(t)\,dxdt$ as $m \to \infty$.*

Indeed, from (f_3), (4.11) and using Hölder's inequality for each measurable set $V \subset \Omega \times [0,T]$, we have

$$\int_V f(u_m)u_m\,dxdt \le CT\left(3|V|^{1/2}\|u_0\|_2 + |V|^{\frac{p^*}{p^*-q}}\left(\frac{\theta p d_p}{\theta - p}\right)^{q/p}\right)$$

for m large enough. In view of (4.15), we have $f(u_m(x,t))u_m(x,t) \to f(u(x,t))u(x,t)$ a.e. in $\Omega \times [0,T]$. Thus, the desired result follows from the Vitali's convergence theorem.
Replacing (3.28) in (4.20) yields

$$\int_\Omega u_t(t)v\,dx + \int_\Omega |\nabla u(t)|^{p-2}\nabla u(t).\nabla v\,dx = \int_\Omega f(u(t))v\,dx, \text{ a.e. in } (0,T), \quad \forall v \in W_0^{1,p}(\Omega)\cap L^2(\Omega). \tag{4.22}$$

As $T > 0$ is arbitrary, it follows that

$$\int_\Omega u_t(t)v\,dx + \int_\Omega |\nabla u(t)|^{p-2}\nabla u(t).\nabla v\,dx = \int_\Omega f(u(t))v\,dx, \text{ a.e. in } (0,+\infty), \tag{4.23}$$

for all $v \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$.

4.1. Existence of solution for the original problem

In what follows, we set $p_m \to 1^+$ and $u_m = u_{p_m}$ the solution obtained in the last version, that is,

$$u_m \in L^\infty(0,+\infty; W_0^{1,p_m}(\Omega) \cap L^2(\Omega)),$$
$$u_{mt} \in L^2(0,+\infty; L^2(\Omega)),$$

and

$$\int_\Omega u_{mt}(t)v\,dx + \int_\Omega |\nabla u_m(t)|^{p_m-2}\nabla u_m(t).\nabla v\,dx = \int_\Omega f(u_m(t))v\,dx, \text{ a.e. in } (0,+\infty), \tag{4.24}$$

for all $v \in W_0^{1,p_m}(\Omega) \cap L^2(\Omega)$.
Moreover, we also have

$$\int_\Omega |\nabla u_m(t)|^{p_m}\,dx \le \frac{\theta p_m d_{p_m}}{\theta - p_m}, \quad \int_0^t \|u_{ms}(s)\|_2^2\,ds \le d_{p_m}, \quad \|u_m(t)\|_2 \le \|u_0\|, \quad t \in [0,+\infty). \tag{4.25}$$

Since $\theta > 1$, $p_m \to 1^+$, and (d_{p_m}) is a bounded sequence by Lemma 3.1, there is $C_1 > 0$ such that

$$\int_\Omega |\nabla u_m(t)|^{p_m}\,dx < C_1, \quad \int_0^t \|u_{ms}(s)\|_2^2\,ds < C_1, \quad \|u_m(t)\|_2 \le \|u_0\| \ , \quad \forall m \in \mathbb{N}. \tag{4.26}$$

By Young inequality,

$$\int_\Omega |\nabla u_m(t)|\,dx \le \frac{1}{p_m}\int_\Omega |\nabla u_m(t)|^{p_m}\,dx + \frac{p_m - 1}{p_m}|\Omega|, \ \ t \in [0, \infty), \quad \forall m \in \mathbb{N}.$$

Hence, there is $C_2 > 0$ such that

$$\int_\Omega |\nabla u_m(t)|\,dx < C_2, \ \ t \in [0, +\infty), \quad \forall m \in \mathbb{N}. \tag{4.27}$$

Claim 4.2. *(u_m) is a bounded sequence in $L^\infty(0, +\infty; L^1(\Omega))$. Hence, (u_m) is a bounded sequence in $L^\infty(0, +\infty; BV(\Omega))$ and $(f(u_m))$ is a bounded sequence in $L^\infty(0, +\infty; L^2(\Omega))$.*

Indeed, from Hölder's inequality and (4.26),

$$\|u_m(t)\|_1 \le |\Omega|^{1/2}\|u_m\|_2^2 \le |\Omega|^{1/2}\|u_0\|_2^2, \quad \forall t \in [0, +\infty) \quad \text{and} \quad \forall m \in \mathbb{N} \tag{4.28}$$

showing that (u_m) is a bounded sequence in $L^\infty(0, +\infty; L^1(\Omega))$. Recalling that the usual norm in $BV(\Omega)$ is

$$\|u\| = \int_\Omega |Du| + |u|_1, \quad \forall u \in BV(\Omega),$$

it follows from (4.25) and (4.28) that there is $C_4 > 0$ such that

$$\int_\Omega |\nabla u_m|\,dx + \int_\Omega |u_m|\,dx \le C_4, \quad \forall t \in (0, +\infty)$$

showing that (u_m) is bounded in $L^\infty(0, T; BV(\Omega))$. Since $2q - 2 < 2$, the second part of the claim follows directly from (4.17). Hence, $(f(u_m))$ is a bounded sequence in $L^\infty(0, +\infty; L^2(\Omega))$.

The last claim permits to conclude that

$$\begin{cases} u_m \overset{*}{\rightharpoonup} u & \text{in} \quad L^\infty(0, T; BV(\Omega) \cap L^2(\Omega)), \\ u_{mt} \rightharpoonup u_t & \text{in} \quad L^2(0, T; L^2(\Omega)). \end{cases} \tag{4.29}$$

By [34, Corollary 4, p. 85], for any $T > 0$ we have

$$u_m \to u \ \text{ in } C([0, T]; L^\kappa(\Omega)), \ \forall \kappa \in [1, 1^*), \tag{4.30}$$

and

$$u_m(x, t) \to u(x, t) \ \text{ a.e. } (x, t) \in \Omega \times [0, T]. \tag{4.31}$$

Using (4.30), we infer that $u_m(t) \to u(t)$ in $L^1(\Omega)$ for all $t \in [0, +\infty)$. Hence, $u(t) \in BV(\Omega)$ for all $t \in [0, +\infty)$ and

$$\liminf_{m\to+\infty}\int_\Omega \varphi|\nabla u_m(t)|\,dx \geq \int_\Omega \varphi|Du(t)|, \quad \forall 0\leq\varphi\in C_0^1(\Omega) \text{ and } \quad t\in[0,+\infty) \text{ (see [1])}$$

and

$$\liminf_{m\to+\infty}\int_\Omega |\nabla u_m(t)|^{p_m}\,dx \geq \liminf_{m\to+\infty}\left(\int_\Omega |\nabla u_m(t)|\,dx + \int_{\partial\Omega}|u_m|\,d\mathcal{H}^{N-1}\right) \geq \|u(t)\|, \quad \forall t\in[0,+\infty)$$

Moreover, by (4.31) we get

$$f(u_m) \overset{*}{\rightharpoonup} f(u) \text{ in } L^\infty(0,T;L^2(\Omega)), \quad \forall T>0. \tag{4.32}$$

Claim 4.3. *There exists a vector field $z\in L^\infty(0,+\infty;L^\infty(\Omega))$ with $div\, z(t)\in L^2(\Omega)$ such that, up to subsequence,*

(i) $$|\nabla u_m|^{p_m-2}\nabla u_m \rightharpoonup z \quad in \quad L^s(0,T;L^s(\Omega)), \quad \forall s>1 \quad and \quad \forall T>0,$$

(ii) $$|z(t)|_\infty \leq 1, \quad \forall t>0,$$

(iii) $$(z(t),Du(t)) = |Du(t)|, \quad as\ measures\ on \quad \Omega,\ a.e.\ in\ (0,+\infty),$$

(iv) $$[z(t),\nu]\in sign(-u(t)) \ \ \mathcal{H}^{N-1}-a.e \ \ on\ \partial\Omega,$$

and

(v) $$u_t(t) - div\, z(t) = f(u(t)), \quad in \quad \mathcal{D}'(\Omega),\ a.e.\ in\ (0,+\infty).$$

Now the proof of this claim follows as in the proof of Claim 3.4. Therefore the proof of Theorem 1.2 is now complete.

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