



Research Article

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A symbolic approach to multiple Hurwitz zeta values at non-positive integers

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Abstract: In this article, we give another method to calculate the values of multiple Hurwitz zeta function at non-positive integers by means of Raabe's formula and the Bernoulli numbers and we simplify these values by symbolic computation techniques.

Keywords: multiple Hurwitz zeta function, integral representation, special values, Raabe's formula, Bernoulli numbers and symbols

MSC 2020: 11M32, 11M35

1 Introduction

Let n be a positive integer, let $\underline{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ be complex variables, and let $\alpha \neq 0, -1, -2, \dots$ be a real number. The multiple Hurwitz zeta function is defined by

$$\zeta_n(\alpha; s_1, \dots, s_n) = \sum_{\underline{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \frac{1}{(m_1 + \alpha)^{s_1} \cdots (m_1 + \cdots + m_n + \alpha)^{s_n}}. \quad (1.1)$$

This multiple series is convergent absolutely if $\Re(s_{n-k+1} + \cdots + s_n) > k$ for $1 \leq k \leq n$ (see [1]).

For $n = 1$, the series (1.1) is just the classical Hurwitz zeta function

$$\zeta(\alpha; s) = \sum_{n=0}^{+\infty} \frac{1}{(n + \alpha)^s}. \quad (1.2)$$

In recent years, multiple Hurwitz zeta functions have attracted wide attention from many scholars. They are not only important for the general zeta function theory but also appear in different mathematical fields, for example, but not limited, the theory of special functions (see [2]), holomorphic dynamics (see [3]), renormalization theory (see [4]), knot theory (see [5]), and quantum field theory (see [6]).

In [7], Katsurada and Matsumoto proved that for $n = 2$, the series (1.1) can be continued meromorphically to the whole space \mathbb{C}^2 . For the general case n , the meromorphic continuation of (1.1) for general r was established by Akiyama and Ishikawa [8].

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For $\alpha = 1$, the series (1.1) becomes the multiple zeta function, which has been studied by many researchers (see [5,9],...). When $n = 2$, (1.1) is the classical Euler sum, and its analytic continuation was first obtained by Atkinson [10]. In the case $n > 1$, the analytic continuation of (1.1) was proved by Zhao [11] and proved independently by Akiyama and Ishikawa [8]. For more details on the history of the problem of the series (1.1), the readers are referred, without limitation, to ([1,12,13],...).

In this article, we give the values at the non-positive integers of a type of the multiple Hurwitz zeta function (see (3.1)) by the use of the “*Raabe* formula”,¹ which expresses an integral in terms of sum. Encouraged by the simplification that symbolic computation may bring to the manipulation of complicated sums, we use here symbols which is simply a scaled version of that introduced by Gessel [16]; these symbols simplify these values and give a general recursion of this series on their depth.

2 The symbols \mathcal{B} and \mathcal{C}

2.1 The generalized Bernoulli symbols \mathcal{B}

In what follows, we introduce the generalized Bernoulli symbol \mathcal{B} (see, e.g., [16,17]), with the evaluation rule, for all $\alpha \in \mathbb{R} \setminus \{0\}$ and for all $n \in \mathbb{N}$, we have:

$$\mathcal{B}^n = \alpha^{-n} B_n \Leftrightarrow B_n = (\alpha \mathcal{B})^n, \quad (2.1)$$

where B_n is the n th Bernoulli number, defined by the generating function

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n = \frac{z}{e^z - 1}. \quad (2.2)$$

This definition gives

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{(\alpha \mathcal{B})^n}{n!} z^n = e^{\alpha \mathcal{B} z}. \quad (2.3)$$

Moreover, the Bernoulli polynomials $B_n(x)$ with generating function

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = \frac{z}{e^z - 1} e^{zx}, \quad (2.4)$$

which gives

$$\sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n = e^{\alpha \mathcal{B} z} e^{zx}. \quad (2.5)$$

So, we have the simple symbolic expression

$$B_n(x) = (\alpha \mathcal{B} + x)^n, \quad (2.6)$$

since

$$B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k = \sum_{k=0}^n \binom{n}{k} (\alpha \mathcal{B})^{n-k} x^k = (\alpha \mathcal{B} + x)^n, \quad (2.7)$$

which produces, for $x = \alpha$

$$(\mathcal{B} + 1)^n = \alpha^{-n} B_n(\alpha) := \mathcal{D}^n, \quad (2.8)$$

¹ *Raabe*'s 1843 formula is $\int_0^1 \log \Gamma(x+t) / \sqrt{2} dt = x \log x$. See [14, p. 367] for the connection between the sum and the integral. A p-adic version of *Raabe*'s formula was given in [15].

where

$$\mathcal{D}^n := \alpha^{-n} B_n(\alpha). \quad (2.9)$$

Additionally, for distinct generalized Bernoulli symbols $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_n$, for all $p_1, p_2, \dots, p_n \geq 1$, we have

$$\mathcal{B}_1^{p_1} \mathcal{B}_2^{p_2} \dots \mathcal{B}_n^{p_n} = \left(\alpha^{-\sum_{k=1}^n p_k} \right) B_{p_1} B_{p_2} \dots B_{p_n}. \quad (2.10)$$

In the exceptional case where the generalized symbols $\mathcal{B}_k = \mathcal{B}_l$, then we have

$$\mathcal{B}_1^{p_1} \dots \mathcal{B}_k^{p_k} \dots \mathcal{B}_l^{p_l} \dots \mathcal{B}_n^{p_n} = \left(\alpha^{-\sum_{k=1}^n p_k} \right) B_{p_1} \dots B_{p_k+p_l} \dots B_{p_n}. \quad (2.11)$$

Remark 2.1. In the case $\alpha = 1$, we find the Bernoulli symbols used in [18] where we can find an example.

2.2 The C symbols

Now, we introduce the symbols $C_{1,2,\dots,k}$ defined recursively in terms of the generalized Bernoulli symbols $\mathcal{B}_1, \dots, \mathcal{B}_{k+1}$ as

$$C_1^n := \frac{\mathcal{D}_1^n}{n}, \quad C_{1,2}^n := \frac{(C_1 + \mathcal{D}_2)^n}{n}, \dots \text{ and } C_{1,2,\dots,k+1}^n := \frac{(C_{1,2,\dots,k} + \mathcal{D}_{k+1})^n}{n}, \quad (2.12)$$

where, for all $1 \leq j \leq k+1$, the symbol \mathcal{D}_j is defined by $\mathcal{D}_j := (\mathcal{B}_j + 1)$.

The expressions can be expanded to obtain formulas involving only generalized Bernoulli symbols \mathcal{B}_k . For example, the term

$$\begin{aligned} C_1^{n_1} C_{1,2}^{n_2} &= C_1^{n_1} \frac{(C_1 + \mathcal{D}_2)^{n_2}}{n_2} = \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{C_1^{n_1+k} (\mathcal{D}_2)^{n_2-k}}{n_2} \\ &= \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{(\mathcal{D}_1)^{n_1+k} (\mathcal{D}_2)^{n_2-k}}{(n_1+k)n_2} = \alpha^{-n_1-n_2} \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{(\alpha \mathcal{D}_1)^{n_1+k} (\alpha \mathcal{D}_2)^{n_2-k}}{(n_1+k)n_2} \end{aligned}$$

is evaluated as (using (2.9))

$$C_1^{n_1} C_{1,2}^{n_2} = \frac{\alpha^{-n_1-n_2}}{n_2} \sum_{k=0}^{n_2} \binom{n_2}{k} \frac{B_{n_1+k}(\alpha)}{(n_1+k)} B_{n_2-k}(\alpha). \quad (2.13)$$

For more details of these symbols, the reader can see [18].

3 The main results

For real numbers $\alpha \in \mathbb{R}$, such that: $\alpha \neq 0, -1, -2, \dots$, and complex n -tuples $\underline{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$, we define the following type of the multiple Hurwitz zeta function

$$\zeta_n(\alpha; \underline{s}) := \zeta_n(\alpha; s_1, \dots, s_n) \quad (3.1)$$

$$= \sum_{\underline{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \prod_{i=1}^n \frac{1}{(m_1 + \dots + m_i + i\alpha)^{s_i}} \quad (3.2)$$

$$= \sum_{m_1 > \dots > m_n > 0} \prod_{i=1}^n \frac{1}{(m_i + i\alpha)^{s_i}}, \quad (3.3)$$

and the corresponding integral function associated with the multiple Hurwitz zeta function is defined by

$$Y_n(\alpha; \mathbf{s}) = \int_{[0,+\infty]^n} \prod_{i=1}^n \frac{1}{(x_1 + \dots + x_i + i\alpha)^{s_i}} d\mathbf{x}. \quad (3.4)$$

Remark 3.1. We remark that, if $\alpha = 1$, then the series (3.1) is corresponding to the multiple zeta function.

For the meromorphic continuation of the integral (3.4) and the series (3.1), we refer the reader to the work of [11,13,19–21].

We first give the well-known elementary result for the integral function.

Proposition 3.1. Let $\mathbf{N} = (N_1, \dots, N_n)$ be a point of \mathbb{N}^n ,

(1) The point $(\mathbf{s} = -\mathbf{N})$ is a polar divisor for the function $Y_n(\alpha; \mathbf{s})$ if and only if there exists a $\mathbf{k} = (k_2, \dots, k_n) \in \mathbb{N}^{n-1}$ such that

$$(s_n - 1)(s_n + s_{n-1} - 2 + k_n) \cdots \left(\sum_{i=1}^n s_i - n + \sum_{i=2}^n k_i \right) = \prod_{j=1}^n \left(\sum_{i=j}^n s_i - n + j - 1 + \sum_{i=j+1}^n k_i \right) = 0. \quad (3.5)$$

(2) If $(\mathbf{s} = -\mathbf{N})$ is not a polar divisor for the integral function, then the value of this function at this point exists and is given by

$$Y_n(\alpha; -\mathbf{N}) = (-1)^n \alpha^{(\sum_{i=1}^n N_i + n)} \prod_{k=1}^n \mathcal{T}_{1, \dots, k}^{N_k + 1}, \quad (3.6)$$

where the symbols $\mathcal{T}_{1,2, \dots, k}$ are defined recursively as

$$\mathcal{T}_1^n := \frac{1}{n}, \quad \mathcal{T}_{1,2}^n := \frac{(\mathcal{T}_1 + 1)^n}{n}, \dots \text{ and } \mathcal{T}_{1,2, \dots, k+1}^n := \frac{(\mathcal{T}_{1,2, \dots, k} + 1)^n}{n}.$$

Now, we define the following function which is the key of the proof of the main result.

For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_+^n$ and $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$, we define the “shifted” function

$$Y_{n,\mathbf{a}}(\alpha; \mathbf{s}) = \int_{[\alpha, +\infty]^n} \prod_{i=1}^n (x_1 + \dots + x_i + a_1 + \dots + a_i)^{-s_i} d\mathbf{x}. \quad (3.7)$$

So, we have

Proposition 3.2. Let $\mathbf{N} = (N_1, \dots, N_n)$ be a point of \mathbb{N}^n , then we have for $\mathbf{a} \in \mathbb{R}_+^n$

$$Y_{n,\mathbf{a}}(\alpha; -\mathbf{N}) = (-1)^n \alpha^{\bar{N} + n} \prod_{k=1}^n \mathcal{A}_{1, \dots, k}^{N_k + 1}, \quad (3.8)$$

where the symbols $\mathcal{A}_{1,2, \dots, k}$ are defined recursively as

$$\mathcal{A}_1^n := \frac{\left(\frac{a_1}{\alpha} + 1\right)^n}{n}, \quad \mathcal{A}_{1,2}^n := \frac{\left(\mathcal{A}_1 + \left(\frac{a_2}{\alpha} + 1\right)\right)^n}{n}, \dots \text{ and } \mathcal{A}_{1,2, \dots, k}^n := \frac{\left(\mathcal{A}_{1,2, \dots, k-1} + \left(\frac{a_k}{\alpha} + 1\right)\right)^n}{n}.$$

Finally, by using the Raabe formula we give the main result for the multiple Hurwitz zeta function.

Theorem 1. Let $\mathbf{N} = (N_1, \dots, N_n)$ be a point of \mathbb{N}^n , if the point $(\mathbf{s} = -\mathbf{N})$ is not a polar divisor for the integral function $Y_n(\alpha; \mathbf{s})$, then the value of the multiple Hurwitz zeta function $\zeta_n(\alpha; \mathbf{s})$ at the point $(\mathbf{s} = -\mathbf{N})$ exists and is given by

$$\zeta_n(\alpha; -\mathbf{N}) = (-1)^n \alpha^{(\bar{N} + n)} \prod_{k=1}^n \mathcal{C}_{1, \dots, k}^{N_k + 1}, \quad (3.9)$$

with $\bar{N} = \sum_{i=1}^n N_i$.

3.1 Application of Theorem 1 in the case $n = 1$ and $\alpha = 1$

In this section, we give an application of our main result for $n = 1$ and $\alpha = 1$. We have

$$\zeta(s) = \sum_{n \geq 0} \frac{1}{(n+1)^s} = \sum_{n > 0} \frac{1}{n^s}, \quad (3.10)$$

which is the classical zeta function.

For $a \in \mathbb{R}_+$ we set

$$Y_a(s) = \int_1^{+\infty} (x+a)^{-s} dx. \quad (3.11)$$

So, for $\Re(s) > 1$ gives

$$Y_a(s) = \int_1^{+\infty} (x+a)^{-s} dx = \frac{(1+a)^{-s+1}}{s-1}. \quad (3.12)$$

Thus, for all $N \in \mathbb{N}$:

$$Y_a(-N) = -\frac{(1+a)^{N+1}}{N+1} = \frac{-1}{N+1} \sum_{k=0}^{N+1} \binom{N+1}{k} a^k. \quad (3.13)$$

Then, Proposition 5.2 of Section 5 shows that

$$\zeta(-N) = \frac{-1}{N+1} \sum_{k=0}^{N+1} \binom{N+1}{k} B_k, \quad (3.14)$$

where B_k is the k th Bernoulli number.

Now, we recall the elementary result

$$(N+1)B_N = -\sum_{k=0}^{N-1} \binom{N+1}{k} B_k. \quad (3.15)$$

Finally, we obtain the known result

$$\zeta(-N) = -\frac{B_{N+1}}{N+1}, \quad (3.16)$$

and equations (2.1) and (2.12) give the formula of Theorem 1:

$$\zeta(-N) = -\frac{B_{N+1}}{N+1} = -C_1^{N+1}. \quad (3.17)$$

4 Values of the integral representation of the multiple Hurwitz zeta function

This section is devoted to the proof of Proposition 3.1. Let the integral function

$$Y_n(\alpha; \mathbf{s}) = \int_{[0, +\infty[^n} \prod_{i=1}^n (x_i + \dots + x_i + i\alpha)^{-s_i} d\mathbf{x}. \quad (4.1)$$

If we use the following change of variables:

$$y_i = x_i + \alpha \quad (4.2)$$

for all $1 \leq i \leq n$, we find

$$Y_n(\alpha; \mathbf{s}) = \int_{[\alpha, +\infty]^n} \prod_{i=1}^n (y_1 + \dots + y_i)^{-s_i} d\mathbf{y}. \quad (4.3)$$

Now, using the following change of variables:

$$z_i = y_1 + \dots + y_i - (i-1)\alpha \quad (4.4)$$

for all $1 \leq i \leq n$. This change gives

$$\begin{cases} y_1 = z_1, \\ y_i = z_i - z_{i-1} + \alpha, \quad \forall 2 \leq i \leq n. \end{cases} \quad (4.5)$$

Since $\mathbf{y} = (y_1, \dots, y_n) \in [\alpha, +\infty]^n$, this gives

$$\mathbf{z} \in V_n = \{\mathbf{z} \in \mathbb{R}^n : \alpha \leq z_1 \leq z_2 \leq \dots \leq z_n\} \quad (4.6)$$

and we find

$$Y_n(\alpha; \mathbf{s}) = \int_{V_n} \prod_{i=1}^n (z_i + (i-1)\alpha)^{-s_i} d\mathbf{z}. \quad (4.7)$$

This integral can be rewritten as follows:

$$Y_n(\alpha; \mathbf{s}) = \int_{V_{n-1}} \prod_{i=1}^{n-1} (z_i + (i-1)\alpha)^{-s_i} \left(\int_{z_{n-1}}^{+\infty} (z_n + (n-1)\alpha)^{-s_n} dz_n \right) dz_1 \dots dz_{n-1}, \quad (4.8)$$

with

$$\begin{aligned} \int_{z_{n-1}}^{+\infty} (z_n + (n-1)\alpha)^{-s_n} dz_n &= \frac{(z_{n-1} + (n-2)\alpha)^{-s_n+1} \left(1 + \frac{\alpha}{z_{n-1} + (n-2)\alpha} \right)^{-s_n+1}}{s_n - 1} \\ &= \sum_{k_n \in \mathbb{N}} \binom{-s_n+1}{k_n} \frac{(z_{n-1} + (n-2)\alpha)^{-s_n+1-k_n}}{s_n - 1} \alpha^{k_n} \end{aligned} \quad (4.9)$$

if and only if $\Re(s_n) - 1 > 0$.

Inductively on n , we find

$$\begin{aligned} Y_n(\alpha; \mathbf{s}) &= \sum_{k=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \frac{\binom{-s_n+1}{k_n} \binom{-s_n-s_{n-1}+2-k_n}{k_{n-1}} \dots \binom{-\sum_{i=2}^n s_i + n - \sum_{i=3}^n k_i}{k_2} \alpha^{(-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i)}}{(s_n - 1)(s_n + s_{n-1} - 2 + k_n) \dots \left(\sum_{i=1}^n s_i - n + \sum_{i=2}^n k_i \right)} \alpha^{\sum_{i=2}^n k_i} \\ &= \sum_{k=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \frac{\binom{-s_n+1}{k_n} \binom{-s_n-s_{n-1}+2-k_n}{k_{n-1}} \dots \binom{-\sum_{i=2}^n s_i + n - \sum_{i=3}^n k_i}{k_2} \alpha^{(-\sum_{i=1}^n s_i + n)}}{(s_n - 1)(s_n + s_{n-1} - 2 + k_n) \dots \left(\sum_{i=1}^n s_i - n + \sum_{i=2}^n k_i \right)} \end{aligned}$$

if and only if for all $1 \leq i \leq n-1$

$$\Re\left(\sum_{i=1}^n s_i\right) - n + j - 1 + \sum_{i=2}^n k_i > 0 \quad (4.10)$$

and

$$\Re(s_n) - 1 > 0. \quad (4.11)$$

Therefore, for any point $\mathbf{N} = (N_1, \dots, N_n) \in \mathbb{N}^n$

(1) The point $(\mathbf{s} = -\mathbf{N})$ is a polar divisor for the function $Y_n(\alpha; \mathbf{s})$ if there exists a $\mathbf{k} = (k_2, \dots, k_n) \in \mathbb{N}^{n-1}$ such that

$$(s_n - 1)(s_n + s_{n-1} - 2 + k_n) \cdots \left(\sum_{i=1}^n s_i - n + \sum_{i=2}^n k_i \right) = \prod_{j=1}^n \left(\sum_{i=j}^n s_i - n + j - 1 + \sum_{i=j+1}^n k_i \right) = 0. \quad (4.12)$$

(2) If $(\mathbf{s} = -\mathbf{N})$ is not a polar divisor we obtain

$$\binom{N_n + 1}{k_n} \cdots \binom{\sum_{i=2}^n N_i + n - \sum_{i=3}^n k_i}{k_2} = \prod_{j=2}^n \binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} = 0 \quad (4.13)$$

if and only if there exists a $\mathbf{k} = (k_2, \dots, k_n) \in \mathbb{N}^{n-1}$ and $2 \leq j \leq n$, such that

$$k_j > \sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i.$$

Let

$$T(\mathbf{N}) := \left\{ \mathbf{k} = (k_2, \dots, k_n) \in \mathbb{N}^{n-1} : 0 \leq k_j \leq \sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i, \forall 2 \leq j \leq n \right\},$$

which is finite, then

$$\begin{aligned} Y_n(\alpha; -\mathbf{N}) &= (-1)^n \alpha^{\left(\sum_{i=1}^n N_i + n\right)} \sum_{\mathbf{k} \in T(\mathbf{N})} \frac{1}{\left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)} \prod_{j=2}^n \frac{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)} \\ &= (-1)^n \alpha^{\left(\sum_{i=1}^n N_i + n\right)} \sum_{\mathbf{k} \in T(\mathbf{N})} \frac{\prod_{j=2}^n \binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\prod_{j=1}^n \left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)}. \end{aligned}$$

Using the following symbols $\mathcal{T}_{1,2,\dots,k}$, defined recursively as

$$\mathcal{T}_1^n := \frac{1}{n}, \quad \mathcal{T}_{1,2}^n := \frac{(\mathcal{T}_1 + 1)^n}{n}, \dots \quad \text{and} \quad \mathcal{T}_{1,2,\dots,k+1}^n := \frac{(\mathcal{T}_{1,2,\dots,k} + 1)^n}{n}$$

to obtain

$$Y_n(\alpha; -\mathbf{N}) = (-1)^n \alpha^{\left(\sum_{i=1}^n N_i + n\right)} \prod_{k=1}^n \mathcal{T}_{1,\dots,k}^{N_k+1}. \quad (4.14)$$

4.1 An intermediate estimate

In this section, we show Proposition 3.2.

Let $\underline{\alpha} \in \mathbb{R}_+^n$, such that for all $\underline{x} = (x_1, \dots, x_n) \in [\alpha, +\infty[^n$ and for all $1 \leq i \leq n$

$$\frac{\alpha + a_i}{x_1 + \cdots + x_{i-1} + \alpha_1 + \cdots + \alpha_{i-1}} < 1, \quad (4.15)$$

the relation (4.3) shows that

$$Y_{n,\underline{\alpha}}(\alpha; \mathbf{s}) = \int_{[\alpha, +\infty[^n} \prod_{i=1}^n (x_1 + \cdots + x_i + \alpha_1 + \cdots + \alpha_i)^{-s_i} d\underline{x}. \quad (4.16)$$

This integral can be written as follows:

$$Y_{n,g}(\alpha; \underline{s}) = \int_{[\alpha, +\infty]^{n-1}} \prod_{i=1}^{n-1} (x_1 + \cdots + x_i + a_1 + \cdots + a_i)^{-s_i} \\ \times \left(\int_{\alpha}^{+\infty} (x_1 + \cdots + x_n + a_1 + \cdots + a_n)^{-s_n} dx_n \right) dx_1 \cdots dx_{n-1}.$$

Since for $\Re(s_n) > 1$, we have

$$\int_{\alpha}^{+\infty} (x_1 + \cdots + x_n + a_1 + \cdots + a_n)^{-s_n} dx_n = \frac{(x_1 + \cdots + x_{n-1} + a_1 + \cdots + a_{n-1} + \alpha + a_n)^{-s_n+1}}{s_n - 1},$$

and condition (4.15) yields

$$\int_{\alpha}^{+\infty} (x_1 + \cdots + x_n + a_1 + \cdots + a_n)^{-s_n} dx_n = \sum_{k_n \in \mathbb{N}} \binom{-s_n + 1}{k_n} \frac{(\alpha + a_n)^{k_n}}{s_n - 1} (x_1 + \cdots + x_{n-1} + a_1 + \cdots + a_{n-1})^{-s_n+1-k_n}.$$

If for $1 \leq j \leq n-1$

$$\left(\sum_{i=j}^n \Re(s_i) - n + j - 1 + \sum_{i=j+1}^n k_i \right) > 0, \quad (4.17)$$

then inductively we find

$$Y_{n,g}(\alpha; \underline{s}) = (-1)^n \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \frac{(\alpha + a_1)^{-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i}}{\left(-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i \right)} \\ \times \prod_{j=2}^n \left(\frac{-\sum_{i=j}^n s_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \right) \frac{(\alpha + a_j)^{k_j}}{\left(-\sum_{i=j}^n s_i + n - j + 1 - \sum_{i=j+1}^n k_i \right)}. \quad (4.18)$$

But, for all $2 \leq j \leq n$ we have

$$(\alpha + a_j)^{k_j} = \sum_{\substack{v_j \in \mathbb{N} \\ v_j \leq k_j}} \binom{k_j}{v_j} \alpha^{k_j - v_j} a_j^{v_j} \quad (4.19)$$

and

$$(\alpha + a_1)^{-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i} = \sum_{\substack{v_1 \in \mathbb{N} \\ v_1 \leq (-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i)}} \binom{-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i}{v_1} \alpha^{(-\sum_{i=1}^n s_i + n - v_1 - \sum_{i=2}^n k_i)} a_1^{v_1},$$

which yields

$$Y_{n,g}(\alpha; \underline{s}) = (-1)^n \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n \\ v_j \leq k_j \ \forall 2 \leq j \leq n; v_1 \leq (-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i)}} \frac{A(\underline{s}) a_1^{v_1}}{\left(-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i \right)} \prod_{j=2}^n \frac{a_j^{v_j}}{\left(-\sum_{i=j}^n s_i + n - j + 1 - \sum_{i=j+1}^n k_i \right)}, \quad (4.20)$$

with

$$A(\underline{s}) = \binom{-\sum_{i=1}^n s_i + n - \sum_{i=2}^n k_i}{v_1} \alpha \binom{-\sum_{i=1}^n s_i + n - \sum_{i=1}^n v_i}{v_1} \prod_{j=2}^n \binom{-\sum_{i=j}^n s_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j}.$$

Setting $\underline{s} = -\underline{N} = -(N_1, \dots, N_n) \in \mathbb{N}^n$ gives

$$Y_{n,\underline{a}}(\alpha; -\underline{N}) = (-1)^n \sum_{\substack{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1} \\ v_j \leq k_j \ \forall 2 \leq j \leq n; v_1 \leq \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n \\ v_j \leq k_j \ \forall 2 \leq j \leq n; v_1 \leq \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} \frac{A(-\underline{N}) a_1^{v_1}}{\left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)} \prod_{j=2}^n \frac{a_j^{v_j}}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)}, \quad (4.21)$$

with

$$A(-\underline{N}) = \binom{\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i}{v_1} \alpha \binom{\sum_{i=1}^n N_i + n - \sum_{i=1}^n v_i}{v_1} \prod_{j=2}^n \binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j}.$$

We will simplify these values and observe that

$$\begin{aligned} & \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n: v_j \leq k_j \ \forall 2 \leq j \leq n \\ \text{and } v_1 \leq \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} \binom{\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i}{v_1} \left(\prod_{j=2}^n \binom{k_j}{v_j} \right) \alpha^{\left(\sum_{i=1}^n N_i + n - \sum_{i=1}^n v_i\right)} \prod_{j=1}^n a_j^{v_j} \\ &= \alpha^{\left(\sum_{i=1}^n N_i + n\right)} \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n: v_j \leq k_j \ \forall 2 \leq j \leq n \\ \text{and } v_1 \leq \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} \binom{\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i}{v_1} \left(\prod_{j=2}^n \binom{k_j}{v_j} \right) \prod_{j=1}^n \alpha^{-v_j} a_j^{v_j} \\ &= \alpha^{\left(\sum_{i=1}^n N_i + n\right)} \left(\frac{a_1}{\alpha} + 1 \right)^{\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i} \left(\frac{a_2}{\alpha} + 1 \right)^{k_2} \dots \left(\frac{a_n}{\alpha} + 1 \right)^{k_n}. \end{aligned}$$

Now, we put for all $1 \leq j \leq n$

$$\mathcal{F}_j := \left(\frac{a_j}{\alpha} + 1 \right) \quad (4.22)$$

and

$$\bar{N} := \sum_{i=1}^n N_i. \quad (4.23)$$

So, we find

$$Y_{n,\underline{a}}(\alpha; -\underline{N}) = (-1)^n \alpha^{\bar{N}+n} \sum_{\substack{\underline{k}=(k_2, \dots, k_n) \in T(\underline{N}) \\ \bar{N} + n - \sum_{i=2}^n k_i}} \frac{\mathcal{F}_1^{\bar{N}+n - \sum_{i=2}^n k_i}}{\bar{N} + n - \sum_{i=2}^n k_i} \prod_{j=2}^n \frac{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \mathcal{F}_j^{k_j}}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)},$$

finally, substituting the symbols $\mathcal{A}_{1,2,\dots,k}$, defined recursively as

$$\mathcal{A}_1^n := \frac{\mathcal{F}_1^n}{n} := \frac{\left(\frac{a_1}{\alpha} + 1\right)^n}{n}, \quad \mathcal{A}_{1,2}^n := \frac{(\mathcal{A}_1 + \mathcal{F}_2)^n}{n} := \frac{\left(\mathcal{A}_1 + \left(\frac{a_2}{\alpha} + 1\right)\right)^n}{n}, \dots,$$

$$\mathcal{A}_{1,2,\dots,k}^n := \frac{(\mathcal{A}_{1,2,\dots;k-1} + \mathcal{F}_k)^n}{n} := \frac{\left(\mathcal{A}_{1,2,\dots,k-1} + \left(\frac{a_k}{\alpha} + 1\right)\right)^n}{n},$$

and summing over the indices k_2, \dots, k_n successively, we obtain

$$Y_{n,\underline{a}}(\alpha; -\mathbf{N}) = (-1)^n \alpha^{\bar{N}+n} \prod_{k=1}^n \mathcal{A}_{1,\dots,k}^{N_k+1}, \quad (4.24)$$

which ends the proof of Proposition 3.2.

5 Raabe formula

In this section, we proceed to the proof of the main Theorem 1. The proof relies on the *Raabe* formula [14], which expresses the integral in terms of the sum.

Proposition 5.1.

(1) *Raabe formula: for all $\underline{s} \in \mathbb{C}^n$, outside the possible polar divisors of $Y_n(\alpha; \underline{s})$, we have*

$$Y_n(\alpha; \underline{s}) = \int_{\underline{t} \in [0,1]^n} \zeta_{n,\underline{t}}(\alpha; \underline{s}) d\underline{t}, \quad (5.1)$$

where

$$\zeta_{n,\underline{t}}(\alpha; \underline{s}) = \sum_{\underline{m} \in \mathbb{N}^n} \prod_{i=1}^n \frac{1}{((m_i + t_i + \alpha) + \dots + (m_i + t_i + \alpha))^{s_i}}$$

and $d\underline{t}$ is the Lebesgue measure on \mathbb{R}^n .

(2) For a fixed point $\underline{N} = (N_1, \dots, N_n)$ in \mathbb{N}^n the maps $\underline{a} \mapsto Y_{n,\underline{a}}(\alpha; -\underline{N})$ and $\underline{a} \mapsto \zeta_{n,\underline{t}}(\alpha; -\underline{N})$ are polynomials in $\underline{a} = (a_1, \dots, a_n) \in \mathbb{R}_+^n$.

Proof.

(1) Let $\underline{s} \in \mathbb{C}^n$ be chosen in such a way that the integral function and the multiple Hurwitz zeta function are absolutely convergent.

Thus, for $\underline{t} \in \mathbb{R}_+^n$, we have

$$\begin{aligned} \int_{[0,1]^n} \zeta_{n,\underline{t}}(\alpha; \underline{s}) d\underline{t} &= \int_{[0,1]^n} \sum_{\underline{m}=(m_1, \dots, m_n) \in \mathbb{N}^n} \prod_{i=1}^n (t_i + \dots + t_i + m_i + \dots + m_i + i\alpha)^{-s_i} d\underline{t} \\ &= \sum_{\underline{m} \in \mathbb{N}^n} \int_{\prod_{i=1}^n [m_i, m_i+1]} \prod_{i=1}^n (t_i + \dots + t_i + m_i + \dots + m_i + i\alpha)^{-s_i} d\underline{t} \\ &= \int_{[0,+\infty[^n} \prod_{i=1}^n (x_i + \dots + x_i + i\alpha)^{-s_i} d\underline{x} = Y_n(\alpha; \underline{s}). \end{aligned}$$

This last equality which is verified for all $\underline{s} \in \mathbb{C}^n$ follows by analytic continuation outside the polar divisors.

(2) Follows from (4.21) combined with the *Raabe* formula. \square

Lemma 5.1. *Let P and Q be two polynomials in n variables linked by*

$$P(\underline{a}) = \int_{\mathbf{t} \in [0,1]^n} Q(\underline{a} + \mathbf{t}) \, d\mathbf{t}. \quad (5.2)$$

Write out

$$P(\underline{a}) = P(a_1, \dots, a_n) = \sum_{\underline{L}} h_{\underline{L}} \prod_{i=1}^n a_i^{L_i}, \quad (5.3)$$

where $h_{\underline{L}} \in \mathbb{C}$ and $\underline{L} = (L_1, \dots, L_n) \in \mathbb{N}^n$ ranges over a finite set of multi-index. Then

$$Q(\underline{a}) = Q(a_1, \dots, a_n) = \sum_{\underline{L}} h_{\underline{L}} \prod_{i=1}^n B_{L_i}(a_i), \quad (5.4)$$

where the $B_{L_i}(a_i)$ are the Bernoulli polynomials [22].

Conversely, if Q is given by (5.4), then the relations (5.2) and (5.3) yield equivalent formulas for the polynomial P .

Proof. Let $V := V_{m,n}$ be the finite-dimensional complex space of polynomials in n variables $\underline{a} = (a_1, \dots, a_n)$ with complex coefficients and having degree at most m . Note that both $\{\underline{a}^{\underline{L}}\}_{\underline{L}}$ and $\{B_{\underline{L}}(\underline{a})\}_{\underline{L}}$ are \mathbb{C} -bases of V . Here $\underline{L} = (L_1, \dots, L_n)$ ranges over all multi-indices with $|\underline{L}| := \sum_{i=1}^n L_i \leq m$ and $\underline{a}^{\underline{L}} := \prod_{i=1}^n a_i^{L_i}$. That $\{B_{\underline{L}}(\underline{a})\}_{\underline{L}}$ is a basis of $V := V_{m,n}$ can be proved by induction on m , since $\underline{a}^{\underline{L}} - B_{\underline{L}}(\underline{a})$ has degree strictly less than $|\underline{L}|$. Let $f: V \rightarrow V$ be the \mathbb{C} -map taking $Q = Q(\underline{a}) \in V$ to

$$f(Q)(\underline{a}) := \int_{\mathbf{t} \in [0,1]^n} Q(\underline{a} + \mathbf{t}) \, d\mathbf{t}.$$

The lemma can be restated as saying that the inverse map to f exists and takes $\underline{a}^{\underline{L}}$ to $B_{\underline{L}}(\underline{a})$.

Hence, it will suffice to show that $f(B_{\underline{L}}(\underline{a})) = \underline{a}^{\underline{L}}$, for then f is an isomorphism (it takes one basis to another). Using [15, p. 4] and [23, pp. 66–67]

$$\frac{d}{dx} B_{j+1}(x) = (j+1)B_j(x) \quad \text{and} \quad B_j(x+1) - B_j(x) = jx^{j-1},$$

we calculate

$$\begin{aligned} f(B_{\underline{L}}(\underline{a})) &= \int_{\mathbf{t} \in [0,1]^n} B_{\underline{L}}(\underline{a} + \mathbf{t}) \, d\mathbf{t} = \prod_{i=1}^n \int_0^1 B_{L_i}(a_i + t_i) \, dt_i \\ &= \prod_{i=1}^n \frac{1}{L_i + 1} (B_{L_i+1}(a_i + 1) - B_{L_i+1}(a_i)) = \prod_{i=1}^n a_i^{L_i}, \end{aligned}$$

which concludes the proof of the lemma. □

Proposition 5.2. *If we write out the polynomial $Y_{\underline{a}}(\alpha; -\mathbf{N})$ as a sum of monomials,*

$$Y_{\underline{a}}(\alpha; -\mathbf{N}) = \sum_{\underline{L}} G_{\underline{L}} \underline{a}^{\underline{L}}$$

with $\underline{a}^{\underline{L}} = \prod_{i=1}^n a_i^{L_i}$ and $G_{\underline{L}} = G_{\underline{L}}(\mathbf{N}) \in \mathbb{C}$.

Then

$$\zeta_n(\alpha; -\mathbf{N}) = \sum_{\underline{L}} G_{\underline{L}} B_{\underline{L}},$$

where $B_{\underline{L}} = \prod_{i=1}^n B_{L_i}$ is a product of Bernoulli numbers.

More generally, for $\underline{a} = (a_1, \dots, a_n) \in \mathbb{R}_+^n$, we have

$$\zeta_{n,\underline{a}}(\alpha; -\mathbf{N}) = \sum_{\underline{L}} G_{\underline{L}} B_{\underline{L}}(\underline{a}),$$

where $B_{\underline{L}}(\underline{a}) = \prod_{i=1}^n B_{L_i}(a_i)$ is a product of Bernoulli numbers.

Proof. It follows from the above lemma, with $P(\underline{a}) = Y_{n,\underline{a}}(\alpha; -\mathbf{N})$ and $Q(\underline{a}) = \zeta_{n,\underline{a}}(\alpha; -\mathbf{N})$. \square

6 Values of the multiple Hurwitz zeta function at non-positive integers

In this section, we give the proof of our main result (Theorem 1).

Relation (4.21) shows that for all $\underline{a} \in \mathbb{R}_+^n$

$$\begin{aligned} Y_{n,\underline{a}}(\alpha; -\mathbf{N}) &= (-1)^n \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n \\ v_j \leq k_j \ \forall 2 \leq j \leq n; v_1 \leq \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} \\ &\times A(-\mathbf{N}) \prod_{j=1}^n \frac{a_j^{v_j}}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)}, \end{aligned} \quad (6.1)$$

with

$$A(-\mathbf{N}) = \binom{\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i}{v_1} \alpha^{\left(\sum_{i=1}^n N_i + n - \sum_{i=1}^n v_i\right)} \prod_{j=2}^n \binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j},$$

and

$$T(\mathbf{N}) := \left\{ \underline{k} = (k_2, \dots, k_n) \in \mathbb{N}^{n-1} : 0 \leq k_j \leq \sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i, \ \forall 2 \leq j \leq n \right\}.$$

Setting

$$\underline{a}^{\underline{v}} = \prod_{j=1}^n a_j^{v_j}, \quad (6.2)$$

this gives

$$\begin{aligned} Y_{n,\underline{a}}(\alpha; -\mathbf{N}) &= (-1)^n \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n \\ v_j \leq k_j \ \forall 2 \leq j \leq n; v_1 \leq \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} \\ &\times A(-\mathbf{N}) \underline{a}^{\underline{v}} \prod_{j=1}^n \frac{1}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)}. \end{aligned} \quad (6.3)$$

It follows from Proposition 5.2 that

$$\begin{aligned} \zeta_n(\alpha; -\mathbf{N}) &= (-1)^n \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n \\ v_j \leq k_j \ \forall 2 \leq j \leq n; v_1 \leq \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} \\ &\times A(-\mathbf{N}) B_{\underline{v}} \prod_{j=1}^n \frac{1}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)}, \end{aligned} \quad (6.4)$$

with

$$B_{\underline{v}} = \prod_{j=1}^n B_{v_j}$$

and B_{v_j} is the v_j th Bernoulli number.

Now, using the generalized Bernoulli symbols, we find

$$\alpha^{(\bar{N}+n)} \sum_{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n} \binom{\bar{N} + n - \bar{k}}{v_1} \prod_{j=2}^n \binom{k_j}{v_j} \left(\prod_{j=1}^n \alpha^{-v_j} B_{v_j} \right) = \alpha^{(\bar{N}+n)} (1 + \mathcal{B}_1)^{\bar{N}+n-\bar{k}} (1 + \mathcal{B}_2)^{k_2} \dots (1 + \mathcal{B}_n)^{k_n},$$

where $\bar{N} = \sum_{i=1}^n N_i$ and $\bar{k} = \sum_{i=2}^n k_i$.

The relation (2.8) reduces this to

$$\alpha^{(\bar{N}+n)} \sum_{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n} \binom{\bar{N} + n - \bar{k}}{v_1} \prod_{j=2}^n \binom{k_j}{v_j} \left(\prod_{j=1}^n \alpha^{-v_j} B_{v_j} \right) = \alpha^{(\bar{N}+n)} \mathcal{D}_1^{\bar{N}+n-\bar{k}} \mathcal{D}_2^{k_2} \dots \mathcal{D}_n^{k_n}.$$

It follows that

$$\begin{aligned} \zeta_n(\alpha; -\mathbf{N}) &= (-1)^n \alpha^{(\bar{N}+n)} \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \frac{\mathcal{D}_1^{\bar{N}+n-\bar{k}} \mathcal{D}_2^{k_2} \dots \mathcal{D}_n^{k_n}}{\bar{N} + n - \bar{k}} \prod_{j=2}^n \frac{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}} \\ &= (-1)^n \alpha^{(\bar{N}+n)} \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} C_1^{\bar{N}+n-\bar{k}} \mathcal{D}_2^{k_2} \dots \mathcal{D}_n^{k_n} \prod_{j=2}^n \frac{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}}. \end{aligned}$$

Next, we sum over k_2 and using the definition of symbol $C_{1,2}$, we find

$$\begin{aligned} \zeta_n(\alpha; -\mathbf{N}) &= (-1)^n \alpha^{(\bar{N}+n)} \sum_{(k_3, \dots, k_n) \in \mathbb{N}^{n-2}} C_1^{N_1+1} C_{1,2}^{\left(\sum_{i=2}^n N_i + n - 1 - \sum_{i=3}^n k_i\right)} \mathcal{D}_3^{k_3} \dots \mathcal{D}_n^{k_n} \\ &\quad \times \prod_{j=3}^n \frac{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}}. \end{aligned}$$

By summing over the remaining indices we obtain

$$\zeta_n(\alpha; -\mathbf{N}) = (-1)^n \alpha^{(\bar{N}+n)} C_1^{N_1+1} C_{1,2}^{N_2+1} \dots C_{1,2, \dots, n}^{N_n+1}, \quad (6.5)$$

which ends the proof of Theorem 1.

7 A general recursion formula on the depth for multiple Hurwitz zeta function

1. Let the “shifted” multiple Hurwitz zeta function be defined by

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = \sum_{\underline{m} \in \mathbb{N}^n} \frac{1}{(m_1 + a_1 + \alpha)^{N_1} \dots (m_n + a_n + \alpha)^{N_n}}, \quad (7.1)$$

for all $\underline{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$.

From the *Raabe* formula (relation (5.1) and relation (5.4)), we have

$$\begin{aligned} \zeta_n(\alpha; -\mathbf{N}; \underline{a}) &= (-1)^n \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \sum_{\substack{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n \\ v_j \leq k_j \ \forall 2 \leq j \leq n; \ v_1 \leq \left(\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i\right)}} \\ &\times A(-\mathbf{N}) B_{\underline{v}}(\underline{a}) \prod_{j=1}^n \frac{1}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)}, \end{aligned} \quad (7.2)$$

with

$$A(-\mathbf{N}) = \binom{\sum_{i=1}^n N_i + n - \sum_{i=2}^n k_i}{v_1} \alpha^{\left(\sum_{i=1}^n N_i + n - \sum_{i=1}^n v_i\right)} \prod_{j=2}^n \binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j} \binom{k_j}{v_j}$$

and

$$B_{\underline{v}}(\underline{a}) = \prod_{j=1}^n B_{v_j}(a_j).$$

Using relation (2.7), we find

$$B_{v_j}(a_j) = (\alpha \mathcal{B}_j + a_j)^{v_j} := (\alpha \mathcal{R}_j)^{v_j}. \quad (7.3)$$

It follows that

$$\alpha^{(\bar{N}+n)} \sum_{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n} \binom{\bar{N} + n - \bar{k}}{v_1} \prod_{j=2}^n \binom{k_j}{v_j} \left(\prod_{j=1}^n \alpha^{-v_j} B_{v_j}(a_j) \right) = \alpha^{(\bar{N}+n)} (1 + \mathcal{R}_1)^{\bar{N}+n-\bar{k}} (1 + \mathcal{R}_2)^{k_2} \dots (1 + \mathcal{R}_n)^{k_n},$$

where $\bar{N} = \sum_{i=1}^n N_i$ and $\bar{k} = \sum_{i=2}^n k_i$. We put, for all $1 \leq j \leq n$

$$\mathcal{Y}_j^{k_j} := (1 + \mathcal{R}_j)^{k_j} = (1 + \alpha B_j(a_j))^{k_j}, \quad (7.4)$$

which gives

$$\alpha^{(\bar{N}+n)} \sum_{\underline{v}=(v_1, \dots, v_n) \in \mathbb{N}^n} \binom{\bar{N} + n - \bar{k}}{v_1} \prod_{j=2}^n \binom{k_j}{v_j} \left(\prod_{j=1}^n \alpha^{-v_j} B_{v_j}(a_j) \right) = \alpha^{(\bar{N}+n)} \mathcal{Y}_1^{\bar{N}+n-\bar{k}} \mathcal{Y}_2^{k_2} \dots \mathcal{Y}_n^{k_n}.$$

So, formula (7.2) reduces to

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = (-1)^n \alpha^{(\bar{N}+n)} \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} \frac{\mathcal{Y}_1^{\bar{N}+n-\bar{k}} \mathcal{Y}_2^{k_2} \dots \mathcal{Y}_n^{k_n}}{\bar{N} + n - \bar{k}} \prod_{j=2}^n \frac{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\left(\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i\right)}.$$

Now, we introduce the symbols C_{1, \dots, a_k}^n defined recursively as

$$\begin{aligned} C_1^n(a_1) &:= \frac{\mathcal{Y}_1^n}{n} := \frac{\left(1 + \mathcal{B}_1 + \frac{a_1}{\alpha}\right)^n}{n}, \\ C_{1,2}^n(a_1, a_2) &:= \frac{(C_1(a_1) + \mathcal{Y}_2)^n}{n} := \frac{\left(C_1(a_1) + 1 + \mathcal{B}_2 + \frac{a_2}{\alpha}\right)^n}{n}, \dots, \\ C_{1,2, \dots, k}^n(a_1, \dots, a_k) &:= \frac{(C_{1,2, \dots, k-1}(a_1, \dots, a_{k-1}) + \mathcal{Y}_k)^n}{n} := \frac{\left(C_{1,2, \dots, k-1}(a_1, \dots, a_{k-1}) + 1 + \mathcal{B}_k + \frac{a_k}{\alpha}\right)^n}{n}, \end{aligned}$$

which gives

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = (-1)^n \alpha^{(\bar{N}+n)} \sum_{\underline{k}=(k_2, \dots, k_n) \in \mathbb{N}^{n-1}} C_1^{\bar{N}+n-\bar{k}}(a_1) \mathcal{Y}_2^{k_2} \cdots \mathcal{Y}_n^{k_n} \prod_{j=2}^n \frac{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}{k_j}}{\binom{\sum_{i=j}^n N_i + n - j + 1 - \sum_{i=j+1}^n k_i}}{k_j}$$

by summing over the indices k_2, \dots, k_n successively, we obtain

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = (-1)^n \alpha^{(\bar{N}+n)} C_1^{N_1+1}(a_1) C_{1,2}^{N_2+1}(a_1, a_2) \cdots C_{1,2,\dots,n}^{N_n+1}(a_1, \dots, a_n). \quad (7.5)$$

Thus, we proved the following theorem.

Theorem 2. Let $\underline{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, then for all $\mathbf{N} = (N_1, \dots, N_n)$ a point of \mathbb{N}^n which is not a divisor point for the integral function $Y_n(\alpha; \underline{s}, \underline{a})$, then the value of the “shifted” multiple Hurwitz zeta function $\zeta_n(\alpha; \underline{s}, \underline{a})$ at the point $(\underline{s} = -\mathbf{N})$ exists and is given by

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = (-1)^n \alpha^{(\bar{N}+n)} \prod_{k=1}^n C_{1,2,\dots,k}^{N_k+1}(a_1, \dots, a_k), \quad (7.6)$$

with

$$C_1^{N_1}(a_1) := \frac{\mathcal{Y}_1^n}{n} := \frac{\left(1 + \mathcal{B}_1 + \frac{a_1}{\alpha}\right)^n}{n}, \quad C_{1,2}^{N_2}(a_1, a_2) := \frac{(C_1(a_1) + \mathcal{Y}_2)^n}{n} := \frac{\left(C_1(a_1) + 1 + \mathcal{B}_2 + \frac{a_2}{\alpha}\right)^n}{n}, \dots,$$

and

$$C_{1,2,\dots,k}^{N_k}(a_1, \dots, a_k) := \frac{(C_{1,2,\dots,k-1}(a_1, \dots, a_{k-1}) + \mathcal{Y}_k)^n}{n} := \frac{\left(C_{1,2,\dots,k-1}(a_1, \dots, a_{k-1}) + 1 + \mathcal{B}_k + \frac{a_k}{\alpha}\right)^n}{n}.$$

2. Now, if we put for $(k, j) \in \mathbb{N}^2$

$$\mathcal{B}_j^k(a_j) := \left(1 + \mathcal{B}_j + \frac{a_j}{\alpha}\right)^k, \quad (7.7)$$

the relation (2.7) shows that

$$\mathcal{B}_j^k(a_j) := \alpha^{-k} \mathcal{B}_k(\alpha + a_j). \quad (7.8)$$

Let us expand the last term of (7.6)

$$C_{1,2,\dots,n}^{N_n+1}(a_1, \dots, a_n) = \frac{\left(C_{1,2,\dots,k-1}(a_1, \dots, a_{n-1}) + 1 + \mathcal{B}_n + \frac{a_n}{\alpha}\right)^{N_n+1}}{N_n + 1} = \frac{(C_{1,2,\dots,k-1}(a_1, \dots, a_{n-1}) + \mathcal{B}_n(a_n))^{N_n+1}}{N_n + 1},$$

and the binomial theorem produces

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = \frac{(-1)^n \alpha^{(\bar{N}+n)}}{N_n + 1} \sum_{l=0}^{N_n+1} \binom{N_n+1}{l} \left(\prod_{k=1}^{n-1} C_{1,2,\dots,k}^{N_k+1}(a_1, \dots, a_k) \right) C_{1,2,\dots,n-1}^l(a_1, \dots, a_{n-1}) \mathcal{B}_n^{N_n+1-l}(a_n).$$

Then identify

$$(-1)^{n-1} \alpha^{\sum_{i=1}^{n-1} N_i} \left(\prod_{k=1}^{n-1} C_{1,2,\dots,k}^{N_k+1}(a_1, \dots, a_k) \right) C_{1,2,\dots,n-1}^l(a_1, \dots, a_{n-1}) = \frac{1}{\alpha^l} \zeta_{n-1}(\alpha; -N_1, \dots, N_{n-2}, -N_{n-1} - l; \underline{a}),$$

which gives

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = \frac{-\alpha^{N_n+1}}{N_n + 1} \sum_{l=0}^{N_n+1} \binom{N_n+1}{l} \frac{1}{\alpha^l} \zeta_{n-1}(\alpha; -N_1, \dots, N_{n-2}, -N_{n-1} - l; \underline{a}) \mathcal{B}_n^{N_n+1-l}(a_n).$$

Relation (7.8) yields

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = \frac{-1}{N_n + 1} \sum_{l=0}^{N_n+1} \binom{N_n + 1}{l} \zeta_{n-1}(\alpha; -N_1, \dots, N_{n-2}, -N_{n-1} - l; \underline{a}) B_{N_n+1-l}(\alpha + a_n).$$

Thus, we obtain the following theorem.

Theorem 3. *The multiple Hurwitz zeta function satisfies the recursion formula*

$$\zeta_n(\alpha; -\mathbf{N}; \underline{a}) = \frac{-1}{N_n + 1} \sum_{l=0}^{N_n+1} \binom{N_n + 1}{l} \zeta_{n-1}(\alpha; -N_1, \dots, N_{n-2}, -N_{n-1} - l; \underline{a}) B_{N_n+1-l}(\alpha + a_n). \quad (7.9)$$

8 Some examples of specific values at non-positive integers

In this section, we give some examples of the evaluation of the series (3.1) at non-positive integers.

Example 8.1. For depth $n = 2$, Theorem 1 and the relationship (2.13) give, for all α and $\mathbf{N} = (N_1, N_2)$

$$\zeta_2(\alpha; -\mathbf{N}) = \alpha^{(N_1+N_2+2)} C_1^{N_1+1} C_{1,2}^{N_2+1}.$$

So,

– If $\alpha = 1$ and $\mathbf{N} = (N, 0)$, we find

$$\zeta_2(1; -(N, 0)) = \frac{B_{N+2}(1)}{N+2} - \frac{B_{N+1}(1)}{2(N+1)}.$$

– If $\alpha = 1$ and $\mathbf{N} = (0, N)$, we find

$$\zeta_2(1; -(0, N)) = \frac{1}{N+1} \sum_{k=0}^{N+1} \binom{N+1}{k} \frac{B_{1+k}(1)}{(1+k)} B_{N+1-k}(1).$$

Example 8.2. For depth $n = 3$, Theorem 1 and the relationship (2.13) give, for all α and $\mathbf{N} = (N_1, N_2, N_3)$

$$\zeta_3(\alpha; -\mathbf{N}) = -\alpha^{(N_1+N_2+N_3+3)} C_1^{N_1+1} C_{1,2}^{N_2+1} C_{1,2,3}^{N_3+1}.$$

So,

– If $\alpha = 1$ and $\mathbf{N} = (N, 0, 0)$, we find

$$\zeta_3(1; -(N, 0, 0)) = -\left[\frac{B_{N+3}(1)}{n+3} - \frac{3}{2} \frac{B_{N+2}(1)}{N+2} + \frac{5}{12} \frac{B_{N+1}(1)}{N+1} \right].$$

– If $\alpha = 1$ and $\mathbf{N} = (0, N, 0)$, we find

$$\zeta_3(1; -(0, N, 0)) = \frac{-1}{N+2} \sum_{k=0}^{N+2} \binom{N+2}{k} \frac{B_{1+k}(1)}{(1+k)} B_{N+2-k}(1) + \frac{1}{2} \frac{1}{N+1} \sum_{k=0}^{N+1} \binom{N+1}{k} \frac{B_{1+k}(1)}{(1+k)} B_{N+1-k}(1).$$

Example 8.3. Now, we use the recursion formula (7.9) of Theorem 3, to compute the value $\zeta_3(1; -(0, 0, 2))$, with $\alpha = 1$, $\mathbf{N} = (0, 0, 2)$ and $\underline{a} = (0, 0, 0)$, we find

$$\begin{aligned}
\zeta_3(1; -(0, 0, 2)) &= \zeta_3(1; -(0, 0, 2); (0, 0, 0)) \\
&= \frac{-1}{3} \sum_{l=0}^3 \binom{3}{l} \zeta_2(1; -(0, l); (0, 0)) B_{3-l}(1) \\
&= \frac{-1}{3} \sum_{l=0}^3 \binom{3}{l} \zeta_2(1; -(0, l)) B_{3-l}(1) \\
&= \frac{-1}{3} [B_3(1)\zeta_2(1; -(0, 0)) + 3B_2(1)\zeta_2(1; -(0, 1)) + 3B_1(1)\zeta_2(1; -(0, 2)) + B_0(1)\zeta_2(1; -(0, 3))] \\
&= \frac{-1}{3} \left[\frac{3}{72} + \frac{3}{180} - \frac{1}{120} \right] = -\frac{1}{60},
\end{aligned}$$

which confirmed the values given in [18] and [9].

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