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Ulam-Hyers stability of nonlinear* Volterra *integro-differential equation

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Dedication

My dissertation is dedicated to:

To my dear parents Badreddin and Assia, to my brothers Abdou Lah, Yacine, Makhloud and Abd El djalil who support me to continue my studies until the end.

To my dears Sofia, Bouchra, Widdad, Ahlem, Zineb and Lamia ,all my friends and everyone who encouraged me.

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Abstract

In this thesis, we are interested to study the Ulam-Hyers stability in the general case and particular case for a type of nonlinear Volterra itegro-differential equation using the Banach fixed point.

Keywords:

Ulam−Hyers , Ulam−Hyers−Rassias stability, Nonlinear Volterra integro-deferentail equation,Theorem of Banach fixed point.

Résumé

Dans cet mèmoire, nous sommes intéressés à étudier la stabilité ulam-Hyers dans le cas général et cas particulier pour un type d'une équation integro-différentiel de Volterra non lineaire en utilisant le théorème de point fixe de Banach.

Mots clés:

Stabilité au sens d' Ulam−Hyers, Stabilité au sens d'Ulam−Hyers −Rassias, Equation integro-differentiel de Volterra, Theoreme de point fixe de Banach.

Notations

- $\frac{\partial}{\partial x}$ partial derivative operator .

- $\int_a^b$ integral operator .

- $\mathbb{N}$ the set of positive integers, that is $\mathbb{N} = \{0, 1, 2, \cdots\}$.

- $\mathbb{R}$ the set of real numbers.

- $\mathbb{C}$ the set of complex numbers.

- $\mathbb{R}^n$ is the real space of dimension n.

- $C^m(\Omega)$ space of m times continuously differentiable functions on Ω, $m \in \mathbb{N}$.

- $C(\mathbb{R}^d)$ the space of continuous function on $\mathbb{R}^d$.

- $C^\infty(\Omega) = \bigcap_{m\in\mathbb{N}} C^m(\Omega)$.

- $C_0^\infty(\Omega)$ the space of $C^\infty(\Omega)$ functions with compact support in Ω.

- $L^p(\Omega)$ Lebesgue space with norm $\|\cdot\|_p$.

CONTENTS

GENERAL INTRODUCTION

In this thesis, we are interested to study the stability of ***Volterra integro-differential equations***. This type of equations come out to light in 1896 when the Italian mathematician *Vito Volterra* generalized Abel's result, when Abel investigated a special case of integral equation then volterra trated the general case and he got second kind of integral equation which will be defined later[6]. From this interesting result, which he opened a new horizon to the universe of mathematics and to a new type of integro differential equations of Volterra (IDEVs).

Integro-differential equation is a type of differential equations that combine both of differential an integral operator. The IDEVs are defined as an equation of the form
$L[u] = f(x)$, where L is an integro-differential operator, $u(x)$ is the unknown function and $f(x)$ is a given function.

However, these types of equations have a vast utilization in various scientific and engineering such us in heat and mass transfer theory, electric circuit problems electromagnetic theory, fluid dynamics, neuron transport theory neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating.
In addition, the Volterra integral equations are difficult to solve analytically because of their complexity. Consequently, a variety of numerical methods have been used to solve these issues in VIDEs. Though numerical methods can be sensitive to errors and perturbation in initial conditions and parameters.

The solution of Volterra integral equation should be treated by studying the perturbation of solution using one of mathematical technique in our thesis we are talking about the Ulam-Hyers stability witch is a concept in analysis that addresses the sensitivity of solutions to small perturbations. Specifically, it refers to the stability of an approximate solution under small perturbations of the initial conditions and parameters.

Ulam-Heyrs stability has been studied extensively for ordinary differential equations and partial differential equations. However, it's application to integro-differential equations is relatively new. If this technique for Ulam-Heyrs stability analysis can be provide it has made an important insights into the behavior of numerical methods and their accuracy in solving these equations, and that is the goal of our thesis.

Plan of thesis

This manuscript includes a general introduction and three chapters.

The first chapter:

The first chapter is an introduction to the integral equation and their classification, also we will cite the Banach space, which play an important role in our study.

The second chapter:

The second chapter is interesting to study the existence of the solution of an integral equation using a Banach fixed point then we treat the stability of the solution using the Ulam-Heyrs stability after we will generalize the stability of the integral equation using the general method of stability named with Ulam-Heyres-Rassias stability.

The third chapter:

The last chapter consist to use the same concept of the second chapter, but we will use it and generalized on an integro-differential equation of Volterra. First we will study the existence and uniqueness of the solution with two different conditions then we are going to studding the stability with Ulam-Heyrs and Ulam-Hyers-Rassias methods.

CHAPTER 1

PRELIMINARIES

*I*n this chapter we will introduce the concept of *integral equations* and *integro-differential equations* and their classification also we will define the mathematical tools using in solving our thesis problem with given definitions, theorems and examples to make clear our reasoning and the understanding of readers.

1.1 Banach Space

Banach space has the name of Stefan Banach(1892-1945). A Polish mathematician who is known as one of the founders of functional analysis[14].

Definition 1.1 [17] Consider that X is a linear vector space (of finite or infinite dimension over $\mathbb{R}$ or $\mathbb{C}$). A mapping of X into $[0, +\infty[$ satisfying the following norm axioms is a norm $\| x \|_X = \| x \|$:

1. It is positive on nonzero vectors, that is

$$\| x \| = 0 \quad onlly \quad if \quad x = 0, \tag{1.1}$$

2. For every vector$x$ and every scalar λ:

$$\| \lambda x \| = |\lambda| \, \| x \| \quad for\, all \quad x \in X \quad and \quad \lambda \in \mathbb{R}\, (or\, \mathbb{C}), \tag{1.2}$$

3. The *triangle inequality* holds; that is, for every vectors x and y

$$\| x + y \| \leq ||x|| + ||y|| \quad for\, all \quad x, y \in X. \tag{1.3}$$

Remark 1.1 The pair$(X, ||.||)$ represents a normed vector space. We use the symbol of the space as a subscript for simplicity if there are multiple vector spaces or if the choice of the norm is not evident, for example, $||.||_X$, $||.||_Y$.

Definition 1.2 [17] The normed linear space $(X, ||.||)$ is called complete, if any Cauchy sequence in X is also convergent

$$\lim_{m,n\to\infty} ||U_m - U_n|| = 0.$$

Acauchy sequence is defined to be convergent so X is complete.

Definition 1.3 [17] A complete normed linear space is called a ***Banach space***. Examples of ***Banach spaces*** that are simple to understand include finite-dimensional vector spaces $\mathbb{R}$ with the maximum norm (1.4) or the Euclidean norm (1.5) for the vectors

$$||x||_\infty = max|x_i|, 1 \leq i \leq d, \tag{1.4}$$

$$||x||_2 = (\sum_{i=1}^{d} |x_i|)^{\frac{1}{2}}. \tag{1.5}$$

Remark 1.2 Two norms $||.||_1$ and $||.||_2$ in the finite-dimensional are called equivalent if they give the same topology, which is equivalent to the existence of constants ***c1*** and ***C2*** such that:

$$||\boldsymbol{v}||_1 \leq \boldsymbol{c1}||\boldsymbol{v}||_2,$$

and

$$||\boldsymbol{v}||_2 \leq \boldsymbol{C2}||\boldsymbol{v}||_1,$$

hold for all $\boldsymbol{v}$.

1.2 Linear Bounded Operators

Definition 1.4 [11] An operwtor $A : X \to Y$ mapping a linear space X into a linear space Y is named linear if:

$$A(\alpha x_1 + \beta x_2) = \alpha(x_1) + \beta A(x_2), \tag{1.6}$$

for all $x_1, x_2 \in X$ and $\alpha, \beta \in \mathbb{R}$or $(\mathbb{C})$.

Definition 1.5 [11] A linear operator $A : X \to Y$ from a normed space X into a normed space Y is called bounded if there exists a positive number γ such

$$||A(x)|| \leq \gamma||x||,$$

for all $x \in X$. A number γ for which this inequality holds is called a bound for the operator A.

Theorem 1.1 *[11] For a linear operator $A : X \times Y$ mapping a normed space X into a normed space Y the following properties are equivalent:*

i) A is continuous at one element.

ii) A is continuous.

iii) A is bounded.

Proof.

1. *[i] $\Longrightarrow$ [ii], let A be continuous at $x_0 \in X$ then for every $x \in X$ and every sequence (x_n) with $x_n \to x, n \to \infty$, we have:*
   $Ax_n = A(x_n - x + x_0) + A(x - x_0) \to A(x_0) + A(x - x_0) = A(x), n \to \infty,$
   since
   $x_n - x + x_0 \to x_0, n \to \infty.$
   Therefore, A is continuous at all $x \in X$.

2. *[ii] $\Longrightarrow$ [iii], let A be continuous and assume there is no $\gamma > 0$ such that $||Ax|| \leq \gamma||x||$ for all $x \in X$. Then there exists a sequence (x_n) in X with $||x_n|| = 1$ and $||x_n|| = 1$ and $||Ax_n|| \geq$......*
   Consider the sequence $y_n := ||Ax_n||^{-1}x_n$. Then $y_n \to 0$, $n \to \infty$, since A is continues $Ay_n \to A(0) = 0, n \to \infty$. This is a contradiction to $||Ay_n|| = 1$ for all n.

3. *[iii] $\Longrightarrow$ [i] Let A be bounded and let (x_n) be a sequence in X with $x_n \to 0, n \to \infty$. Then from$||Ax_n|| \leq \gamma||x_n||$it follows that $Ax_n \to 0, n \to \infty$. Thus, A is continuous at $x = 0$.*

■

Theorem 1.2 *[11] The linear space $L(X, Y)$ of bounded linear operators from a normed space X into a normed space Y is a normed space with the norm $||A|| = \sup_{x \leq 1} ||Ax|| < \infty$. If Y is a Banach space then $L(X, Y)$ also is a Banach space.*

Proof. The proof consists in carrying over the norm axioms and the completeness from Y onto $L(X, Y)$
. For the second part, let (A_n) be a Cauchy sequence in $L(X, Y)$,i.e., $||A_m - A_n|| \to 0, m, n \to \infty$. Then for each $x \in X$ the sequence $(A_n x)$ is a Cauchy sequence in Y and converges, since Y is *complete.* Then $Ax = \lim_{n \to \infty} A_n x$ defines a bounded linear operator $A : X \to Y$, which is the limit of the sequence (A_n), i.e., $||An - A|| \to 0, n \to \infty$.

■

1.2.1 Integral Operators

An important class of operators will now be defined using integrated operators, whose field of integration is a domain that can be measured in $\mathbb{R}^d$.

Definition 1.6 [1] Let $k(x,t)$ be a measurable function on $\Gamma\times\Gamma$. Then the general form of an integral linear operator A, is formally given by the expression:

$$Au(x) = \int_{\Gamma} k(x,t)u(t)dt. \tag{1.7}$$

Au is defined once that integral exists.

Remark 1.3 The equation (1.7) is called an integral operator with continuous kernel K. It is a bounded linear operator with:

$$||A||_{\infty} = \max_{x\in\Gamma} \int_{\Gamma} |k(x,t)|dt \qquad ,x\in\Gamma, \tag{1.8}$$

Example 1.1 Let $E = C(I)$ where I is a compact interval($I\subset\mathbb{R}$).The integral linear operator A of E is taken to be defined by:

$$Au(x) = \int_{I} |k(x,t)u(t)|dt, \tag{1.9}$$

where k a function of continuous real values in square e $I\times I$. To determine ||A||, we have

$$|Au(x)| \leq ||u|| \int_{I} |k(x,t)u(t)|dx, \tag{1.10}$$

as well as,

$$||A||_{\infty} \leq \max_{x\in I} \int_{I} |k(x,t)|dx, \tag{1.11}$$

such as,

$$U(x) = \max_{x\in I} \int_{I} |k(x,t)|dx, \tag{1.12}$$

is continue on I,U reaches its maximum at some point $t_0\in I$.let define

$$\xi(x) = \begin{cases} \frac{|k(t_0,x)|}{|k(t_0,x)|} & if \quad k(t_0,x)\neq 0 \\ 0 & if \qquad not, \end{cases} \tag{1.13}$$

it clear that ξ is a integrable function on I, because it is a bounded and measurable. So exist a mapping ξ_n in $C(I)$ such that $||\xi_n||\leq 1$ and ξ_n converge to ξ in $L^1(I)$. Because $C(I)$ dense on $L^1(I)$, $1\leq p\leq\infty$). So,

$$||A|| \geq ||A\xi|| \geq A\xi(t_0) \to \int_{I} k(t_0,x)\xi(t)dx, \tag{1.14}$$

hence,

$$||A|| \geq ||A\xi|| \geq A\xi(t_0) \geq \int_I k(t_0,x)\xi(t)dx \geq \int_I |k(t_0,x)|dx \geq \max_{x\in I}\int_I k(x,t)dx, \quad (1.15)$$

Thus,according to (1.11) and (1.15)

$$||A||_\infty \leq \max_{x\in I}\int_I |k(x,t)|dx, \quad (1.16)$$

Example 1.2 let L an integral operator defined by:

$$Ls : [0,1] \to \mathbb{C} : x \to (Ls)(x) = \int_0^1 e^{tx^3} s(t)dt, \quad (1.17)$$

it is an integral operator with kernel $k(x,t) = e^{tx^3}$.

1.2.2 Compact Operators

To provide the tools for establishing the existence of solutions to a wider class of integral equations we now turn to the introduction and investigation of the compact Operators.

Definition 1.7 [11] let X and Y a normed space, A a linear operator $A : X \to Y$ is called *compact* if it maps each bounded set in X into a relatively compact set in Y.
Since a subset U of a normed space Y is relatively compact if each sequence in U contains a subsequence that converges in Y,(for the proof see [11]).

In the rest we will give equivalents conditions for a compact operator.

Theorem 1.3 *[11] A linear operator $A : X \to Y$ is compact if and only if for each bounded sequence (u_n) in X the sequence (Au_n) contains a convergent subsequence in Y.*

the proof of this theorem based on the basic properties of compact operators.(see the proof in [11])

Theorem 1.4 *[11]. Compact linear operators are bounded.*

Theorem 1.5 *[11] Let X be a normed space and Y be a Banach space. Let the sequence $A_n : X \to Y$ of compact linear operators be norm convergent to a linear operator $A : X \to Y$, i.e.,$||A_n - A|| = 0\,, n \to \infty$. Then A is compact.*

Proof. Let (u_m) be a bounded sequence in X, i.e., $||u_m| \leq C$ for all $m \in \mathbb{N}$ and some $C > 0$. Because the A_n are compact, by the standard diagonalization procedure (see the proof of Theorem [11]), we can choose a subsequence $(u_{m(k)})$ such that $(A_n u_{m(k)})$ converges for every fixed n as $k \to \infty$. Given $\epsilon > 0$, since $||A_n - A|| \to 0\, n \to \infty$, there exists $n_0 \in \mathbb{N}$such that$||A_{n_0} - A|| < \epsilon/3C$. Because $(A_{n_0}u_{m(k)})$ converges, there exists $N(\epsilon) \in \mathbb{N}$ such that:

$$||A_{n_0}u_{m(k)} - A_{n_0}u_{m(l)}|| \leq \frac{\epsilon}{3}, \tag{1.18}$$

for all $k, l \geq N(\epsilon)$, But thwn we have

$$||Au_{m(k)} - Au_{m(l)}|| \leq ||Au_{m(k)} - A_{n_0}u_{m(l)}|| + ||A_{n_0}u_{m(k)} - A_{n_0}u_{m(l)}|| + A_{n_0}u_{m(k)} - u_{m(l)}||, \tag{1.19}$$

Thus $(Au_{m(k)})$ is a Cauchy sequence, and therefore it is convergent in the Banach space Y. ■

Theorem 1.6 *[11] Let $A : X \to Y$ be a bounded linear operator with finite-dimensional range $A(X)$. Then A is compact.*

Theorem 1.7 *[11] A compact linear operator $A : X \to Y$ cannot have a bounded inverse unless X has* ***finite dimension***

Theorem 1.8 *[11] Integral operators with continuous kernel are compact linear operators on $C(G)$.*

Proof. Let $U \subset C(G)$ be bounded, i.e.,$||U||_\infty \leq C$ for all $x \in U$ and some $C > 0$. Then for the integral operator A defined by

$$|(AU)|(x) \leq C|G| \max_{x,y \in G} |k(x, y)|, \tag{1.20}$$

for all $x \in G$and all$U \in U$,, i.e., A(U) is bounded. Since K is uniformly continuous on the compact set $G \times G$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|k(x, z) - k(x, y)| < \frac{\epsilon}{C|G|}, \tag{1.21}$$

For all x,y,z $\in G$ with $|x - y| < \delta$,

$$|AU(x) - AU(y)| \leq \epsilon. \tag{1.22}$$

For allx, $y \in G$ with $|x - y| < \delta$ and all $U \in U$, i.e., $A(U)$ is *equicontinuous*(definition). Hence A is compact by Arzela - Ascoli Theorem cited in [11]). ■

1.3 L^p spaces

Definition 1.8 [16] Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set. In the study of $L^p(\Omega)$ spaces, we identify functions (i.e. such functions are considered identical) which are equal a.e. on Ω. For $p \in [1, \infty)$, $L^p(\Omega)$ is the linear space of measurable functions $v : \Omega \to \mathbb{R}$ such that

$$\|v\|_{L^p(\Omega)} = \left[\int_\Omega |v(\boldsymbol{x})|^p dx \right]^{1/p} < \infty$$

The space $L^\infty(\Omega)$ consists of all essentially bounded measurable functions $v : \Omega \to \mathbb{R}$

$$\|v\|_{L^\infty(\Omega)} = \inf_{\text{meas}(\Omega')=0} \sup_{\boldsymbol{x} \in \Omega \setminus \Omega'} |v(\boldsymbol{x})| < \infty$$

For $p = 1, 2, \infty$, it is quite straightforward to show $\| \cdot \|_{L^p(\Omega)}$ is a norm.

Theorem 1.9 *[16] Let Ω be an open bounded set in $\mathbb{R}^d$.*

(a) For $p \in [1, \infty]$, $L^p(\Omega)$ is a Banach space.

(b) For $p \in [1, \infty]$, every Cauchy sequence in $L^p(\Omega)$ has a subsequence which converges pointwise a.e. on Ω.

(c) If $1 \leq p \leq q \leq \infty$, then $L^q(\Omega) \subset L^p(\Omega)$,

$$\|v\|_{L^p(\Omega)} \leq \text{meas}(\Omega)^{1/p - 1/q} \|v\|_{L^q(\Omega)} \quad \forall v \in L^q(\Omega)$$

and

$$\|v\|_{L^\infty(\Omega)} = \lim_{p \to \infty} \|v\|_{L^p(\Omega)} \quad \forall v \in L^\infty(\Omega)$$

(d) If $1 \leq p \leq r \leq q \leq \infty$ and we choose $\theta \in [0, 1]$ such that

$$\frac{1}{r} = \frac{\theta}{p} + \frac{(1-\theta)}{q}$$

then

$$\|v\|_{L^r(\Omega)} \leq \|v\|^\theta_{L^p(\Omega)} \|v\|^{1-\theta}_{L^q(\Omega)} \quad \forall v \in L^q(\Omega)$$

In (c), when $q = \infty, 1/q$ is understood to be 0 . The result (d) *is called an interpolation property of the L^p spaces. We can use the Hölder inequality to prove (c) and (d) (see the proof in [16]).*

For $p \in (1, \infty)$, we have the following Clarkson inequalities. Let $u, v \in L^p(\Omega)$. If $2 \leq p < \infty$, then

$$\left\| \frac{u+v}{2} \right\|^p_{L^p(\Omega)} + \left\| \frac{u-v}{2} \right\|^p_{L^p(\Omega)} \leq \frac{1}{2} \|u\|^p_{L^p(\Omega)} + \frac{1}{2} \|v\|^p_{L^p(\Omega)}.$$

If $1 < p \leq 2$, then

$$\left\|\frac{u+v}{2}\right\|^q_{L^p(\Omega)} + \left\|\frac{u-v}{2}\right\|^q_{L^p(\Omega)} \leq \left[\frac{1}{2}\|u\|^p_{L^p(\Omega)} + \frac{1}{2}\|v\|^p_{L^p(\Omega)}\right]^{q-1}$$

where $q = p/(p-1)$ is the conjugate exponent of p. A proof of these inequalities can be found in [16]. Smooth functions are dense in $L^p(\Omega), 1 \leq p < \infty$

Theorem 1.10 *[16] Let $\Omega \subset \mathbb{R}^d$ be an open set, $1 \leq p < \infty$. Then the space $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$; in other words, for any $v \in L^p(\Omega)$, there exists a sequence $\{v_n\} \subset C_0^\infty(\Omega)$ such that*

$$\|v_n - v\|_{L^p(\Omega)} \to 0 \quad \text{as } n \to \infty$$

For any $m \in \mathbb{Z}_+$, by noting the inclusions $C_0^\infty(\Omega) \subset C^m(\bar{\Omega}) \subset L^p(\Omega)$, we see that the space $C^m(\bar{\Omega})$ is also dense in $L^p(\Omega)$.

1.4 Integral Equations

Definition 1.9 [16] An integral equation is an equation for an unknown function u, where u appears also under the integral sign. The integral equation generally used to resolve an ordinary differential equations.Consider the initial value problem

$$u'(x) = f(x, u(x)) \quad for\, x > x_0, \quad u(x_0) = u_0. \tag{1.23}$$

Integration from x_0 to x reduces this to the integral equation.

$$u(x) = u_0 + \int_{x_0}^{x} f(s, u(s))ds\, for\, x > x_0. \tag{1.24}$$

The reformulation (1.24) is interesting for a variety of reasons, one of which is that it is better suited than (1.23) to show that a solution exists and is unique.

1.5 Integral Equation Classification

Definition 1.10 [2] There are many kinds of integral equations. The types mostly depend on the kernel of the problem and the integration boundaries. We will focus on the following categories of integral equations in this section.

The ordinary form of a linear integral equation is given by

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dx, \tag{1.25}$$

where g(x) and h(x) are the integration limits, λ is a constant parameter, where $k(x,t)$ is named by *the kernel* or *the nucleus*[2], a well-known function of the two variables x and t. The unknown function $u(x)$ that will be determined appears inside the integral sign.The unknown function $u(x)$ frequently appears both inside and outside the integral sign. In advance, the functions $k(x,t)$ and $f(x)$ are given. We can classing the integral equations depending the boundaries.

1.5.1 Fredholm integral equation

Definition 1.11 [2] Fredholm integral equations, the boundaries of integration are fixed $g(x) = b$ and $h(x) = a$.

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dx, \tag{1.26}$$

1. if, the unknown function $u(x)$ appears only inside integral sign the Fredholm integral equation called Fredholm integral equation of the *first kind* and it is given by the form:

$$f(x) = \lambda \int_a^b k(x,t)u(t)dt, \tag{1.27}$$

2. if the unknown function $u(x)$ appears inside and outside the integral sign. the Fredholm integral equation called Fredholm integral equation of *second space*.

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \tag{1.28}$$

Example 1.3 Examples of the two kinds are given by:

$$sin(x) - xcosx = \int_0^1 k(x,t)u(t)dt, \tag{1.29}$$

$$u(x) = ln(x) + \int_1^2 k(x,t)u(t)dt, \tag{1.30}$$

1.5.2 Volterra integral equation

Definition 1.12 [2] The integral equation called Volterra integral equation if the integral has at least one limit that is a variable, $h(x) = x$ and $g(x) = 0$. It given by the form

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dx, \tag{1.31}$$

1. if the unknown function $u(x)$ appears only inside integral sign in the form the Volterra equation called Volterra equation of the *first kind* and it is given by the for

$$f(x) = \lambda \int_0^x k(x,t)u(t)dx, \tag{1.32}$$

2. if the unknown function $u(x)$ appears inside and out side integral sign in the Volterra equation called Volterra equation of the *second kind* and it is given in the form:

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dx, \tag{1.33}$$

Example 1.4 1. Examples of the Volterra integral equations of the first kind are:

$$xe^{-x} = \int_0^x e^{t-x}dt, \tag{1.34}$$

$$5x^2 + x^3 = \int_0^x u(t)dt, \tag{1.35}$$

2. examples of the Volterra integral equations of the second kind are:

$$u(x) = 1 - \int_0^x tu(t)dt, \tag{1.36}$$

$$u(x) = x^3 + \int_0^x (t-x)u(t)dt, \tag{1.37}$$

1.5.3 Fredholm-Volterra Integral Equation

Definition 1.13 [2] The Volterra-Fredholm integral equations come from parabolic boundary value problems, mathematical modeling of the spatio-temporal development of an epidemic, and a variety of physical and biological models .

The Volterra-Fredholm integral equations can be found in two different formats in the literature:

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt, \tag{1.38}$$

and

$$u(x,t) = f(x,t) + \lambda \textstyle\int_0^t \int_\omega F(x, t\xi, \tau, u(\xi,\tau))d\xi d\tau \quad (x,t) \in \Omega \times [o, \Gamma], \tag{1.39}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on $D = \Gamma \times [0, \Gamma]$, and Ω is a closed subset of $\mathbb{R}^n$, $n = 1, 2, 3...$ It is interesting to note that (1.38) contains disjoint

Volterra and Fredholm integral equations, whereas (1.39) contains mixed Volterra and Fredholm integral equations. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of the first kind, but will not be examined in this text.

Example 1.5 we give two exemples of the tow types:

$$u(x) = 6x + 3x^2 + 2 - \int_0^x xu(x)dt - \int_0^1 tu(x)dt, \tag{1.40}$$

$$u(x) = 6x + t^3 + \frac{1}{2}t^2 - \int_0^t \int_0^1 (\tau - \xi)dt, \tag{1.41}$$

1.5.4 Singular Integral Equations

Integral equations of the first kind:

$$f(x) = \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dx, \tag{1.42}$$

or of the second kind:

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)u(t)dx, \tag{1.43}$$

are defined as *singular* when one of the integration boundaries, g(x), h(x), or both are infinite.The previous two equations are called singular if the kernel $k(x,t)$ becomes unbounded at one or more points in the interval of integration.

Example 1.6

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \tag{1.44}$$

$$u(x) = f(x) \int_0^x \frac{1}{(x-t)^\alpha} u(t)dt, \tag{1.45}$$

in this example the two equations(1.44)(1.45) are called generalized *Abel's* integral equation and weakly singular integral equations respectively if $\alpha = \frac{1}{2}$:

$$f(x) = \int_0^x \frac{1}{(x-t)^{\frac{1}{2}}} u(t)dt, \tag{1.46}$$

is called the *Abel' s* singular integral equation. It is to be noted that the kernel in each equation becomes infinity at the upper limit $t = x$.

1.6 Classification of Integro-Differential Equations

Definition 1.14 [2] Integro-differential equations are used in many scientific applications, especially when integral equations are used to convert initial value or boundary value problems. Differential and integral operators are both present in the integro-differential equations.The unknown function´ s derivatives could arrive in any order. In classifying integro-differential equations.

In classifying integro-differential equations, we will follow the same category used before.

1.6.1 Fredholm Integro-Differential Equations

Fredholm integro-differential equations appear when we convert differential equations to integral equations. The Fredholm integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(x)}(x)$, $n \geq 1$ inside and outside the integral sign respectively. In this case, the integration's bounds are fixed, similarly to the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions.The *Fredholm integro-differential equation* appears in the form:

$$u^{(n)}(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \tag{1.47}$$

where $u^{(n)}$ represents the nth derivative of $u(x)$. With $u^{(n)}$ at the left side, other derivatives of lower order might also appear.

Example 1.7

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xu(t)dt\, u(0) = 0, \tag{1.48}$$

$$u''(x) + u'(x) = x - sin(x) - \int_0^{\frac{\pi}{2}} xtu(x)dt, \tag{1.49}$$

1.6.2 Volterra Integro-Differential Equations

In Volterra Integro-Differential Equations. The Volterra integro-differential equation contains the unknown function $u(x)$ and one of its derivatives $u^{(n)}(x)$, $n \geq 1$ inside and outside the integral sign. At least one of the limits of integration this case is a variable as in the Volterra integral equations. The equation is called integro-differential because differential and integral operators are involved in the same equation. It is important to note that initial conditions should be given for Volterra integro-differential equations to determine the particular solutions. *The Volterra integro-differential equation* appears in the form:

$$u^{(n)} = f(x) + \lambda \int_0^x k(x)u(t)dt, \tag{1.50}$$

where $u^{(n)}$ indicates the nth derivative of $u(x)$. Other derivatives of less order may appear with $u^{(n)}$ on the left side.

Example 1.8

$$u'(x) = -1 + \tfrac{1}{2}x^2 + xe^x - \int_0^x tu(t)dt, \quad u(0) = 0, \tag{1.51}$$

$$u''(x) + u'(x) = 1 - x(sin(x) + cos(x)) - \int_0^x tu(x)dt, \quad u(0) = -1, \quad u'(0) = 1, \tag{1.52}$$

1.6.3 Volterra-Fredholm Integro-Differential Equations

The Volterra-Fredholm integro-differential equations are formed similarly to the Volterra-Fredholm integral equations, but with one or more ordinary derivatives in addition to the integral operators. There are two ways that the Volterra-Fredholm integro-differential equations can be found in the literature.

$$u^{(n)} = f(x) + \lambda_1 \int_a^b k_1(x,t)u(t)dt + \lambda_2 \int_a^x k_2(x,t)u(t)dt, \tag{1.53}$$

and,

$$u^{(n)}(x,t) = f(x,t) + \lambda \int_0^t \int_\omega F(x,t,\xi,\tau,u(\xi,\tau))d\xi d\tau, \ (x,t) \in \Omega \times [0,\Gamma], \tag{1.54}$$

where $f(x,t)$ and $F(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on and Ω is a closed subset of R^n, $n = 1,2,3$....It's important to note that (1.53) contains disjoint Volterra and Fredholm integral equations, whereas (1.54)contains mixed integrals. It is also possible for equations with lower-order derivatives to exist. Moreover, the unknown functions $u(x)$ and $u(x,t)$ appear inside and outside the integral signs. This is a characteristic feature of a second kind integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of the first kind. Initial conditions should be given to determine the particular solution.

Example 1.9 Examples of the two types are given by:

$$u'(x) = 24x + x^4 + 3 + \int_0^x (x-t)u(t)dt - \int_0^1 tu(t)dt, \quad u(0) = 0, \tag{1.55}$$

$$u'(x,t) = 1 + t^3 + \tfrac{1}{2}t + \tfrac{1}{3}t^3 \int_0^t \int_0^1 (\tau - \xi)d\xi d\tau, \quad u(0) = t^3, \tag{1.56}$$

1.7 Linearity and Homogeneity

Integral equations and integro-differential equations may also be classified into two categories based on the concepts of *linearity* and *homogeneity*. These two concepts have a significant role in the solutions structure. The definitions of these concepts are highlighted in the phrases that follow.

1.7.1 Linearity Concept

Definition 1.15 [2] The integral equation or the integro-differential equation is referred to as linear if the exponent of the unknown function $u(x)$ inside the integral sign is one [6]. The integral equation or the integro-differential equation is referred to as nonlinear if the unknown function $u(x)$ has an exponent other than one or if the equation involves nonlinear functions of $u(x)$, such as e^u, $sinh(u)$, $cos(u)$, or $ln(1+u)$.

Example 1.10 1. The first two examples are Fredholm and Volterra integral *linear* equations respectively:

$$u(x) = 1 - \int_0^1 (t-x)u(t)dt, \tag{1.57}$$

$$u(x) = ln2 + \int_0^x (t-x)u(t)dt, \tag{1.58}$$

2. The last two examples are *nonlinear* Volterra integral equation and nonlinear Fredholm integro-differential equation respectively.

$$u(x) = 1 + \int_0^x (1+x-t)u^4(t)dt, \tag{1.59}$$

$$u'(x) = 1 + \int_0^1 xte^{u(t)}dt\,, u(0) = a, \tag{1.60}$$

1.7.2 Homogeneity Concept

Definition 1.16 [2] The second type of Volterra or Fredholm integral equations or integro-differential equations are categorized as *homogeneous* if the function $f(x)$ is identically zero. The equation is described as *inhomogeneous* if the function $f(x)$ is not identically zero.

Remark 1.4 We should know that this property holds for equations of the second kind only

Example 1.11 The first equation is an *inhomogeneous* Volterra integral equation

$$u(x) = sin(x) + \int_0^x txu(t)dt, \tag{1.61}$$

the second equation is an *inhomogeneous* Fredholm integro-differential equation

$$u'(x) = x + \int_0^1 (t-x)u(t)dt, \qquad u(0) = 0, \tag{1.62}$$

the third equation is an *homogeneous* Volterra integro-differential equation

$$u'(x) = \int_0^x (t-x)^2 u(t)dt, \qquad u(0) = 0, \tag{1.63}$$

the third equation is an *homogeneous* fredholm integral equation

$$u(x) = \int_0^2 x^2 u(t)dt, \tag{1.64}$$

1.8 About integro-differential equation classification

In this section we are going to speak about the classification of intigro-differential equations depending the position of the derivatives of the unknown function $u^{(n)}(x)$ for $n = 1$ inside and outside the integration sign, the derivative function is inside integration sign and if , the derivative function is outside integration sign. Let the integro-differential equation formula types depending the u' position:

1. **The unknown derivative function $u'(x)$ are inside and out side the integration sign:**

$$u'(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t,u(t),u'(t))dt, \qquad u_0(x) = \alpha, \tag{1.65}$$

(a) If g(x) and h(x) are fixed, the integro-equtionintegro-equation called ***Fredholm integro-differential equation***

$$u'(x) = f(x) + \lambda \int_a^b k(x,t,u(t),u'(t))dt, \qquad u_0(x) = \alpha, \tag{1.66}$$

(b) If g(x) and h(x) are at least one of the integral limit is a variable. The Integro-equtionintegro-equation called ***Volterra integro-differential equation***:

$$u'(x) = f(x) + \lambda \int_a^x k(x,t,u(t),u'(t))dt, \qquad u_0(x) = \alpha, \tag{1.67}$$

(c) If g(x) and h(x) are at least one of the integral limit is a variable in the first integral and the second integral limit is fixed $g(x) = a$ and $h(x) = b$

$$u'(x) = f(x)+\lambda_1 \int_{a_1}^x k_1(x,t,u(t),u'(t))dt+\lambda_2 \int_{a_2}^b k_2(x,t,u(t),u'(t))dt, \; u_0(x) = \alpha, \tag{1.68}$$

and,

$$u'(x,t) = f(x,t) + \lambda \int_0^t \int_\omega k(x,t,\xi,\tau,u(\xi,\tau),u'(\xi,\tau))d\xi d\tau \, , u_0(x) = 0, \tag{1.69}$$

$forall \; (x,t) \in \Omega \times [0,\Gamma]$,
where $f(x,t)$ and $k(x,t,\xi,\tau,u(\xi,\tau))$ are analytic functions on and Ω is a closed subset of R^n, $n = 1,2,3....$

2. **The unknown derivative function $u'(x)$ are inside the integration sign**

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t,u(t),u'(t))dt, \qquad u_0(x) = \alpha, \tag{1.70}$$

(a) If g(x) and f(x) are fixed, the integro-equation called ***Fredholm integro-differential equation***

$$u(x) = f(x) + \lambda \int_a^b k(x,t,u(t),u'(t))dt, \qquad u_0(x) = \alpha, \tag{1.71}$$

(b) If g(x) and f(x) are at least one of the integral limit is a variable. The integro-eqution called ***Volterra integro-differential equation***

$$u(x) = f(x) + \lambda \int_a^x k(x,t,u(t),u'(t))dt, \qquad u_0(x) = \alpha, \tag{1.72}$$

(c) If g(x) and h(x) are at least one of the integral limit is a variable in the first integral and the second integral limit is fixed $g(x) = a$ and $h(x) = b$,

$$u(x) = f(x) + \lambda_1 \int_{a_1}^{x} k_1(x, t, u(t))dt + \lambda_2 \int_{a_2}^{b} k_2(x, t, u(t))dt, \quad u_0(x) = \alpha \tag{1.73}$$

and

$$u(x,t) = f(x,t) + \lambda \int_0^t \int_\omega k(x, t, \xi, \tau, u(\xi, \tau), u'(\xi, \tau))d\xi d\tau \, , u_0(x) = 0, \tag{1.74}$$

$(x, t) \in \Omega \times [0, \Gamma]$,
where $f(x, t)$ and $k(x, t, \xi, \tau, u(\xi, \tau))$ are analytic functions on and Ω is a closed subset of R^n, $n = 1, 2, 3....$

1.9 Banach's Fixed Point Theorem

Banach fixed-point theorem, also known as the principle contraction of Banach or Picard fixed point theorem, has appeared for the first time in 1922 as part of solving an integral equation. Note that this theorem is an abstraction of the classical method successive approximations introduced by Liouville (in 1837) and subsequently developed by Picard (in 1890). Due to its simplicity and utility, this theorem is widely used in several branches of analysis especially, in the branch of differential equations. Banach fixed-point theorem has known various generalizations in different spaces.

Definition 1.17 [17] Let X be a *Banach space* and $A : X \to X$ a contraction it is mean there is a number $0 \leq k < 1$,

$$||A(x) - A(y)|| \leq k||x - y||| \quad for\,all \quad x, y \in \mathbb{X} \tag{1.75}$$

,

Hence, the following is true. The fixed point equation:

$$x^* = A(x^*), \tag{1.76}$$

has exactly one solution $x^* \in X$.

Proof.

1 ***Existence*** :let $(x_n)_{n\in\mathbb{N}}$, defined by:

$$\begin{cases} x_0 \in \mathbb{X} \\ x_{n+1} = Ax_n. \end{cases}$$

Let's prove that $(x_n)_{n\in\mathbb{N}}$ is of Cauchy. Let $m,\ n \in \mathbb{N}$ with $m \leq n$:

$$\begin{aligned}||x_m - x_n|| &\leq ||x_m - x_{m+1}|| + ||x_{m+1} - x_n|| \\ &\leq ||x_m - x_{m+1}|| + ||x_{m+1} - x_{m+2}|| + ... + ||x_{n-1} - x_n|| \\ &\leq k^m||x_m - x_{m+1}|| + k^{m+1}||x_{m+1} - x_{m+2}|| + ... + k^{n-1}||x_{n-1} - x_n|| \\ &\leq (k^m + k^{m+1} + ... + k^{n-1})||x_0 - x_1||.\end{aligned}$$

But

$$(k^m + k^{m+1} + ... + k^{n-1}) = k^m(\frac{1 - k^{n-m}}{1-k} = \frac{k^m}{1-k}(1 - k^{n-m}),$$

Hence

$$||x_m - x_n|| \leq \frac{k^m}{1-k}(1 - k^{n-m})||x_0 - x_1||,$$

we have

$$(1 - k^{n-m}) \leq 1,$$

so

$$||x_m - x_n|| \leq \frac{k^m}{1-k}||x_0 - x_1||,$$

suppose that $||x_0 - x_1|| \neq 0$, for $||x_m - x_n|| \leq \epsilon$, it is enough that ,

$$\frac{k^m}{1-k}||x_0 - x_1|| \leq \epsilon,$$

so $(x_n)_{n\in\mathbb{N}}$ it a Cauchy sequence in X and $(X, ||.||)$ it is a complete space, so $(x_n)_{n\in\mathbb{N}}$ converging in X Let $x^* = \lim_{n\to\infty} x_n$, $x^* \in X$, lets proof that x^* is a fixed point of A. A continuous, so :

$$\forall n \in \mathbb{N}, x_{n+1} = Ax_n,$$

$$\Rightarrow \lim_{n\to\infty} x_n = \lim_{n\to\infty} Ax_n,$$

$$\Rightarrow x^* = A(\lim_{n\to\infty} x_n) = A(x^*).$$

2 **Uniqueness**: Let x^* and x^{**} be two solutions of (1.2).
From (1.1) one concludes $||x^* - x^{**}|| \leq k||x^* - x^{**}||$ with $k < 1$; hence, $||x^* - x^{**}|| = 0$ proving the uniqueness of the solution.

■

CHAPTER 2

ULAM– HYERS AND ULAM– HYERS–RASSIAS STABILITY FOR A CLASS OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS

The stability is an important topic in the applications. The stability theory for functional equations started with a problem related to the stability of group homomorphisms that was considered by S.M. Ulam in 1940 (see [12] and [13]).

Ulam considered the following question:

Let G_1 be a group and let G_2 be a group endowed with a metric d. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta,$$

for all $x, y \in G_1$, then we can find a homomorphism $\theta : G_1 \to G_2$ such that

$$d(h(x), \theta(x)) < \epsilon,$$

for all $x \in G_1$?

An affirmative answer to this equation was given by D. H. Hyers (see [5]) for the case of Banach spaces. This answer, in this case, says that the Cauchy functional equation is stable in the sens of Heyers-Ulam. In 1950.

In 1978, Th. M. Rassias [15] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. In [15], Th. M. Rassias has introduced a new type of stability which is called the Hyers-Ulam-Rassias stability. In general, we may say that the main issue in determining the conditions under which slightly different solutions of an equation must approach the solution of the given equation is the stability of functional equations. For the nonlinear Volterra integral equations of the type, we suggest a *Hyers-Ulam-Rassias stability* research in the current work. For a given continuous function k and a fixed real number a, the integral equation

$$u(x) = \int_a^x k(x, s, u(s))ds, \quad -\infty < a \leq x < +\infty, \tag{2.1}$$

is called a Volterra integral equation of the second kind. We follow the fixed point arguments used in [8] and prove the Hyers-Ulam-Rassias stability and the Hyers-Ulam stability of the Volterra integral equation (2.1) for the case of compact domains.

Definition 2.1 (generalized metric space) [3] Let X be a nonempty set. A function $d : X \times X \to [0, +\infty]$ is called a generalized metric on X if and only if d satisfies the following three propositions:

(P1) $d(x, y) = 0$ if and only if $x = y$.
(P2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(P3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The Banach Fixed Point theorem will play an important role in proving our main stability problem.

Theorem 2.1 *[7] Let (X, d) be a generalized complete metric space and $A : X \to X$ a strictly contractive operator with a Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d\left(A^{k+1}x, A^k x\right) < \infty$ for some $x \in X$, then the following propositions hold true:*

(A) the sequence $(A^n x)_{n\in\mathbb{N}}$ converges to a fixed point x^* of A.

(B) x^* is the unique fixed point of A in

$$X^* = \left\{y \in X \mid d\left(A^k x, y\right) < \infty\right\}, \tag{2.2}$$

(C) if $y \in X^*$, then

$$d\left(y, x^*\right) \leq \frac{1}{1-L} d(Ay, y). \tag{2.3}$$

2.1 The Hyers-Ulam-Rassias stability of the Volterra integral equation

In this part, we will examine if the Volterra integral equation (2.1) accepts the stability of Hyers-Ulam-Rassias. This is assembled put together in the ensuing theorem.

Theorem 2.2 *[8] Let C and L be positive constants with $0 < CL < 1$ and assume that $k : [a, b] \times [a, b] \times \mathbb{C} \to \mathbb{C}$ is a continuous function which additionally satisfies the Lipschitz condition*

$$|k(x, s, u) - k(x, s, z)| =\leq L|u - z|, \tag{2.4}$$

for any $x, s \in [a, b]$ and all $u, z \in \mathbb{C}$.

$$\left|u(x) - \int_a^x k(x, s, u(s))ds\right| \leq \tau(x), \tag{2.5}$$

for all $x \in [a, b]$, and where $s : [a, b] \to (0, \infty)$ is a continuous function with

$$\left|\int_a^x \tau(s)ds\right| \leq C\tau(x), \tag{2.6}$$

for each $x \in [a, b]$, then there exists a unique continuous function $u_0 : [a, b] \to \mathbb{C}$ such that

$$u_0(x) = \int_a^x f\left(x, s, u_0(s)\right) ds, \tag{2.7}$$

$$|u(x) - u_0(x)| \leq \frac{1}{1 - CL}\tau(x), \tag{2.8}$$

for all $x \in [a, b]$.

Proof. We will consider the space of continuous functions

$$X = \{g : [a, b] \to \mathbb{C} \mid g \text{ is continuous }\}, \tag{2.9}$$

endowed with the generalized metric on X defined by:

$$d(g, h) = \inf\{C \in [0, \infty] \mid g(x) - h(x) \mid\leq Cu(x), \text{ for all } x \in [a, b]\}. \tag{2.10}$$

It is evident that (X, d) is a complete generalized metric space introducing the $A : X \to X$ is operator, which is given by:

$$(Ag)(x) = \int_a^x k(x, s, g(s))ds, \tag{2.11}$$

for all $g \in X$ and $x \in [a, b]$. Thus, due to the fact that k is a continuous function, it mean that Ag is also continuous and this ensures that A is a well defined operator. Indeed,

$$\begin{aligned}|(Ag)(x)-(Ag)(x_0)| &= \left|\int_a^x k(x,s,g(s))ds - \int_a^{x_0} k(x_0,s,g(s))\,ds\right| \\ &= |\int_a^x k(x,s,g(s)) - \int_a^x k(x_0,s,g(s))\,ds \\ &\quad + \int_a^x k(x_0,s,g(s)) - \int_a^{x_0} k(x_0,k,g(s))\,ds\,| \\ &\leq \left|\int_a^x k(x,s,g(s)) - \int_a^x k(x_0,s,g(s))\,ds\right| \\ &\quad + \left|\int_a^x k(x_0,s,g(s)) - \int_a^{x_0} k(x_0,s,k(s))\,ds\right| \\ &\leq \int_a^x |k(x,s,g(s)) - k(x_0,s,g(s))|\,ds \\ &\quad + \left|\int_{x_0}^x k(x_0,s,g(s))\,d\tau\right| \stackrel{x\to x_0}{\longrightarrow} 0.\end{aligned}$$

We will now verify that A is strictly contractive on X. For any $g,h\in X$, let us consider $C_{gh}\in[0,\infty]$ such that

$$|g(x)-h(x)| \leq C_{gh}\tau(x), \tag{2.12}$$

for any $x\in[a,b]$. Note that this is always possible due to the definition of (X,d). From the definition of A and (2.1), (**??**) and (2.12), it follows

$$\begin{aligned}|(Ag)(x)-(Ah)(x)| &= \left|\int_a^x [k(x,s,g(s)) - f(x,s,h(s))]ds\right| \\ &\leq \left|\int_a^x\right| f(x,s,g(s)) - k(x,s,h(s))|ds| \\ &\leq L\left|\int_a^x\right| g(s)-h(s)|ds| \\ &\leq LC_{gh}\left|\int_a^x \tau(s)ds\right| \\ &\leq LC_{gh}C\tau(x),\end{aligned}$$

for all $x\in[a,b]$. Therefore, $d(Ag,Ah)\leq LC_{gh}C$. This allows us to say that $dA(g,Ah)\leq LCd(g,h)$ for any $g,h\in X$, and since $CL\in(0,1)$ the (strictly) contraction property is verified. Let us take $g_0\in X$. From the continuous property of g_0 and Ag_0 it means that there exists a constant $C_1\in(0,\infty)$ such that

$$d(Ag_0,g_0)<\infty. \tag{2.13}$$

As a result, we use the Banach Fixed Point theorem and determine that there is a continuous function. $u_0:[a,b]\to\mathbb{C}$ such that

$$A^n g_0 \stackrel{n\to\infty}{\longrightarrow} g_0 \quad \text{in} \quad (X,d), \tag{2.14}$$

and $Au_0 = g_0$. It follows that X may be recast in the new form shown below for any g_0 with the condition (2.13).

$$X = \{g \in X \mid d(g_0, g) < \infty\}. \tag{2.15}$$

Therefore, once again the Banach Fixed Point theorem ensures that u_0 is the unique continuous function with the property (2.7).

Now, from (2.5) it follows that $d(y, Ay) \leq 1$, and so the Banach Fixed Point Theorem leads to

$$d(u, u_0) \leq \frac{1}{1 - CL} d(Au, u) \leq \frac{1}{1 - CL}. \tag{2.16}$$

Inequality (2.8) is produced as a result of the previous inequality and the extended metric definition d. ■

2.2 Hyers-Ulam-Rassias stability of the Volterra integral equation in the infinite interval case

Theorem 2.3 *[8] Let C and L be positive constants with $0 < CL < 1$ and assume that $k : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is a continuous function which additionally satisfies the Lipschitz condition (2.4), for any $x, \tau \in \mathbb{R}$ and all $y, z \in \mathbb{C}$.*

If a continuous function $u : \mathbb{R} \to \mathbb{C}$ satisfies (2.5), for all $x \in \mathbb{R}$ and for some $\delta \in \mathbb{R}$, where $u : \mathbb{R} \to (0, \infty)$ is a continuous function satisfying (2.6), for each $x \in \mathbb{R}$, then there exists a unique continuous function $y_0 : \mathbb{R} \to \mathbb{C}$ which satisfies (2.7) and (2.8) for all $x \in \mathbb{R}$. Proof. **Proof.** First we will prove that u_0 is a continuous function.

For any $n \in \mathbb{N}$, let us define $I_n = [\delta - n, \delta + n]$. According to Theorem (2.2), there exists a unique continuous function $u_{0,n} : I_n \to \mathbb{C}$ such that:

$$u_{0,n}(x) = \int_{\delta}^{x} k(x, s, u_{0,n}(s))\, ds, \tag{2.17}$$

$$|u(x) - u_{0,n}(x)| \leq \frac{1}{1 - CL} \tau(x), \tag{2.18}$$

for all $x \in I_n$. Because $u_{0,n}$ is unique, it implies that if $x \in I_n$, then

$$u_{0,n}(x) = u_{0,n+1}(x) = u_{0,n+2}(x) = \ldots \tag{2.19}$$

For any $x \in \mathbb{R}$, let us define $\zeta(x) \in \mathbb{N}$ as

$$\zeta(x) = \min\{n \in \mathbb{N} \mid x \in I_n\}. \tag{2.20}$$

We also define the following function $u_0 : \mathbb{R} \to \mathbb{C}$ by

$$u_0(x) = u_{0,\zeta(x)}(x), \tag{2.21}$$

and we can say that u_0 is continuous. Indeed, for any $x_1 \in \mathbb{R}$, let $\zeta_1 = n(x_1)$. Then x_1 belongs to the interior of I_{ζ_1+1} and there exists an $\epsilon > 0$ such that $u_0(x) = u_{0,\zeta_1+1}(x)$ for all $x \in (x_1 - \epsilon, x_1 + \epsilon)$. By Theorem (2.2), u_{0,ζ_1+1} is continuous at x_1, so it is u_0. We shall now demonstrate that for any $x \in \mathbb{R}$, u_0 satisfies conditions (2.7) and (2.8). We have to select $\zeta(x)$ for any $x \in \mathbb{R}$. Once x is $\in I_\zeta(x)$, it follows from (2.17) that

$$u_0(x) = u_{0,\zeta(x)}(x) = \int_\delta^x k\left(x, s, u_{0,\zeta(x)}(s)\right) ds = \int_\delta^x k\left(x, s, u_0(s)\right) ds, \tag{2.22}$$

where the last equality is correct since every value of tau within $I_\zeta(x)$ has $\zeta(s) \leq \zeta(x)$, and it is evident from (2.19) that

$$u_0(s) = u_{0,\zeta(s)}(s) = u_{0,\zeta(x)}(s), \tag{2.23}$$

Additionally, (2.18) implies that for any $x \in \mathbb{R}$

$$|u(x) - u_0(x)| = \left|u(x) - u_{0,\zeta(x)}(x)\right| \leq \frac{1}{1 - CL}\tau(x). \tag{2.24}$$

We shall demonstrate that u_0 is unique in the end. Suppose that u_1 is another continuous function which satisfies (2.7) and (2.8), for all $x \in \mathbb{R}$. Since the restrictions $u_0|_{I_{\zeta(x)}} = u_{0,\zeta(x)}$ and $u_1|_{I_{\zeta(x)}}$ both satisfy (2.7) and (2.8) for all $x \in I_{\zeta(x)}$, the uniqueness of $u_0|_{I_{\zeta(x)}} = u_{0,\zeta(x)}$ implies that

$$u_0(x) = u_0|_{I_{\zeta(x)}}(x) = \zeta_1|_{I_{\zeta(x)}}(x) = u_1(x). \tag{2.25}$$

2.3 The Hyers-Ulam stability of the Volterra integral Equation

The Hyers-Ulam stability is attained for the Volterra integral equation under consideration (in the finite interval case) in this final part by applying additional stricter assumptions. ■

Theorem 2.4 *[8] Let $K = b - a$ and consider L>0 constant such that $0 < KL < 1$. Assume that : $[a, b] \times [a, b] \times \mathbb{C} \to \mathbb{C}$ is a continuous function which satisfies the Lipschitz condition*

$$|k(x, s, u) - k(x, s, v)| \leq L|u - v|, \tag{2.26}$$

for any $x, s \in [a, b]$ and $u,\ v \in \mathbb{C}$. If a continuous function $u : [a, b] \to \mathbb{C}$ fulfills

$$\left|u(x) - \int_a^x k(x, s, u(s))ds\right| \leq W, \tag{2.27}$$

for each $x \in [a,b]$ and some $W \geq 0$, then there exists a unique continuous function $u_0 : [a,b] \to \mathbb{C}$ such that:

$$u_0(x) = \int_a^x k\left(x, s, u_0(s)\right) ds, \tag{2.28}$$

$$|u(x) - u_0(x)| \leq \frac{W}{1 - KL}, \tag{2.29}$$

for all $x \in [a,b]$.

Proof. Let's consider the space of continuous functions shown in (2.9) and given with the generalized metric delineated by

$$d(f,g) = \inf\{C \in [0,\infty] || f(x) - g(x) | \leq C, \text{ for all } x \in [a,b]\}. \tag{2.30}$$

Introducing the $A : X \to X$ operator, which is given by:

$$(Af)(x) = \int_a^x k(x, s, f(s)) ds, \tag{2.31}$$

for all $f \in X$ and $x \in [a,b]$. As we've shown before, Af is continuous for each continuous function f. Proving that operator A is strictly contractive on X. For any $f,\ g \in X$, let us consider $C_{f,g} \in [0,\infty]$ such that

$$|f(x) - g(x)| \leq C_{fg}, \tag{2.32}$$

in $[a,b]$ $forall$ x. It follows from the definition of A in clauses (2.26) and (2.28).

$$\begin{aligned} |(Ag)(x) - (Ah)(x)| &= \left| \int_a^x [k(x,s,f(s)) - k(x,s,h(s))] ds \right| \\ &\leq \left| \int_a^x |f(x,s,g(s)) - f(x,s,h(s)) \right| ds \\ &\leq L \left| \int_a^x |g(s) - h(s)| ds \right| \\ &\leq L C_{fg} K. \end{aligned} \tag{2.33}$$

For all $x \in [a,b]$. Hence, $d(Ag, Ah) \leq LC_{gh}K$. This allows us to conclude that $d(Ag, Ah) \leq LKd(g,h)$ for any $f, g \in X$, since $KL \in (0,1)$ the (strict) contraction property is proven. In a manner similar to that used in the demonstration of Theorem (2.2), we can select $f_0 \in X$ using

$$d\left(Af_0, f_0\right) < \infty. \tag{2.34}$$

As a result, we are in a position to use the Banach Fixed Point Theorem to prove that a continuous function $u_0 : [a,b] \to \mathbb{C}$

$$A^n f_0 \overset{n\to\infty}{\longrightarrow} u_0 \quad \text{in} \quad (X,d), \tag{2.35}$$

thus, $Au_0 = u_0$. It follows that X may be transform in the new form shown below for any f_0 with the condition (2.34).

$$X = \{f \in X \mid d\left(f_0, f\right) < \infty\}. \tag{2.36}$$

As a result, the Banach Fixed Point Theorem once more assures that y_0 is the only continuous function with the condition (2.28). Additionally, Theorem (2.1)'s third proposition results in

$$|y(x) - y_0(x)| \leq \frac{\theta}{1 - KL}, \tag{2.37}$$

for all $x \in [a, b]$. ■

CHAPTER 3

ULAM –HYERS AND ULAM – HYERS – RASSIAS STABILITY OF VOLTERRA INTEGRO – DIFFERENTIAL EQUATION

intoduction

In this chapter we are going to study Ulam-Hyers and Ulam-Hyers-Rassias stability of the equation given at the bottom which was posed by Pachpat [4], using the Banach fixed point theorem and the Lipchtiez condition with constant Lipchtiz and variant Lipchtiz condition. Let the integro-differential equation given by:

$$u(x) = f(x) + \int_0^x k\left(x, s, u(s), u'(s)\right) ds, \tag{3.1}$$

for all $x,\ s \in I = [0, 1]$, where $u, fandk$ are n-dimensional real vectors, with the derivative denoted by the symbol'. Let $\mathbb{R}$ represent the set of real numbers and $\mathbb{R}^n$ represent the n-dimensional Euclidean space with the appropriate norm indicated by $|.|$. Let $I = [0, 1]$ be the given subset of $\mathbb{R}$. The functions $f(x)$ and $k(x, s, u(s), u'(s))$ are continuous and are continuIusly differentiable with respect to x.

A continuous function $u(x)$ for $x \in \mathbb{R}$ that is continuously differentiable with respect to x and satisfies the related equation (3.1) is known as a solution of equation (3.1). We denote $|u(x)|_1 = |u(x)| + |u'(x)|$ for every continuous function $u(x)$ in $\mathbb{R}^n$ with its continuous first derivative $u'(x)$ for all $x \in I$. Let $C^1(\mathbb{R}^n)$ be the set of continuous functions $u(x)$ in $C^1(\mathbb{R}^n)$ and it is continuous first derivative $u'(x)$ in $C^1(\mathbb{R}^n)$ that satisfy this condition

$$\|u(x)\|_1 = \sup_{x \in I} |u(x)|_1. \tag{3.2}$$

3.1 Hyers-Ulam stability of Volterra integro− differential equations

We prove the Hyers-Ulam stability of the nonlinear Volterra intego-differential equation (3.1), in this section under many realistic circumstances. The following is the first main result we have:

Theorem 3.1 *(The alternative of fixed point) [7] Suppose we are given a complete generalized metric space (X, d) and a strictly contractive mapping : $X \to X$, with the Lipschitz constant L. Then, for each given point*
(i) $d(A^n x, A^{n+1}x) < \infty$ for all natural number $n \geq k_0$.
(ii) The sequence $\{A^n x\}$ converges to a fixed point y_* of A.
(iii) y_* is the unique fixed point of A in the set

$$Y = \left\{ y \in X : d\left(A^{k_0}x, y\right) < \infty \right\}. \tag{3.3}$$

(iv) If $y \in Y$, then

$$d(y, y_*) \leq \frac{1}{1-L} d(Ay, y). \tag{3.4}$$

Theorem 3.2 *Let set $I := [0,1]$. Let $X = C^1(\mathbb{R}^n)$, it's clear that X is Banach space over the (real or complex) . Let $L = 2max(\gamma_1, \gamma_2) + max(\beta_1, \beta_2)$ be a positive constant with $0 < L < 1$. Assume that $k : I \times I \times \mathbb{R} \times \mathbb{R} \to X$ is a continuous function which fulfills the next Lipschitz condition.*

Assume that the function k in equation (3.1) and its derivative with respect to x satisfy this conditions, such that for any x, $s \in [0,1]$ and u, v ∈X, we have

$$\begin{aligned} &|k(x, s, u(s), u'(s)) - k(x, s, v(s), v'(s))|, \\ &\leq \gamma_1 |u(x) - v(x)| + \gamma_2 |u'(x) - v'(x)| \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &|\frac{\partial}{\partial x} k(x, s, u(s), u'(s)) - \frac{\partial}{\partial x} k(x, s, v(s), v'(s))| \\ &\leq \beta_1 |u(x) - v(x)| + \beta_2 |u'(x) - v'(x)|. \end{aligned} \tag{3.6}$$

Suppose that a continuous function $v : I \in C^1$ satisfies

$$\|v(x) - f(x) - \int_0^x k(x, s, u(x), u'(x)) ds\|_1 \leq \tau, \quad \forall x \in I, \tag{3.7}$$

for some positive number τ. Then a unique continuous function $u_0 : I \in C^1$ exists such that

$$u_0(x) = f(x) + \int_0^x k\left(x, s, u_0(s), u_0'(s)\right) ds, \quad \forall x \in I, \tag{3.8}$$

(As a result, u_0 is a solution to the equation (3.1)) and

$$\|v(x) - u_0(t)\|_1 \leq \frac{\tau}{1-L}, \tag{3.9}$$

for all $x \in [0,1]$.

Proof. Let X be the set of all continuous functions from I to X. For $v, u \in C^1$, we set

$$d_1(v,u) = \inf\{\tau \in [0,\infty] : |v(x) - u(x)|_1 \leq \tau, \forall x \in I\}. \tag{3.10}$$

That much is clear to observe $(C^1(\mathbb{R}^n), d_1)$ is a complete generalized metric space.

Now, consider the operator $A : x \to x$ defined by

$$Au(x) = f(x) + \int_0^x k(x, s, u(s), u'(s))ds, \quad \forall x \in I. \tag{3.11}$$

We prove that A is strictly contractive on the space X. Let $v, u \in\in C^1$ and let $C(v,u) \in [0,\infty]$ be an arbitrary constant such that $d_1(v,u) \leq C(v,u)$.
Then, by (3.11), we have

Let $u(x) \in X$ and define the operator A given by

$$Au = f(x) + \int_0^x k\left(x, s, u(s), u'(s)\right) ds. \tag{3.12}$$

Differentiating both sides of (3.6) with respect to x we get

$$A'u(x) = f(x) + k((x, x, u(x), u'(x)) + \int_0^x \frac{\partial}{\partial x} k\left(x, s, u(s), u'(s)\right) ds. \tag{3.13}$$

Now, we verify that the operator A is a contraction map. Let $u(x)$, $v(x) \in X$. from (3.6) and (3.13) and using the hypotheses we have for any $x \in I$, we verify that the operator A is a contraction map. Let $u(x), v(x) \in X$. From (3.6) and (3.13) and using the hypotheses we have

$$
\begin{aligned}
|Au(x) - Av(x)| &= |f(x) + \int_0^x k\left(x, s, u(s), u'(s)\right) ds - f(x) - \int_0^x k\left(x, s, v(s), v'(s)\right) ds| \\
&= | \int_0^x k\left(x, s, u(s), u'(s)\right) ds - \int_0^x k(x, s, v(s), v'(s)) ds| \\
&\leq \int_0^x |k\left(x, s, u(s), u'(s)\right) - k(x, s, v(s), v'(s))| ds \\
&\leq \int_0^x \gamma_1 |u(s) - v(s)| + \gamma_2 |u'(s) - v'(s)| ds \\
&\leq \int_0^x \max(\gamma_1 + \gamma_2) |u(s) - v(s)|_1 ds \\
&\leq \max(\gamma_1 + \gamma_2) \|u(x) - v(x)\|_1 \qquad (i)
\end{aligned}
$$

and

$$
\begin{aligned}
|A'u(x) - A'v(x)| &= |f(x) + k(x, x, u(x), u'(x)) + \int_0^x \frac{\partial}{\partial x} k\left(x, s, u(s), u'(s)\right) ds - f(x) \\
&- k((x, x, v(x), v'(x)) - \int_0^x \frac{\partial}{\partial x} k\left(x, s, v(s), v'(s)\right) ds| \\
&= |k((x, x, u(x), u'(x)) + \int_0^x \frac{\partial}{\partial x} k\left(x, s, u(s), u'(s)\right) ds - k((x, x, v(x), v'(x)) \\
&- \int_0^x \frac{\partial}{\partial x} k\left(x, s, v(s), v'(s)\right) ds| \\
&\leq |k((x, x, u(x), u'(x)) - k((x, x, v(x), v'(x))| + \int_0^x |\frac{\partial}{\partial x} k\left(x, s, u(s), u'(s)\right) \\
&- \frac{\partial}{\partial x} k\left(x, s, v(s), v'(s)\right) | ds \\
&\leq \gamma_1 |u(x) - v(x)| + \gamma_2 |u'(x) - v'(x)| + \int_0^x \beta_1 |u(x) - v(x)| \\
&+ \beta_2 |u'(x) - v'(x)| ds \\
&\leq \max(\gamma_1 + \gamma_2) |u(x) - v(x)|_1 + \int_0^x max(\beta_1 + \beta_2) |u(x) - v(x)|_1 ds \\
&\leq \max(\gamma_1 + \gamma_2) \|u(x) - v(x)\|_1 + \max(\beta_1 + \beta_2) \|u(x) - v(x)\|_1 ds
\end{aligned}
$$

From (i) and(ii) we have

$$
\|Au(x) - Av(x)\|_1 \leq [2max(\gamma_1, \gamma_2) + max(\beta_1, \beta_2)] \|u(x) - v(x)\|_1,
$$

we suppose that $[2max(\gamma_1, \gamma_2) + max(\beta_1, \beta_2)] < 1$. As a result, the Banach Fixed Point Theorem once more assures that ther existe one and only solution $u_0(x) \in X$.
Let u be any arbitrary element in $C^1(\mathbb{R}^n)$. By continuity of the mappings $v,\ u$ and A , τ

on the compact set I, there exists a constant $\tau \in \mathbb{R}^*$ such that

$$\|Au(x) - u(x)\|_1 = \|f(x) + \int_0^x k(x, s, u(s), u'(s))ds - u(x)\|_1 \leq \tau, \tag{3.14}$$

for all $x \in I$.

We conclude that

$$d_1(u, Au) < +\infty, \quad \forall u \in X. \tag{3.15}$$

Let $v_0 \in X$ be given, then by virtue Banach fixed point, there exists a continuous function u_0 in X such that $Au_0 = u_0$, that is u_0 is a solution to the equation (3.1).

We observe that d is actually a metric. Therefore, $u_0 : I \to X$ is the unique continuous function such that

$$u_0(x) = f(x) + \int_0^x k\left(x, s, u_0(s), u_0'(s)\right) s, \quad \forall x \in I. \tag{3.16}$$

By assumption (3.14), we deduce that $d_1(v, Av) \leq \tau$, thus by virtue of (iv) of Theorem (3.1) , we get the following estimate

$$d_1\left(v, u_0\right) \leq \frac{\tau}{1 - L}, \tag{3.17}$$

which implies that

$$\|v(x) - u_0(x)\|_1 \leq \frac{\tau}{1 - L}. \tag{3.18}$$

Also, by (ii) of Theorem (3.1), the sequence of iterates $\{A^n v\}$ converges to u_0 in the metric space (X, d_1). The proof is finished. ■

3.2 Hyers-Ulam-Rassias stability of Volterra integro–differential equations

Theorem 3.3 *Let set* $I := [0, 1]$. *Let* X *be a Banach space over the (real or complex). Let* L *be positive constants with* $0 < \delta L < 1$. *Let* $\tau : I \to (0, \infty)$ *be a continuous function such that*

$$\int_0^x \tau(s)ds \leq \delta\tau(x), \quad \forall x \in [0, 1]. \tag{3.19}$$

Assume that $k : I \times I \times \mathbb{R} \times \mathbb{R} \to X$ *is continuous and it is continuously differentiable with respect to* x *which satisfies Lipschitz condition in theorem (3.2). Suppose that a continuous function* $v(x)$ *for* $x \in \mathbb{R}$ *that is continuously differentiable with respect to* x*, satisfies:*

$$\|v(x) - f(x) - \int_0^x k(x, s, v(s), v'(s))ds\|_1 \leq \tau(x), \quad \forall x \in I. \tag{3.20}$$

Then a unique continuous function $u_0 : I \to X$ that is continuously differentiable with respect to x exists such that

$$u_0(x) = f(x) + \int_0^x k\left(x, s, u_0(s), u_0'(s)\right) ds, \quad \forall x \in I, \tag{3.21}$$

consequently, u_0 is a solution to the equation (3.1) and

$$\|v(x) - u_0(x)\|_1 \leq \frac{1}{1 - \delta L}\tau(x), \tag{3.22}$$

for all $x \in [0, 1]$.

Proof. We consider the set $X = C^1(\mathbb{R}^n)$ of all continuous functions from I to X. For $v, u \in X$, we set

$$d_1(v, u) := \inf\{\delta \in [0, \infty] : |v(x) - u(x)|_1 \leq \delta\tau(x), \forall x \in I\}, \tag{3.23}$$

the generalized metric space $(C^1(\mathbb{R}^n), d_1)$ is easily observable. Also, it is clear that $(C^1(\mathbb{R}^n), d_1)$ is complete.

Now, we define the operator $A : X \to X$

$$(Av)(x) = (x) + \int_0^x k(x, s, v(x), v'(x))ds, \quad \forall x \in I. \tag{3.24}$$

We prove that A is strictly contractive on the space X. Let $v, u \in X$ and let

$\delta(v, u) \in [0, \infty]$ be an aribitrary constant such that $d_1(v, u) \leq \delta(v, u)$. Then, by(3.23), we have

$$\|v(x) - u(x)\|_1 \leq \delta(v, u)\tau(x), \forall x \in I. \tag{3.25}$$

For any $x \in I$, we have Now, we verify that the operator A is a contraction map. Let

$u(x), v(x) \in X$.From (3.6) and (3.13) and using the hypotheses we have

$$\begin{aligned}
|Au(x) - Av(x)| &= |f(x) + \int_0^x k\,(x, s, u(s), u'(s))\,ds - f(x) - \int_0^x k\,(x, s, v(s), v'(s))\,ds| \\
&= |\int_0^x k\,(x, s, u(s), u'(s))\,ds - \int_0^x k(x, s, v(s), v'(s))ds| \\
&\leq \int_0^x |k\,(x, s, u(s), u'(s)) - k(x, s, v(s), v'(s))|ds \\
&\leq \int_0^x \gamma_1|u(s) - v(s)| + \gamma_2|u'(s) - v'(s)|ds \\
&\leq \int_0^x \max(\gamma_1 + \gamma_2)|u(s) - v(s)|_1 ds \\
&\leq \max(\gamma_1 + \gamma_2)\|u(x) - v(x)\|_1 \qquad (i)
\end{aligned}$$

and

$$\begin{aligned}
|A'u(x) - A'v(x)| &= |f(x) + k(x, x, u(x), u'(x)) + \int_0^x \frac{\partial}{\partial x}k\,(x, s, u(s), u'(s))\,ds - f(x) \\
&- k((x, x, v(x), v'(x)) - \int_0^x \frac{\partial}{\partial x}k\,(x, s, v(s), v'(s))\,ds| \\
&= |k((x, x, u(x), u'(x)) + \int_0^x \frac{\partial}{\partial x}k\,(x, s, u(s), u'(s))\,ds - k((x, x, v(x), v'(x)) \\
&- \int_0^x \frac{\partial}{\partial x}k\,(x, s, v(s), v'(s))\,ds| \\
&\leq |k((x, x, u(x), u'(x)) - k((x, x, v(x), v'(x))| + \int_0^x |\frac{\partial}{\partial x}k\,(x, s, u(s), u'(s)) \\
&- \frac{\partial}{\partial x}k\,(x, s, v(s), v'(s))\,|ds \\
&\leq \gamma_1|u(x) - v(x)| + \gamma_2|u'(x) - v'(x)| + \int_0^x \beta_1|u(x) - v(x)| \\
&+ \beta_2|u'(x) - v'(x)|ds \\
&\leq \max(\gamma_1 + \gamma_2)|u(x) - v(x)|_1 + \int_0^x max(\beta_1 + \beta_2)|u(x) - v(x)|_1 ds \\
&\leq \max(\gamma_1 + \gamma_2)\|u(x) - v(x)\|_1 + \max(\beta_1 + \beta_2)\|u(x) - v(x)\|_1 ds,
\end{aligned}$$

from (i) and (ii) we have

$$\begin{aligned}
|Au(x) - Av(x)|_1 &\leq [2max(\gamma_1, \gamma_2) + max(\beta_1, \beta_2)]\|u(x) - v(x)\|_1 \\
&\leq [2max(\gamma_1, \gamma_2) + max(\beta_1, \beta_2)]\delta(v, u)\tau(x).
\end{aligned}$$

We suppose that $[2max(\gamma_1, \gamma_2)+max(\beta_1, \beta_2)] < 1$. As a result, the Banach Fixed Point Theorem once more assures that ther existe one and only solution $u(x)_0 \in X$.

for all $x \in I$ and $L = [2max(\gamma_1, \gamma_2) + max(\beta_1, \beta_2)]$. Hence we have $d_1(Av, Au) \leq L\tau(x)(v, u)$. We conclude that

$$d_1(Av, Au) \leq \delta L d_1(v, u), \quad \forall v, u \in C^1(\mathbb{R}^n). \tag{3.26}$$

Let u be any arbitrary element in $C^1(\mathbb{R}^n)$. Since $\tau(I) \subset (0, +\infty)$, then by continuity of the mappings v, u, Au and τ on the compact set I, there exists a finite constant $\delta \in (0, \infty)$ such that

$$\|Au(x) - u(x)\|_1 = \|f(x) + \int_0^x k(x, s, u(s), u'(x))ds - u(x)\|_1 \leq \delta\tau(x), \tag{3.27}$$

for all $x \in I$.

We deduce that $d_1(u, Au) < +\infty, \quad \forall u \in X$. Let $u_0 \in X$ be given, then by virtue of Theorem (3.1), there exists a continuous function v_0 in $C^1(\mathbb{R}^n)$ such that the sequence $\{A^n v_0\}$ converges to u_0 and $Au_0 = u_0$, that is u_0 is a solution to the equation (3.1).

Since $\min\{\tau(x) : x \in I\} > 0$, then d_1 is actually a metric. Therefore, $u_0 : I \to X$ is the unique continuous function such that

$$u_0(x) = f(x) + \int_0^x k\left(x, s, u_0(s), u_0'(s)\right) ds, \quad \forall x \in I. \tag{3.28}$$

By assumption (3.20), we know that $d_1(u, Au) \leq 1$, thus by virtue of (iv) of Theorem (3.1), we get the following estimate

$$d_1\left(v, u_0\right) \leq \frac{1}{1 - \delta L}\tau(x),$$

which implies that

$$\|v(x) - u_0(x)\|_1 \leq \frac{1}{1 - \delta L}\tau(x). \tag{3.29}$$

■

3.3 Ulam-Hyers stability and Ulam-Hyers-Rassias stability with variant Lipchtiz conditions

In this section we are going to study the Ulam-Hyers stability and the Ulam-Hyers-Rassias stability with variant Lipchtiz conditions, how was treated in B. G Pachpatte's paper [4] to study the existence, uniqueness and the continuity of Fredholm equation

3.3.1 Existence and uniqueness with variant Lipchtiz conditions

Theorem 3.4 *Consider that:*
(i) the function k in equation (3.1) and its derivative with respect to x achieve the conditions

$$\begin{aligned} |f(x,s,u,v)-f(x,s,u_1,v_1)| &\leq h_1(x,s)[|u-u_1|+|v-v_1|] \\ \left|\frac{\partial}{\partial x}f(x,s,u,v)-\frac{\partial}{\partial x}f(x,s,u_1,v_1)\right| &\leq h_2(x,s)[|u-u_1|+|v-v_1|] \end{aligned} \tag{3.30}$$

where for $i=1,2$ and $a\leq s\leq x<\infty, h_i(x,s)\in C(I^2)$.

(ii)Non-negative constants exist: a_1,a_2 such that $a_1+a_2<1$ and

$$\int_a^x h_1(x,s)ds \leq a_1 \tag{3.31}$$

$$h_1(x,x)+\int_a^x h_2(x,s)ds \leq a_2, \tag{3.32}$$

for $x\in I$.
(iii) there exist nonnegative constants U_1,U_2 such that

$$|f(x)|+\int_\alpha^x |f(x,s,0,0)|ds \leq U_1, \tag{3.33}$$

$$|f'(x)|+|f(x,s,0,0)|+\int_\alpha^x \left|\frac{\partial}{\partial x}k(x,s,0,0)\right| ds \leq U_2, \tag{3.34}$$

where f,k are defined in equation (3.1). Then equation (3.1) has a unique solution $u(x)$ in X.

Proof. Let $u(x)\in X$ and define the operator

$$(Au)(x)=f(x)+\int_\alpha^x k\left(x,s,u(s),u'(s)\right)ds, \tag{3.35}$$

differentiating both sides of (3.35) with respect to x we get

$$(Au)'(x)=f'(x)+k\left(x,x,u(x),u'(x)\right)+\int_\alpha^x \frac{\partial}{\partial x}k\left(x,s,u(s),u'(s)\right)ds. \tag{3.36}$$

Now, we show that Au maps X into itself. Evidently, $(Au),(Au)'$ are continuous on I and $(Au),(Au)'\in X$. We verify that (3.1) is fulfilled. From (3.35), (3.36) and using the hypotheses and (3.35) we have:

$$\begin{aligned} |(Au)(x)| &\leq |f(x)|+\int_\alpha^x \left|k\left(x,s,x(s),u'(s)\right)-k(x,s,0,0)+fk(x,s,0,0)\right|ds \\ &\leq |f(x)|+\int_\alpha^x |k(x,s,0,0)|ds+\int_\alpha^x h_1(x,s)|u(s)|_1 ds \\ &\leq U_1+\|u\|_1\int_\alpha^x h_1(x,s)ds \\ &\leq [U_1+a_1] \end{aligned} \tag{3.37}$$

and

$$
\begin{aligned}
|(Au)'(x)| \leq & |f'(u)| + |k(x,x,u(x),u'(x)) - k(x,x,0,0) + k(x,x,0,0)| \\
& + \int_\alpha^x \left| \frac{\partial}{\partial x} k(x,s,u(s),u'(s)) - \frac{\partial}{\partial x} u(x,s,0,0) + \frac{\partial}{\partial x} k(x,s,0,0) \right| ds \\
\leq & |f'(x)| + |k(x,x,0,0)| + \int_\alpha^x \left| \frac{\partial}{\partial x} k(x,s,0,0) \right| ds + h_1(x,x) \|u(x)\|_1 \\
& + \int_\alpha^x h_2(x,s) |u(s)|_1 ds \\
\leq & U_2 + \|u\|_1 h_1(x,x) + \|u\|_1 \int_\alpha^x h_2(x,s) ds \\
\leq & [U_2 + a_2] .
\end{aligned}
\tag{3.38}
$$

From (3.37) and (3.38) we get

$$
\|(A)(x)\|_1 \leq [U_1 + U_2 + (a_1 + a_2)] . \tag{3.39}
$$

From (3.39) it follows that $Au \in X$. This proves that A maps X into itself.

Now, we verify that the operator A is a contraction map. Let $u(x),\ v(x) \in X$. From (3.35) and (3.36) and using the hypotheses we have

$$
\begin{aligned}
\|(Au)(x) - (Av)(x)\|_1 \leq & \int_\alpha^x |k(x,s,u(s),u'(s)) - k(x,s,v(s),v'(s))| ds \\
\leq & \int_\alpha^x h_1(x,s) |u(s) - v(s)|_1 ds \\
\leq & \|u - v\|_1 \int_\alpha^x h_1(x,s) ds \\
\leq & \|u - v\|_1 a_1
\end{aligned}
\tag{3.40}
$$

and

$$
\begin{aligned}
|(Au)'(x) - (Av)'(x)| \leq & |k(x,x,u(x),u'(x)) - f(x,x,v(x),v'(x))| \\
& + \int_\alpha^x \left| \frac{\partial}{\partial x} k(x,s,x(s),x'(s)) - \frac{\partial}{\partial x} k(x,s,v(s),v'(s)) \right| ds \\
\leq & h_1(x,x) |u(x) - v(x)|_1 + \int_\alpha^x h_2(x,s) |u(s) - v(s)|_1 ds \\
\leq & \|u - v\|_1 h_1(x,x) + \|u - v\|_1 \int_\alpha^x h_2(x,s) ds \\
\leq & \|u - v\|_1 a_2 .
\end{aligned}
\tag{3.41}
$$

From (3.40) and (3.41) we get

$$
\|(Au)(x) - (Av)(x)\|_1 \leq \|x - y\|_1 (a_1 + a_2) . \tag{3.42}
$$

From (3.42) we obtain

$$\|Au - Av\|_1 \leq (a_1 + a_2)\,\|x - y\|_1. \tag{3.43}$$

Since $a_1 + a_2 < 1$, it follows from Banach fixed point theorem that A has a unique solution u_0 in X. The fixed point of A is however a solution of equation (3.1). The proof is complete([4]). ■

3.3.2 Hyers−Ulam−stability with variant Lipchtiz conditions

Theorem 3.5 *Let set $I := [0, 1]$. Let $X = C^1(\mathbb{R}^n)$, it's clear that $\mathbb{R}$ is Banach space over the (real or complex). Let $M = (a_1 + a_2)$ be a positive constant with $0 < M < 1$. Assume that $k : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and are continuously differentiable with respect to x, which fulfills the varinat Lipschitz condition mentioned in (3.30).*
Suppose that a continuous function $v : I \to X$, that satisfies

$$\|v(x) - f(x) - \int_0^x k(x, s, u(x), u'(x))ds\|_1 \leq \Psi, \quad \forall x \in I, \tag{3.44}$$

for some positive number Ψ. Then a unique continuous function $u_0 : I \to X$ exists such that

$$u_0(x) = f(x) + \int_0^x k\,(x, s, u_0(s), u_0'(s))\,ds, \quad \forall x \in I, \tag{3.45}$$

(As a result, u_0 is a solution to the equation (3.1)) and

$$\|v(x) - u_0(x)\|_1 \leq \frac{\Psi}{1 - M}, \tag{3.46}$$

for all $x \in [0, 1]$.

Proof. Let X represent the set of all continuous real-valued functions on I initially. Moreover, we construct a generalized metric on X by:

$$d_1(v, u) = \inf\{\zeta \in [0, \infty] : |v(x) - u(x)|_1 \leq \zeta\Psi, \forall x \in I\}. \tag{3.47}$$

It is clear to see that (X, d_1) is a complete generalized metric space now, we define the operator $A : X \to X$ by

$$Au(x) = f(x) + \int_0^x k(x, s, u(s), u'(s))ds, \quad \forall x \in I, \tag{3.48}$$

for all u $\in X$.

Now, should we prove the cntraction of the operator A how was proved before in theorem(3.4) by Pachpat (*in*[4]).

We conclude that

$$d_1(Av, Au) \leq M\zeta\Psi(v, u) \qquad \forall\ u,\ v \in X. \tag{3.49}$$

For all $u \in X$. Let v_0 be any arbitrary element in X.

$$\|Av_0 - v_0\|_1 = \|f(x) + \int_0^x k(x, s, v(s), v'(s))ds - v_0\|_1 \leq \Psi. \tag{3.50}$$

Thus, (3.50) implies that $d(Av_0, v_0) \leq \infty$. Consequently, theorem (3.1) (i) implies that a continuous function u_0 exists. Such that $A_n v_0 \to u_0$ in (X, d_1) as $n \to \infty$, and such that $Au_0 = u_0$, that is, u_0 satisfies equation (3.45) for all $x \in I$. We conclude that u_0 given by (3.45), is the unique continuous function from theorem (3.1) (ii).

Lastly, theorem (3.1) (iii) and (3.49) imply that

$$d_1(u - u_0) \leq \frac{\Psi}{1 - M}, \tag{3.51}$$

which implies that

$$\|v(x) - u_0(x)\|_1 \leq \frac{1}{1 - M}\Psi. \tag{3.52}$$

the inequality (3.6) holds true for all $x \in I$ this case. ■

3.3.3 Ulam−Hyers−Rassias stability with variant Lipchtiz conditions

This section will demonstrate the Hyers-Ulam-Rassias stability of the nonlinear Volterra integro-differential equation given in our thesis with variant Lipchtiz condition.

Theorem 3.6 *Let set $I := [0, 1]$. Let $X = C^1(\mathbb{R}^n)$, it's clear that X is Banach space over the (real or complex) . Let $M = (a_1 + a_2)$ be a positive constant with $0 < \zeta M < 1$. Assume that $k : I \times I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function witch is continuously differentiable with respect to x , which fulfills the varinat Lipschitz condition mentioned in (3.30). Suppose that a continuous function $v : I \to X$ satisfies*

$$\|v(x) - f(x) - \int_0^x k(x, s, u(x), u'(x))ds\|_1 \leq \Psi(x), \quad \forall x \in I, \tag{3.53}$$

for a continuous function $\Psi(x)$, where $\Psi : I \to X$ is a continuous function with

$$\int_0^x \Psi(s)ds \leq \zeta\Psi(x), \quad \forall x \in [0, 1]. \tag{3.54}$$

For each $x \in I$, then there exists a unique continuous function $u_0 : I \to x$ such that

$$u_0(x) = f(x) + \int_0^x k\left(x, s, u_0(s), u_0'(s)\right) ds, \quad \forall x \in I. \tag{3.55}$$

Proof. Let X denote the set of functions once derivable I. For $u, v \in X$ we set

$$d_1(v,u) = \inf\{\zeta) \in [0,\infty[: v(x) - u(x)|_1 \leq \zeta\Psi(x),\ \forall x \in I\}. \tag{3.56}$$

It is clear that (X, d_1) represents a complete generalized metric space.

Now, consider the operator $A : X \to X$ defined by

$$Au(x) = f(x) + \int_0^x k(x,s,u(s),u'(s))ds, \quad \forall x \in I, \tag{3.57}$$

for all u $\in X$.

We have from [4] that the operator A is strictly contractive on X. that is, $d_1(Av, Au) \leq \zeta M\Psi(x)(u,v)$. Hence, we can conclude that $d_1(Au, Av) \leq \zeta M d_1(u,v)$ for any $u,\ v \in X$, where we note that $0 < \zeta M < 1$. It follows from (3.55) that for arbitrary $v_0 \in X$, there exists a continuous function $\Psi(x)$ with

$$\|v(x) - f(x) - \int_0^x k(x,s,v_0(x),v_0'(x))ds\|_1 \leq \Psi(x), \quad \forall x \in I, \tag{3.58}$$

Thus, (3.56) implies that

$$d_1(Av, u_0) \leq \infty. \tag{3.59}$$

As a result, Theorem (3.1)(i) states that, there exists a continuous function $u_0 : I \to X$ such that $A_n v_0 \to u_0$ in (X, d_1) and $Au_0 = u_0$, that is, u_0 satisfies equation (3.55) for every $x \in I$. From Theorem 3.1 (ii), we deduce that u_0, given by (3.55), is the unique continuous function. From (3.56), we have

$$d_1(v, u_0) \leq \frac{1}{1-\zeta M}\Psi(x),$$

which implies that

$$\|v(x) - u_0(x)\|_1 \leq \frac{1}{1-\zeta M}\Psi(x). \tag{3.60}$$

We have investigated at the Hyers-Ulam-Rassias stability of the Volterra integro-differential equation (3.1), which is based on Theorem 3.6. ■

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