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Semi-Hurewicz spaces

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Abstract

In this paper we study some covering properties in topological spaces by using semi-open covers. We introduce and investigate the properties of s-Hurewicz and almost s-Hurewicz spaces and their star versions.

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1. Introduction

The properties of Menger and Hurewicz, which are the basic and oldest selection principles, take their origin in papers [12] and [6]. Both of them appeared as counterparts of σ-compactness. A topological space X has the *Menger* (resp. *Hurewicz*) *property*, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that every $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and the family $\bigcup\{V : V \in \mathcal{V}_n, n \in \mathbb{N}\}$ is a cover of X (resp. each $x \in X$ belongs to $\bigcup \mathcal{V}_n = \bigcup\{V : V \in \mathcal{V}_n\}$ for all but finitely many n). Clearly, every σ-compact space X has the Hurewicz property and every Hurewicz space has the Menger property. Every Menger space is Lindelöf. As a generalization of Hurewicz spaces,

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the authors of [18] defined a topological space X to be *almost Hurewicz* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \bigcup\{\mathrm{Cl}(V) : V \in \mathcal{V}_n\}$ for all but finitely many n. Clearly, the Hurewicz property implies the almost Hurewicz property. The authors [18] showed that every regular almost Hurewicz space is Hurewicz and gave an example that there exists a Urysohn almost Hurewicz space that is not Hurewicz. On the study of Hurewicz and almost Hurewicz spaces (and other weaker versions) the readers can see the references [6, 7, 9, 16, 17, 18].

In 1963, N. Levine [11] defined semi-open sets in topological spaces. A set A in a topological space X is *semi-open* if and only if there exists an open set $O \subset X$ such that $O \subset A \subset \mathrm{Cl}(O)$, where $\mathrm{Cl}(O)$ denotes the closure of the set O. If A is semi-open, then its complement is called *semi-closed* [2]. The collection of all semi-open subsets of X is denoted by $\mathsf{SO}(X)$. The union of any collection of semi-open sets is semi-open, while the intersection of two semi-open sets need not be semi-open. It happens if X is an extremally disconnected space [13]. The intersection of open and semi-open set is semi-open. According to [2], the semi-closure and semi-interior were defined analogously to the closure and interior: the *semi-interior* $\mathrm{sInt}(A)$ of a set $A \subset X$ is the union of all semi-open subsets of A; the *semi-closure* $\mathrm{sCl}(A)$ of $A \subset X$ is the intersection of all semi-closed sets containing A. A set A is semi-open if and only if $\mathrm{sInt}(A) = A$, and A is semi-closed if and only if $\mathrm{sCl}(A) = A$. Note that for any subset A of X

$$\mathrm{Int}(A) \subset \mathrm{sInt}(A) \subset A \subset \mathrm{sCl}(A) \subset \mathrm{Cl}(A).$$

The n-th power of a semi-open set in X is a semi-open set in X^n, whereas a semi-open set in X^n may not be written as a product of semi-open sets of X. A subset A of a topological space X is called a *semi-regular set* if it is semi-open as well as semi-closed or equivalently, $A = \mathrm{sCl}(\mathrm{sInt}(A))$ or $A = \mathrm{sInt}(\mathrm{sCl}(A))$.

A mapping $f : (X, \tau_X) \to (Y, \tau_Y)$ is called:

(1) *semi-continuous* if the preimage of every open set in Y is semi-open in X;
(2) *s-open* [1] if the image of every semi-open set in X is open in Y;
(3) *s-closed* if the image of every semi-closed set in X is closed in Y;
(4) *quasi-irresolute* if for every semi-regular set A in Y the set $f^{\leftarrow}(A)$ is semi-regular in X [4].

For more details on semi-open sets and semi-continuity, we refer to [2, 3, 11].

A space X is *semi-regular* if for each semi-closed set A and $x \notin A$ there exist disjoint semi-open sets U and V such that $x \in U$ and $A \subset V$ [5].

1.1. Lemma. ([5]) *The following are equivalent for a space X:*

(i): *X is a semi-regular space;*
(ii): *For each $x \in X$ and $U \in \mathsf{SO}(X)$ such that $x \in U$, there exists $V \in \mathsf{SO}(X)$ such that $x \in V \subset \mathrm{sCl}(V) \subset U$;*
(iii): *For each $x \in X$ and each $U \in \mathsf{SO}(X)$ with $x \in U$, there is a semi-regular set $V \subset X$ such that $x \in V \subset U$.*

The purpose of this paper is to investigate Hurewicz and almost Hurewicz spaces (and their star versions) and their topological properties using semi-open covers.

2. Preliminaries

A semi-open cover $\mathcal{U}$ of a space X is called;

- an *$s\omega$-cover* if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$;
- an *$s\gamma$-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$;
- *s-groupable* if it can be expressed as a countable union of finite, pairwise disjoint subfamilies $\mathcal{U}_n$, $n \in \mathbb{N}$, such that each $x \in X$ belongs to $\bigcup \mathcal{U}_n$ for all but finitely many n;
- *weakly s-groupable* if it is a countable union of finite, pairwise disjoint sets $\mathcal{U}_n$, $n \in \mathbb{N}$, such that for each finite set $F \subset X$ we have $F \subset \bigcup \mathcal{U}_n$ for some n.

For a topological space X we denote:

- $s\mathcal{O}$ the family of semi-open covers of X;
- $s\Omega$ the family of $s\omega$-covers of X;
- $s\mathcal{O}^{\mathrm{gp}}$ the family of s-groupable covers of X;
- $s\mathcal{O}^{\mathrm{wgp}}$ the family of weakly s-groupable covers of X.

For notation and terminology, we refer the reader to [10, 17].

Let A be a subset of X and $\mathcal{U}$ be a collection of subsets of X, then $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. We usually write $\mathrm{St}(x, \mathcal{U})$ for $\mathrm{St}(\{x\}, \mathcal{U})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be the sets whose elements are covers of a space X.

2.1. Definition. ([8]) $\mathsf{S}_{\mathsf{fin}}{}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}}\{\mathrm{St}(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an element of $\mathcal{B}$.

2.2. Definition. ([8]) $\mathsf{SS}_{\mathsf{fin}}{}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(K_n : n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(K_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

2.3. Definition. ([8]) $\mathsf{S}_1{}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and $\{\mathrm{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

2.4. Definition. ([8]) $\mathsf{U}_{\mathsf{fin}}{}^*(\mathcal{A}, \mathcal{B})$ denotes the selection hypothesis:

For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\{\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in \mathcal{B}$.

2.5. Definition. A space X is said to have:

- [15] *star s-Menger property* if it satisfies $\mathsf{S}_{\mathsf{fin}}{}^*(s\mathcal{O}, s\mathcal{O})$.
- [15] *star s-Rothberger property* if it satisfies $\mathsf{S}_1{}^*(s\mathcal{O}, s\mathcal{O})$.

For the definitions of star-Hurewicz and strongly star-Hurewicz spaces see [10].

3. Semi-Hurewicz and related spaces

3.1. Definition. Call a space X:

- *semi-Hurewicz* (or shortly *s-Hurewicz*) if it satisfies: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $s\mathcal{O}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$ for all but finitely many n, $x \in \bigcup \mathcal{V}_n$;
- *almost s-Hurewicz* if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\mathrm{sCl}(\bigcup \mathcal{V}_n)$ for all but finitely many n.

Evidently, we have

$$\text{Hurewicz} \Leftarrow s\text{–Hurewicz} \Rightarrow \text{almost } s\text{–Hurewicz}.$$

3.2. Example. (1) Every semi-compact space is s-Hurewicz. The converse is not true. The real line $\mathbb{R}$ with the cocountable topology is a T_1 semi-Hurewicz space which is not semi-compact.

(2) The Sorgenfrey line $\mathbb{S}$ and the space of irrationals with the usual metric topology are not semi-Hurewicz (because they are not Hurewicz, as it is well known).

3.3. Example. There is a Hurewicz space which is not s-Hurewicz,

The real line with the usual metric topology is a Hurewicz space. On the other hand, it is not an s-Hurewicz space, because from a sequence of covers whose elements are sets of the form $[a, b)$, $a, b \in \mathbb{R}$, one cannot choose finite subfamilies whose union covers $\mathbb{R}$. It follows from the fact that the sets $[a, b)$, $a, b \in \mathbb{R}$, form a base for the Sorgenfrey line $\mathbb{S}$ which is not a Hurewicz space.

3.4. Theorem. *Let X be a semi-regular space. If X is an almost s-Hurewicz space, then X is s-Hurewicz.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Since X is a semi-regular space, by Lemma 1.1, there exists for each n a semi-open cover $\mathcal{V}_n$ of X such that $\{\mathrm{sCl}(V) : V \in \mathcal{V}_n\}$ forms a refinement of $\mathcal{U}_n$. By assumption, applied to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, there exists a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ and each $x \in X$ belongs to $\bigcup\{\mathrm{sCl}(W) : W \in \mathcal{W}_n\}$. For every $n \in \mathbb{N}$ and every $W \in \mathcal{W}_n$ we can choose $U_W \in \mathcal{U}_n$ such that $\mathrm{sCl}(W) \subset U_W$. Let $\mathcal{U}'_n = \{U_W : W \in \mathcal{W}_n\}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$, $n \in \mathbb{N}$. It is easy to see that each $x \in X$ belongs to $\bigcup \mathcal{U}'_n$ all but finitely many n, which means that X is s-Hurewicz. □

3.5. Theorem. *A space X is almost s-Hurewicz if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-regular sets, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$ for all but finitely many $n \in \mathbb{N}$.*

Proof. Let X be an almost s-Hurewicz space and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-regular sets. Since every semi-regular set is semi-open, $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of X. By assumption, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$ for all but finitely many n.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Let $(\mathcal{U}'_n : n \in \mathbb{N})$ be the sequence defined by $\mathcal{U}'_n = \{\mathrm{sCl}(U) : U \in \mathcal{U}_n\}$. Then elements of each $\mathcal{U}'_n$ are semi-regular sets and thus, by assumption, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$ belongs to $\bigcup \mathcal{V}_n$ for all but finitely many n. For each $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ there exists $U_V \in \mathcal{U}_n$ such that $V = \mathrm{sCl}(U_V)$. Hence, $x \in \mathrm{sCl}(\bigcup\{U_V) : V \in \mathcal{V}_n\}$ for all but finitely many n. So X is an almost s-Hurewicz space. □

3.6. Theorem. *Every semi-regular subspace of an s-Hurewicz* (*almost s-Hurewicz*) *space is s-Hurewicz* (*almost s-Hurewicz*).

Proof. Because the proofs for both case are similar, we consider only the almost s-Hurewicz case. Let A be a semi-regular subset of an almost s-Hurewicz space X and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of A. Each semi-open subset of A is semi-open in X [14], so that $\mathcal{V}_n = \mathcal{U}_n \bigcup \{X \setminus A\}$ is a semi-open cover for X for every $n \in \mathbb{N}$. Since X is almost s-Hurewicz, there exist finite subsets $\mathcal{W}_n$ of $\mathcal{V}_n$, $n \in \mathbb{N}$, such that each $x \in X$ belongs to $\mathrm{sCl}(\bigcup \mathcal{W}_n)$ for all but finitely many $n \in \mathbb{N}$. By semi-regularity of A, $\mathrm{sCl}(X \setminus A) = X \setminus A$ and thus each $a \in A$, belongs to $\mathrm{sCl}(\bigcup(\mathcal{W}_n \setminus (X \setminus A)))$ for all but finitely many n, i.e. the sequence $(\mathcal{W}_n \setminus (X \setminus A) : n \in \mathbb{N})$ witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that A is almost s-Hurewicz. □

Now we consider preservation (in the image or preimage direction) of the properties we consider under some kinds of mappings.

The proof of the next theorem is easy, obtained by applying definitions, and thus is omitted.

3.7. Theorem. *Let $f : X \to Y$ be a semi-continuous surjection. If X is an s-Hurewicz space, then Y is a Hurewicz space.*

3.8. Corollary. *Let $f : X \to Y$ be a continuous surjection. If X is an s-Hurewicz space, then Y is a Hurewicz space.*

We define now the notion of (strong) θ-semi-continuity which is an important slight generalization of semi-continuity.

A mapping $f : X \to Y$ is *θ-semi-continuous* (resp. *strongly θ-semi-continuous*) if for each $x \in X$ and each semi-open set $V \subset Y$ containing $f(x)$ there is a semi-open set $U \subset X$ containing x such that $f(\mathrm{sCl}(U)) \subset \mathrm{sCl}(V)$ (resp. $f(\mathrm{sCl}(U)) \subset V$).

Evidently, each strongly θ-semi-continuous mapping is θ-semi-continuous.

Call a space X an *almost semi-γ-set* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of s-ω-covers of X there is a sequence $(U_n : n \in \mathbb{N})$ such that $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ and $\{U_n : n \in \mathbb{N}\}$ is an s-γ-cover of X.

3.9. Theorem. *A θ-semi-continuous image of an almost semi-γ-set is an almost semi-Hurewicz space.*

Proof. Let $f : X \to Y$ be a θ-semi-continuous mapping of a semi-γ-set X to a space Y. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of Y and $x \in X$. For each $n \in \mathbb{N}$ there is a set $V_{x,n} \in \mathcal{V}_n$ containing $f(x)$. Since f is θ-semi-continuous there is a semi-open set $U_{x,n} \subset X$ containing x such that $f(\mathrm{sCl}(U_{x,n})) \subset \mathrm{sCl}(V_{x,n})$. For each n let $\mathcal{U}_n$ be the set of all finite unions of sets $U_{x,n}$, $x \in X$. Evidently, each $\mathcal{U}_n$

is an s-ω-cover of X. As X is an almost semi-γ-set there is a sequence $(U_n : n \in \mathbb{N})$ such that for each n, $U_n \in \mathcal{U}_n$ and for each $x \in X$ the set $\{n \in \mathbb{N} : x \notin \mathrm{sCl}(U_n)\}$ is finite.

Let $U_n = U_{x_1,n} \bigcup U_{x_2,n} \bigcup \ldots \bigcup U_{x_{i_n},n}$. To each $U_{x_j,n}$, $j \leq i_n$, assign a set $V_{x_j,n} \in \mathcal{V}_n$ with $f(\mathrm{sCl}(U_{x_j,n})) \subset \mathrm{sCl}(V_{x_j,n})$. Let $y = f(x) \in Y$. Then from $x \in \mathrm{sCl}(U_n)$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$, we get $x \in \mathrm{sCl}(U_{x_p,n})$ for some $1 \leq p \leq i_n$ which implies $y \in f(\mathrm{sCl}(U_{x_p,n})) \subset \mathrm{sCl}(V_{x_p,n})$. If we put $\mathcal{W}_n = \bigcup\{V_{x_j,n} : j = 1, 2, \ldots, i_n\}$, we obtain the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite subsets of $\mathcal{V}_n$, $n \in \mathbb{N}$, such that each $y \in Y$ belongs to all but finitely many sets $\bigcup\{\mathrm{sCl}(W) : W \in \mathcal{W}_n\}$. This just means that Y is an almost semi-Hurewicz space. $\square$

3.10. Theorem. *A strongly θ-semi-continuous image Y of an almost semi-Hurewicz space X is a semi-Hurewicz space.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of Y. Let $x \in X$. For each $n \in \mathbb{N}$ there is a set $V_{x,n} \in \mathcal{V}_n$ containing $f(x)$. Since f is strongly θ-semi-continuous there is a semi-open set $U_{x,n} \subset X$ containing x such that $f(\mathrm{sCl}(U_{x,n})) \subset V_{x,n}$. Therefore, for each n the set $\mathcal{U}_n := \{U_{x,n} : x \in X\}$ is a semi-open cover of X. As X is almost semi-Hurewicz, there is a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{F}_n \subset \mathcal{U}_n$ and each $x \in X$ belongs to $\mathrm{sCl}\left(\bigcup \mathcal{F}_n\right)$ for all but finitely many n. To each $F \in \mathcal{F}_n$ assign a set $W_F \in \mathcal{V}_n$ with $f(\mathrm{sCl}(F)) \subset W_F$ and put $\mathcal{W}_n = \{W_F : F \in \mathcal{F}\}$. We obtain the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ of finite subsets of $\mathcal{V}_n$, $n \in \mathbb{N}$, which witnesses for $(\mathcal{V}_n : n \in \mathbb{N})$ that Y is a semi-Hurewicz space, as it is easily checked. $\square$

A mapping $f : X \to Y$ is called *contra-semi-continuous* if the preimage of each semi-open set in Y is semi-closed in X . f is said to be *pre-semi-continuous* if $f^{\leftarrow}(V) \subset \mathrm{sInt}(\mathrm{sCl}(f^{\leftarrow}(V)))$ whenever V is semi-open in Y.

3.11. Theorem. *A contra-semi-continuous, pre-semi-continuous image Y of an almost semi-Hurewicz space X is a semi-Hurewicz space.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of Y. Since f is contra-semi-continuous, for each $n \in \mathbb{N}$ and each $V \in \mathcal{V}_n$ the set $f^{\leftarrow}(V)$ is semi-closed in X. On the other hand, because f is pre-semi-continuous $f^{\leftarrow}(V) \subset \mathrm{sInt}(\mathrm{sCl}(f^{\leftarrow}(V)))$, so that $f^{\leftarrow}(V) \subset \mathrm{sInt}(f^{\leftarrow}(V))$, i.e. $f^{\leftarrow}(V) = \mathrm{sInt}(f^{\leftarrow}(V))$. Therefore, for each n, the set $\mathcal{U}_n = \{f^{\leftarrow}(V) : V \in \mathcal{V}_n\}$ is a cover of X by semi-open sets. As X is an almost semi-Hurewicz space there is a sequence $(\mathcal{G}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup\{\mathrm{sCl}(G) : G \in \mathcal{G}_n\}$. Then $\mathcal{W}_n = \{f(G) : G \in \mathcal{G}_n\}$ is a finite subset of $\mathcal{V}_n$ for each $n \in \mathbb{N}$ and each $z \in Y$ belongs to $\mathrm{sCl}(\bigcup \mathcal{W}_n)$ for all but finitely many n. This means that Y is a semi-Hurewicz space. $\square$

A mapping $f : X \to Y$ is called *weakly semi-continuous* if for each $x \in X$ and each semi-open neighbourhood V of $f(x)$ there is a semi-open neighbourhood U of x such that $f(U) \subset \mathrm{sCl}(V)$.

3.12. Theorem. *A weakly semi-continuous image Y of a semi-Hurewicz space X is an almost semi-Hurewicz space.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. Let $x \in X$. Then for each $n \in \mathbb{N}$ there is a $V \in \mathcal{V}_n$ such that $f(x) \in V$. Since f is weakly semi-continuous there is a semi-open set $U_{x,n}$ in X such that $x \in U_{x,n}$ and $f(U_{x,n}) \subset \mathrm{sCl}(V)$. The set $\mathcal{U}_n := \{U_{x,n} : x \in X\}$ is a semi-open cover of X. Apply the fact that X is a semi-Hurewicz space to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$ and find a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{F}_n \subset \mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup \mathcal{F}_n$ for all but finitely many n. To each n and each $U \in \mathcal{F}_n$ assign a set $V_U \in \mathcal{V}_n$ such that $f(U) \subset \mathrm{sCl}(V_U)$ and put $\mathcal{W}_n = \{V_U : U \in \mathcal{F}_n\}$. Then each $z \in Y$ belongs to $\mathrm{sCl}(\bigcup \mathcal{W}_n)$ for all but finitely many n, i.e. Y is an almost semi-Hurewicz space. $\square$

4. Star semi-Hurewicz property

4.1. Definition. Call a space X:

- *star s-Hurewicz* (shortly denoted SsH) if it satisfies: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathrm{s}\mathcal{O}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each $x \in X$ belongs to $\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n;
- *strongly star s-Hurewicz* (denoted SSsH) if it satisfies: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathrm{s}\mathcal{O}$ there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X, and each $x \in X$ belongs to $\mathrm{St}(A_n, \mathcal{U}_n)$ for all but finitely many n.

Recall that a space X is star semi-compact, denoted SsC, (star semi-Lindelöf, denoted SsL) if for each semi-open cover $\mathcal{U}$ of X there is a finite (countable) $\mathcal{V} \subset \mathcal{U}$ such that $\mathrm{St}(\bigcup \mathcal{V}, \mathcal{U}) = X$. X is strongly star semi-compact, shortly SSsC, (strongly star semi-Lindelöf, SSsL) if for each semi-open cover $\mathcal{U}$ of X there is a finite (countable) $A \subset X$ such that $\mathrm{St}(A, X) = X$.

Evidently, we have the following diagram:

$$
\begin{array}{ccccc}
\mathsf{SSsC} & \Rightarrow & \mathsf{SSsH} & \Rightarrow & \mathsf{SSsL} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\mathsf{SsC} & \Rightarrow & \mathsf{SsH} & \Rightarrow & \mathsf{SsL}
\end{array}
$$

Call a space X *σ-strongly star semi-compact* if it is union of countably many strongly star semi-compact spaces.

4.2. Theorem. *Every σ-strongly star semi-compact space is strongly star s-Hurewicz.*

Proof. Let X be σ-strongly star semi-compact space. Let $X = \bigcup_{n \in \mathbb{N}} X_n$, where each X_n is strongly star semi-compact. Suppose that $X_1 \supset X_2 \supset \ldots \supset X_n \supset \ldots$, because the union of finitely many strongly star semi-compact spaces is strongly star semi-compact. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. For each n let A_n be a finite subset of X_n such that $\mathrm{St}(A_n, \mathcal{U}_n) \supset X_n$. It follows that each point of X belongs to all but finitely many sets $\mathrm{St}(A_n, \mathcal{U}_n)$. That is, the sequence $(A_n : n \in \mathbb{N})$ shows that X is strongly star s-Hurewicz space. $\square$

4.3. Theorem. *Let X be an extremally disconnected space, X is star s-Hurewicz space if and only if X satisfies* $\mathsf{U_{fin}}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$.

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Since X is star s-Hurewicz space, there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and each $x \in X$ belongs to $\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$ for all but finitely many n. This implies that $\{\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\gamma$-cover of X. Since each countable $s\gamma$-cover is s-groupable, $\{\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\} \in s\mathcal{O}^{\mathrm{gp}}$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Let

$$\mathcal{H}_n = \bigwedge\nolimits_{i \le n} \mathcal{U}_i.$$

Then $(\mathcal{H}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of X since X is extremally disconnected. Use now $\mathsf{U}_{\mathsf{fin}}{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$ property of X. For each $\mathcal{H}_n$ and for each $n \in \mathbb{N}$ select a finite set $\mathcal{V}_n \subset \mathcal{H}_n$ such that the set $\{\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{H}_n) : n \in \mathbb{N}\}$ is an s-groupable cover of X. Let $n_1 < n_2 < ... < n_k < \dots$ be a sequence of natural numbers which witnesses this fact, i.e. for each $x \in X$, x belongs to $\bigcup\{\mathrm{St}(\bigcup \mathcal{V}_i, \mathcal{H}_i) : n_k < i \le n_{k+1}\}$ for all but finitely many k. Put

$$\begin{aligned} \mathcal{W}_n &= \bigcup_{i<n} \mathcal{V}_i, \text{ for } n < n_1; \\ \mathcal{W}_n &= \bigcup_{n_k < i \le n_{k+1}} \mathcal{V}_i, \text{ for } n_k \le n < n_{k+1}. \end{aligned}$$

Then the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ shows that X satisfies star s-Hurewicz property because each $x \in X$ belongs to all but finitely many $\mathrm{St}(\bigcup \mathcal{W}_n, \mathcal{U}_n)$. □

4.4. Definition. A space X is said to satisfy $\mathsf{SsH}_{\le \mathsf{n}}$ if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $s\mathcal{O}$ there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \in [\mathcal{U}_n]^{\le n}$, and the set $\{\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\gamma$-cover.

4.5. Theorem. *Let X be an extremally disconnected space satisfying $\mathsf{SsH}_{\le \mathsf{n}}$. Then X satisfies $\mathsf{S}_1{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. For each n define

$$\mathcal{V}_n = \bigwedge\{\mathcal{U}_i : (n-1)n/2 < i \le n(n+1)/2\}.$$

As X is extremally disconnected, each $\mathcal{V}_n$ is a semi-open cover of X. By applying $\mathsf{SsH}_{\le \mathsf{n}}$ to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, we can find a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{W}_n$ is a subset of $\mathcal{V}_n$ having $\le n$ elements, and $\{\mathrm{St}(\bigcup \mathcal{W}_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is an sγ-cover of X. Write $\mathcal{W}_n = \{W_i : (n-1)n/2 < i \le n(n+1)/2\}$, and each $W_i \in \mathcal{W}_n$ as the intersection of some elements from $\mathcal{U}_j$, $(n-1)n/2 < j \le n(n+1)/2$. For each W_i take also the set $U_j \in \mathcal{U}_j$ which is a term in the above representation of W_i. The set $\{\mathrm{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is a semi-open groupable cover of X. For, consider the sequence $n_1 < n_2 < \dots < n_k < \dots$ in $\mathbb{N}$, where $n_k = k(k-1)/2$. Then, as it is easily checked, for each $x \in X$ the fact $x \in \bigcup_{n_k < i \le n_{k+1}} \mathrm{St}(W_i, \mathcal{U}_i)$ for all but finitely many k, implies that the cover $\{\mathrm{St}(U_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is s-groupable, i.e. that X satisfies $\mathsf{S}_1{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$. □

In a similar way we prove the following theorem.

4.6. Theorem. *Let an extremally disconnected space X satisfies the condition* $\mathsf{SSsH}_{\leq \mathsf{n}}$*: For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X there is a sequence $(S_n : n \in \mathbb{N})$ of subsets of X such that for each n $|S_n| \leq n$ and $\{\mathrm{St}(S_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an s-γ cover of X. Then X satisfies* $\mathsf{SS}_1{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$.

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. For each n, as in the proof of the previous theorem, let

$$\mathcal{V}_n = \bigwedge\nolimits_{\frac{(n-1)n}{2} < i \leq \frac{n(n+1)}{2}} \mathcal{U}_t.$$

Apply now $\mathsf{SSsH}_{\leq \mathsf{n}}$ to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$, and find a sequence $(F_n : n \in \mathbb{N})$ of subsets of X such that for each n, $|F_n| \leq n$ and $\{\mathrm{St}(F_n, \mathcal{V}_n) : n \in \mathbb{N}\}$ is an sγ cover of X. For every $x \in X$ there exists positive integer n_0 such that $x \in \mathrm{St}(F_n, \mathcal{V}_n)$ for all $n > n_0$. Write for each n, $F_n = \{x_j : \frac{(n-1)n}{2} < j \leq \frac{n(n+1)}{2}\}$. The sequence $n_1 < n_2 < \ldots < n_k < \ldots$ of natural numbers defined by $n_k = \frac{k(k-1)}{2}$, witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that X satisfies $\mathsf{SS}_1{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$. Indeed, it is evident that each $x \in X$ belongs to $\bigcup_{n_k < j \leq n_{k+1}} \mathrm{St}(x_j, \mathcal{U}_j)$ for all but finitely many k. □

4.7. Theorem. *If a space X satisfies* $\mathsf{U}_{\mathsf{fin}}{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{wgp}})$*, then any open, semi-closed subset of X satisfies* $\mathsf{U}_{\mathsf{fin}}{}^*(s\mathcal{O}, s\Omega)$.

Proof. Let A be an open and semi-closed subset of X and let $(\mathcal{H}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of A. Because A is open, hence semi-open, for each n and each $H \in \mathcal{H}_n$ the set H is semi-open in X. Therefore, setting $\mathcal{S}_n = \mathcal{H}_n \bigcup (X \setminus A)$, we get a sequence $(\mathcal{S}_n : n \in \mathbb{N})$ of semi-open covers of X. Applying $\mathsf{U}_{\mathsf{fin}}{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{wgp}})$ for X we find a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{W}_n$ is a finite subset of $\mathcal{S}_n$ and $\{\mathrm{St}(\bigcup \mathcal{W}_n, \mathcal{S}_n) : n \in \mathbb{N}\}$ is an s-weakly groupable cover of X, i.e. there is a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of natural numbers such that for each finite set F in X one has

$$F \subset \bigcup \{\mathrm{St}(\cup \mathcal{W}_i, \mathcal{S}_i) : n_k < i \leq n_{k+1}\}$$

for some k. For each $n \in \mathbb{N}$ put $\mathcal{K}_n = \mathcal{W}_n \setminus \{X \setminus A\}$. Then each $\mathcal{K}_n$ is a finite subset of $\mathcal{H}_n$. Define now

$$\begin{aligned} \mathcal{G}_n &= \bigcup_{i<n} \mathcal{K}_i, \text{ for } n < n_1, \\ \mathcal{G}_n &= \bigcup_{n_k < i \leq n_{k+1}} \mathcal{K}_i, \text{ for } n_k < n \leq n_{k+1}. \end{aligned}$$

Each $\mathcal{G}_n$ is a finite subset of $\mathcal{H}_n$ and for each finite $E \subset A$ we have $E \subset \mathrm{St}(\cup \mathcal{G}_i, \mathcal{U}_i)$. Hence, A satisfies $\mathsf{U}_{\mathsf{fin}}{}^*(s\mathcal{O}, s\Omega)$. □

4.8. Theorem. *For an extremally disconnected space X the following are equivalent:*

(1) X has the strongly star-s-Hurewicz property;
(2) X satisfies $\mathsf{SS}_{\mathsf{fin}}{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$.

Proof. (1) $\Rightarrow$ (2): It is obvious because countable $s\gamma$-covers are s-groupable and $\mathsf{SS}_{\mathsf{fin}}{}^*$ is monotone in the second variable.

(2) $\Rightarrow$ (1): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Let for each n,

$$\mathcal{W}_n = \bigwedge\nolimits_{i \leq n} \mathcal{U}_i.$$

Each $\mathcal{W}_n$ is a semi-open cover of X. Apply $\mathsf{SS}_{\mathsf{fin}}{}^*(s\mathcal{O}, s\mathcal{O}^{\mathrm{gp}})$ to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$. We find a sequence $(B_n : n \in \mathbb{N})$ of finite subsets of X such that $\{\mathrm{St}(B_n, \mathcal{W}_n) : n \in \mathbb{N}\}$ is an s-groupable cover of X. Let $n_1 < n_2 < \cdots < n_k < \cdots$ be sequence of natural numbers such that for every y in X, we have

$$y \in \bigcup\nolimits_{n_k \leq n < n_{k+1}} \mathrm{St}(B_i, \mathcal{W}_i)$$

for all but finitely many $k \in \mathbb{N}$. For each n, let

$$\begin{aligned} \mathcal{S}_n &= \bigcup_{i<n_1} B_i, \text{ for } n < n_1; \\ \mathcal{S}_n &= \bigcup_{n_k \leq i \leq n_{k+1}} B_i, \text{ for } n_k \leq n < n_{k+1}. \end{aligned}$$

Each $\mathcal{S}_n$ is a finite subset of X. We claim that the set $\{\mathrm{St}(\mathcal{S}_n, \mathcal{W}_n) : n \in \mathbb{N}\}$ is an $s\gamma$-cover of X.

Let $x \in X$. There exist $t \in \mathbb{N}$ such that $x \in \bigcup_{n_k \leq n < n_{k+1}} B_i$ for all $k > t$. Since $\mathrm{St}(B_i, \mathcal{W}_i) \subset \mathrm{St}(\mathcal{S}_i, \mathcal{U}_i)$ for all i with $n_k \leq i < n_{k+1}$, we have that for each $k > t$, $x \in \mathrm{St}(\mathcal{S}_k, \mathcal{U}_k)$, that is $\{\mathrm{St}(\mathcal{S}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\gamma$-cover of X. $\square$

Another characterization of strongly star s-Hurewicz spaces is given in the next theorem.

4.9. Theorem. *A space X is a strongly star s-Hurewicz space if and only if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X there is a sequence $(S_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$, $\mathrm{St}(x, \mathcal{U}_n) \cap S_n \neq \emptyset$ for all but finitely many n.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. There exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to $\mathrm{St}(F_n, \mathcal{U}_n)$ for all but finitely many n. In other words, for each $x \in X$ there exists $n_0(x) \in \mathbb{N}$ such that $x \in \mathrm{St}(F_n, \mathcal{U}_n)$ for all $n > n_0$. $\mathrm{St}(F_n, \mathcal{U}_n)$ is the union of those elements of $\mathcal{U}_n$ which intersect F_n. $\mathrm{St}(\{x\}, \mathcal{U}_n)$ is the union of those elements of $\mathcal{U}_n$ which contains x. This implies $\mathrm{St}(\{x\}, \mathcal{U}_n) \cap F_n \neq \emptyset$ for all $n > n_0$.

Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Then, by assumption, there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every $x \in X$ there exists $n_0 \in \mathbb{N}$ such that $\mathrm{St}(\{x\}, \mathcal{U}_n) \cap A_n \neq \emptyset$ for all $n > n_0$. This implies $x \in \mathrm{St}(A_n, \mathcal{U}_n)$ for all but finitely many n. Therefore, $x \in \mathrm{St}(A_n, \mathcal{U}_n)$ for all but finitely many n, i.e. X is strongly star s-Hurewicz. $\square$

Now we consider preservation of (stronly) star s-Hurewicz property under usual topological operations.

4.10. Theorem. *A semi-open F_σ-subset of a strongly star s-Hurewicz space is strongly star s-Hurewicz.*

Proof. Let X be a strongly star s-Hurewicz space and let $A = \bigcup\{M_n : n \in \mathbb{N}\}$ be a semi-open F_σ-subset of X, where each M_n is closed in X for each $n \in \mathbb{N}$. Without loss of generality, we can assume that $M_n \subset M_{n+1}$ for each $n \in \mathbb{N}$. Now we show that A is strongly star s-Hurewicz space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of A. We need to find a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of A such that for each $a \in A$, $a \in \mathrm{St}(F_n, \mathcal{U}_n)$ for all but finitely many n. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \mathcal{U}_n \bigcup\{X \setminus M_n\}$. Then $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of semi-open covers of X. There exists a sequence $(F'_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, $x \in \mathrm{St}(F'_n, \mathcal{V}_n)$ for all but finitely many n, since X is a strongly star s-Hurewicz space. For each $n \in \mathbb{N}$, let $F_n = F'_n \cap A$. Thus $(F_n : n \in \mathbb{N})$ is a sequence of finite subsets of A. For every $a \in A$, there exists $k \in \mathbb{N}$ such that $a \in F_n$ and $a \in \mathrm{St}(F'_n, \mathcal{V}_n)$ for each $n > k$. Hence $a \in \mathrm{St}(F_n, \mathcal{U}_n)$ for $n > k$, which shows that A is strongly star s-Hurewicz. □

4.11. Theorem. *If each finite power of a space X is star s-Hurewicz, then X satisfies $\mathsf{U_{fin}}^*(s\mathcal{O}, s\Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Let $\mathbb{N} = N_1 \bigcup N_2 \bigcup \cdots$ be a partition of $\mathbb{N}$ into infinitely many infinite pairwise disjoint sets. For every $k \in \mathbb{N}$ and every $t \in N_k$ let $\mathcal{W}_t = \{U_1 \times U_2 \times \cdots \times U_k : U_1, \dots U_k \in \mathcal{U}_t\} = \mathcal{U}_t^k$. Then $(\mathcal{W}_t : t \in N_k)$ is a sequence of semi-open covers of X^k, and since X^k is a star s-Hurewicz space, we can choose a sequence $(\mathcal{H}_t : t \in N_k)$ such that for each t, $\mathcal{H}_t$ is a finite subset of $\mathcal{W}_t$ and $\bigcup_{t\in N_k}\{\mathrm{St}(H, \mathcal{W}_t) : H \in \mathcal{H}_t\}$ is a semi-open cover of X^k. For every $t \in N_k$ and every $H \in \mathcal{H}_t$ we have $H = U_1(H) \times U_2(H) \times \cdots \times U_k(H)$, where $U_i(H) \in \mathcal{U}_t$ for every $i \leq k$. Set $\mathcal{V}_t = \{U_i(H) : i \leq k, H \in \mathcal{H}_t\}$. Then for each $t \in N_k$, $\mathcal{V}_t$ is a finite subset of $\mathcal{U}_t$.

We claim that $\{St(\bigcup \mathcal{V}_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\omega$-cover of X. Let $F = \{x_1, ..., x_p\}$ be a finite subset of X. Then $y = (x_1, ..., x_p) \in X^p$ so that there is an $n \in N_p$ such that $y \in \mathrm{St}(H, \mathcal{W}_n)$ for some $H \in \mathcal{H}_n$. But $H = U_1(H) \times U_2(H) \times \cdots \times U_p(H)$, where $U_1(H), U_2(H), \dots, U_p(H) \in \mathcal{V}_n$. The point y belongs to some $W \in \mathcal{W}_n$ of the form $V_1 \times V_2 \times \cdots \times V_p$, $V_i \in \mathcal{U}_n$ for each $i \leq p$, which meets $U_1(H) \times U_2(H) \times \cdots \times U_p(H)$. This implies that for each $i \leq p$, we have $x_i \in \mathrm{St}(U_i(H), \mathcal{U}_n) \subset \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$, that is, $F \subset \mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n)$. Hence, X satisfies $\mathsf{U_{fin}}^*(s\mathcal{O}, s\Omega)$. □

In a similar way one proves the following theorem.

4.12. Theorem. *If all finite powers of a space X are strongly star s-Hurewicz, then X satisfies $\mathsf{SS_{fin}}^*(s\mathcal{O}, s\Omega)$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of X by semi-open sets. Let $\mathbb{N} = N_1 \bigcup N_2 \bigcup ...$ be a partition of $\mathbb{N}$ into infinite pairwise disjoint sets. For every $k \in \mathbb{N}$ and every $t \in N_k$ let $\mathcal{W}_t = \mathcal{U}_t^k$. Then $(\mathcal{W}_t : t \in N_k)$ is a sequence of semi open covers of X^k. Applying strongly star s-Hurewicz property of X^k we can get a sequence $(V_t : t \in N_k)$ of finite subsets of X^k such that each $x \in X^k$ belongs to $\mathrm{St}(V_t, \mathcal{W}_t) : t \in N_k$, for all but finitely many t. For each t consider A_t a finite subset of X such that $V_t \subset A_t^k$.

We show that $\{\mathrm{St}(A_n, \mathcal{U}_n) : n \in \mathbb{N}\}$ is an $s\omega$-cover of X. Let $F = \{x_1, ..., x_p\}$ be a finite subset of X. Then $(x_1, ..., x_p) \in X^p$ such that there is $n \in \mathbb{N}_p$ and $(x_1, ..., x_p) \in \mathrm{St}(V_n, \mathcal{W}_n) \subset \mathrm{St}(A_n^p, \mathcal{W}_n)$. Consequently, $F \subset \mathrm{St}(A_n, \mathcal{U}_n)$. □

The following two theorems give relations between strongly star s-Hurewicz spaces and s-Hurewicz and s-Lindelöf spaces.

A space X is called *meta semi-compact* if every semi-open cover $\mathcal{U}$ of X has a point-finite semi-open refinement $\mathcal{V}$ (that is, every point of X belongs to at most finitely many members of $\mathcal{V}$).

4.13. Theorem. *Every strongly star s-Hurewicz meta semi-compact space is s-Hurewicz space.*

Proof. Let X be a strongly star s-Hurewicz meta semi-compact space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be a point-finite semi-open refinement of $\mathcal{U}_n$. Since X is strongly star s-Hurewicz, there is a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that each $x \in X$ belongs to $\mathrm{St}(F_n, \mathcal{V}_n)$ for all but finitely many n.

Since $\mathcal{V}_n$ is a point-finite refinement and each F_n is finite, elements of each F_n belong to finitely many members of $\mathcal{V}_n$ say $V_{n_1}, V_{n_2}, V_{n_3}, \ldots, V_{n_k}$. Let $\mathcal{V}'_n = \{V_{n_1}, V_{n_2}, V_{n_3}, \ldots, V_{n_k}\}$. Then $\mathrm{St}(F_n, \mathcal{V}_n) = \bigcup \mathcal{V}'_n$ for each $n \in \mathbb{N}$. We have that each $x \in X$ belongs to $\bigcup \mathcal{V}'_n$ for all but finitely many n. For every $V \in \mathcal{V}'_n$ choose $U_V \in \mathcal{U}_n$ such that $V \subset U_V$. Then, for every n, $\{U_V : V \in \mathcal{V}'_n\} = \mathcal{W}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\bigcup \mathcal{W}_n$ for all but finitely many n, that is X is an s-Hurewicz space. □

4.14. Definition. ([15]) A space X is said to be *meta semi-Lindelöf* if every semi-open cover $\mathcal{U}$ of X has a point-countable semi-open refinement $\mathcal{V}$.

4.15. Theorem. *Every strongly star s-Hurewicz meta semi-Lindelöf space is a semi-Lindelöf space.*

Proof. Let X be a strongly star s-Hurewicz meta semi-Lindelof space. Let $\mathcal{U}$ be a semi-open cover of X then there exists $\mathcal{V}$, a point-countable semi-open refinement of $\mathcal{U}$. Let $\mathcal{V}_n = \mathcal{V}$ for each $n \in \mathbb{N}$. Since X is strongly star s-Hurewicz, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$, $x \in \mathrm{St}(F_n, \mathcal{V}_n)$ for all but finitely many n.

For every $n \in \mathbb{N}$ denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ which intersect F_n. Since $\mathcal{V}$ is point-countable and F_n is finite, $\mathcal{W}_n$ is countable. So, the set $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a countable subset of $\mathcal{V}$ and is a cover of X. For every $W \in \mathcal{W}$ pick a member $U_W \in \mathcal{U}$ such that $W \subset U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$. Hence, X is a semi-Lindelöf space. □

We end this section with few observations on almost star s-Hurewicz spaces.

4.16. Definition. Call a space X *almost star s-Hurewicz* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of semi-open covers of X there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n))$ for all but finitely many n.

4.17. Theorem. *A space X is an almost star s-Hurewicz space if and only if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of covers of X by semi-regular sets there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$ belongs to $\mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n))$ for all but finitely many $n \in \mathbb{N}$.*

Proof. ($\Longrightarrow$) Since every semi-regular set is semi-open, it is obvious.

($\Longleftarrow$) Conversely, let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of X. Let $\mathcal{U}'_n = \{\mathrm{sCl}(U) : U \in \mathcal{U}_n\}$. Then $\mathcal{U}'_n$ is a cover of X by semi-regular sets. By assumption there exists a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$ there is $n_0(x) \in \mathbb{N}$ such that $x \in \mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}'_n))$ for all $n \geq n_0(x)$.

For each $V \in \mathcal{V}_n$ we can find $U_V \in \mathcal{U}_n$ such that $V = \mathrm{sCl}(U_V)$. Let $\mathcal{V}'_n = \{U_V : V \in \mathcal{V}_n\}$. It is easy to see now that x belongs to $\mathrm{sCl}(\bigcup \mathcal{V}'_n, \mathcal{U}_n)$ for all $n \geq n_0(x)$. □

4.18. Theorem. *A quasi-irresolute image of an almost star s-Hurewicz space is an almost star s-Hurewicz space.*

Proof. Let X be an almost star s-Hurewicz space and Y be a topological space. Let $f : X \to Y$ be a quasi-irresolute surjection and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of covers of Y by semi-regular sets. Let $\mathcal{U}'_n = \{f^{\leftarrow}(U) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}'_n$ is a cover of X by semi-regular sets since f is quasi-irresolute. Since X is an almost star s-Hurewicz space, there exists a sequence $(\mathcal{V}'_n : n \in \mathbb{N})$ such that for every $n \in \mathbb{N}$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$ belongs to $\mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}'_n, \mathcal{U}'_n))$ for all but finitely many n.

It is not hard to verify that setting $\mathcal{V}_n = \{f(V) : V \in \mathcal{V}'_n\}$, each $y \in Y$ belongs to all but finitely many sets $\mathrm{sCl}(\mathrm{St}(\bigcup \mathcal{V}_n, \mathcal{U}_n))$ which means that Y is an almost s-Hurewicz space. □

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