The First Integral Approach in Stability Problem of Large Scale Nonlinear Dynamical Systems

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***Abstract*—**In analyzing large scale nonlinear dynamical systems, it is often desirable to treat the overall system as a collection of interconnected subsystems. Solutions properties of the large scale system are then deduced from the solution properties of the individual subsystems and the nature of the interconnections. In this paper a new approach is proposed for the stability analysis of large scale systems, which is based upon the concept of vector Lyapunov functions and the decomposition methods. The present results make use of graph theoretic decomposition techniques in which the overall system is partitioned into a hierarchy of strongly connected components. We show then, that under very reasonable assumptions, the overall system is stable once the strongly connected subsystems are stables. Finally an example is given to illustrate the constructive methodology proposed.

***Keywords*—**Comparison principle, First integral, Large scale system, Lyapunov stability.

I. Introduction

MOST of the present day stability studies are done by simulation on an analogue or digital computer. In this way, the nonlinear differential equations of the system, for a given initial operating condition are integrated numerically. However, to find the boundary of the stability region for a large scale nonlinear system, the simulation method is slow and expensive.

The Lyapunov function method appeared one of the most powerful methods for stability studies of large scale nonlinear systems [10], [14], [30]. However, this method did not seem suitable, owing to the continuous increase in size and complexity of composite systems under study; in particular and when the problem of stability domain estimate of the composite system is attacked. But, despite its elegance, it is still in general impossible to find it for composite system because of no universal and systematic procedure available to tell us how to find the required Lyapunov function. Consequently, the problem of finding the necessary and sufficient conditions for the stability of nonlinear system is a formidable one and as yet an unsolved problem. In particular, finding a Lyapunov function for a system under investigation formidable one and as yet an unsolved problem. In particular, finding a Lyapunov function for a system under investigation is a nontrivial task. It is often the case that such a function can only be found using a method of systemized trial and error.

Attempts to overcome the drawbacks of the Lyapunov approach have led to the decomposition-aggregation method, which is based on Bellman's concept of vector Lyapunov function [4]. Application of the method to a large scale nonlinear system is carried out by decomposing the composite system into a number of subsystems. It is well know from the literature, a large scale system can be modelled as an interlocking set of lumped subsystems or elements [1], [3]. Certain of the subsystems can be detached for analysis from the whole complex by inspection, but for systems composed of many elements, one of the first and most difficult tasks is to break the system down into manageable subsystems. By a proper ordering of the subsystems the dimensionality of the original model can be significantly reduced [7], [16]. This task can be made less subjective and more systematic by suitable algorithm for decomposition. In this paper we follow a graph theoretic approach to develop a decomposition tool which exploits the structure of the directed graphs associated with the nonlinear dynamic systems [5], [9]. In the author's view, however, the structure of the directed graph associated with the dynamic system under consideration is the most crucial of all factors which may contribute for the overall complexity of a large scale nonlinear system since it is this structure which determines whether a system of equations must be treated as a whole or as a number of autonomous subsystems with a given precedence ordering. To do so, we focus in graph theory decomposition based on identifying the strongly connected components (scc) [20, 21]. We carry out an idea explicitly stated by Kevorkian: The stability analysis of a large dynamic system must begin by the exploitation of its digraph [19]. The algorithmic extraction of strongly connected components of a digraph was done by many authors; however the algorithm of Tarjan [31] is apparently more efficient using depth-first search.

The impetus behind this decomposition is to assemble subsystems out of the overall system in some scheme so that each subsystem can be treated independently. The stability of such systems can often then be accomplished in terms of the subsystems and in terms of the interconnecting structure of such composite system [13], [15]. This method may be advantageous to adopt an approach in which the composite system is decomposed into appropriately interconnected subsystems. In this way, difficulties which usually arise in the qualitative analysis of complex dynamical systems may frequently be circumvented.

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II. DECOMPOSITION

In this subsection, we discuss the technique for decomposing a given large scale system into an interconnection of several simpler subsystems. Most of the currently known results concerning the stability of large scale systems either in the Lyapunov sense or the input-output sense assume that the system at hand has already been decomposed into a form suitable for applying the theory therein. The purpose here is to tackle this issue.

In what follows, we describe the decomposition technique. In the first, the system equations are rearranged into a lower triangular form by renumbering and regrouping the systems variables as needed. This technique is very powerful since it enables one to infer the stability properties of the overall system by examining the so-called strongly connected components (SCC) alone.

This technique was applied to Lyapunov stability by Michel [18]. It requires some background of the theory of directed graphs, or digraphs, and this given next.

A directed graph (or digraph) is an ordered pair (V, E) where $V = \{v_1, v_2, \cdots, v_n\}$ a finite set is and E is a subset of $V \times V$. The set V is referred to as the vertex set and E is referred to as the edge set. If $(v_i, v_j) \in E$, then we say that there is an edge from v_i to v_j. We sometimes use a pictorial representation of a digraph (V, E), whereby n points in the plane are labeled as $v_1, v_2, \cdots, v_n$ and an arc is drawn from v_i to v_j with an arrowhead directed towards v_j whenever $(v_i, v_j) \in E$. We say that a vertex v_j is reachable from another vertex v_i if there is a path from v_i to v_j.

Given a digraph (V, E), we define a binary relation $\Re$ on the vertex set V by letting $\Re$ consist of those vertex pairs (v_i, v_j) such that v_j is reachable from v_i. In this case, we write $v_i \;\Re\; v_j$.

*Definition1***:** Given a digraph (V, E), we say that a pair of vertices (v_i, v_j) is strongly connected if $v_i \;\Re\; v_j$ and $v_j \;\Re\; v_i$. We say that the digraph itself is strongly connected if every pair of vertices is strongly connected.

To introduce the notion of decomposition into strongly connected components, we define another binary relation T on the vertex set V by letting a pair (v_i, v_j) belong to T if and only if the pair of vertices (v_i, v_j) is strongly connected. It is then easy to show that T is an equivalence relation on the set V. Hence it possible to partition V into its equivalence classes under T. Clearly, the digraph is strongly connected if and only if V it is a single equivalence class under T.

Theorem1 [9, 29]: Given a digraph (V, E), one can partition the vertex set V into disjoint Union of subsets $V_1, V_2, \cdots, V_k$ in such a way that

i) each V_i is an equivalence class under the relation T

ii) if $v_a \in V_i$, $v_b \in V_j$, and $(v_a, v_b) \in E$, then $i \geq j$

Equivalently, if we renumber the vertices in V in order, beginning with those in V_1 and ending with those in V_k, then the adjacency matrix of the graph is block triangular form.

Therefore, for a given digraph, if we let $V_1, V_2, \cdots, V_k$ denote the equivalence class of V under the relation T, ordered so as to satisfy condition ii) of the above theorem, then the digraph $(V_i, (V_i \times V_i) \cap E)$ is called the ith strongly connected component (SCC). It should be noted that there exist very efficient algorithms for carrying out decomposition into (SCC) The objective (SCC) decomposition is to determine the stability properties of the original large scale system by studying the SCC's and the interconnecting subsystems alone. The result in this direction can be portioned into those that deal with Lyapunov stability.

Up to now, the emphasis has been on the decomposition of a given large scale system into hierarchical interconnection of several lower order subsystems. The potential weakness of this method is that, in some cases, the system digraph is strongly connected, and as result no simplification can be achieved using this technique.

III. PROBLEM FORMULATION

Consider a system described in state-space form by the set of equations

$$\dot{z}_i(t) = \phi_i(z_1(t), \cdots, z_k(t)), \quad i = 1,2,\cdots,k \tag{1}$$

where $z_i(t)$ is the state of the ith subsystem and k is the number of subsystems. Define a digraph as follows: the vertex set consists of $\{1,2,\cdots,k\}$, and there is an edge from vertex j to vertex i if and only if the function ϕ_i depends explicitly on the variable z_j. Now carry out a decomposition of this digraph into its strongly connected components, and let $V_1, V_2, \cdots, V_k$ denote the equivalence classes of vertices satisfying the conditions of the above theorem. If we define the new vector variables

$$x_i(t) = \{z_a(t) : a \in V_i\} \tag{2}$$

Then condition ii) of the theorem1 implies that the state equations now have the triangular form

$$\dot{x}_i(t) = \varphi_i(x_1(t), \cdots, x_k(t)) \qquad i = 1,2,\cdots,k \tag{3}$$

The noteworthy feature of (3) is that the differential equation for the variable x_i depends only on the variables $x_1, \cdots, x_i$, and not on the variables $x_{i+1}, \cdots, x_k$.

It is frequently possible to view system (3) as an interconnection of subsystems. The process of the decomposition into an appropriate form is by no means a trivial task, and it is usually influenced by mathematical convenience to overcome technical difficulties [26],[27]. We study systems described by (3) and we further assume that the digraph associated with it is strongly connected. We have in mind system (3) which can be modeled equivalently by equations of the form:

$$\dot{x}_i(t) = g_i(x_i(t)) + f_i(x(t)) = h_i(x(t)) \quad i = 1,2,\cdots,k \tag{4}$$

where $x_i(t) \in R^{n_i}$ is the state vector of the ith subsystem; $x^T(t) = [x_1(t), \cdots, x_k(t)]^T$, and $g_i : R_+ \times R^{n_i} \to R^{n_i}$; $g_i : R_+ \times R^n \to R^{n_i}$ are continuous function. It is assumed that the set of differential equations (4) has a unique solution for each initial condition, which depends continuously on the initial condition. The system

$$\dot{x}_i(t) = g_i(x_i(t)) \tag{5}$$

is called the ith isolated subsystem, and the term $f_i(x(t))$ is called the ith interaction term.

IV. Lyapunov Stability

The function $h(x)$ is assumed to satisfy the necessary smoothness requirements for the existence, and continuity of the solution for every $x_0 \in R^n$ and $t_0 \in R^+$ so that there exists one and only one solution $x(t; x_0, t_0)$ for $t_0 \geq 0$ and also to satisfy $h(0) = 0$ so that $x = 0$ is an equilibrium point for the interconnected system.

Lyapunov stability results involve the existence of functions $V : D \to R$. We assume that such functions are continuous on their respective domains and that they satisfy locally a Lipschitz condition with respect to x and $V(0) = 0$. The upper right hand derivative of V with respect to t along the solution of (4) is given by

$$DV_{(4)}(x) = \lim_{h \to 0^+} \sup \frac{1}{h} \{V[x(t+h; x, t)] - V[x]\} \tag{6}$$

If V is continuously differentiable with respect to all its arguments, then the total derivative of V with respect to t along the solution of (4) is given by

$$DV_{(4)}(x) = \nabla_i v_i(x_i)^T \{g_i(x_i) + f_i(x_1, x_2, \cdots, x_l)\} \tag{7}$$

where $\nabla_i v_i$ denotes the gradient vector of $v_i(x_i)$ with respect to x_i given by

$$\nabla_i v_i = \left(\frac{\partial v_i}{\partial x_{i1}}, \cdots, \frac{\partial v_i}{\partial x_{im_i}} \right)^T \tag{8}$$

In the analysis of large scale nonlinear systems, the problem of determining the stability region of a stable equilibrium point is very important in many applications. Basically, there are two approaches to estimate the stability regions of large scale systems; namely the scalar Lyapunov approach and the vector Lyapunov function approach. In the scalar Lyapunov function approach, the stability region is estimated via a Lyapunov function, which is a function of the Lyapunov function for each isolated subsystem. In the vector Lyapunov functions approach, an estimated stability region of the so-called comparison system [2] is found first, based on which the stability region of the overall system is then determined. It is not clear at the current stage which approach offers better results. The vector Lyapunov function approach provides an estimated stability region of each subsystem independent of other subsystems; a feature not shared by the scalar Lyapunov function approach. It is well known that both the scalar Lyapunov function approach and vector Lyapunov function approach give very conservative results in estimating the stability regions. This undesirable fact is partly due to the nature of Lyapunov function approach and partly to the characteristic of the decomposition-aggregation technique used in analyzing the large scale nonlinear system. In this paper we focus on vector Lyapunov function approach since it is more general in terms of assumptions imposed on the large scale nonlinear system.

A. *Stability of the Isolated Subsystem*

The Lyapunov theory is very powerful. However, in general, great difficulties arise in applying these results to high dimensional systems with complicated structure. The reason for this lies in the fact that there is no universal and systematic procedure available which tell us how to find the required Lyapunov functions. Although converse Lyapunov theorems have been established, these results provide no clue for the construction of the Lyapunov functions. For this reason we will pursue an approach which allows us to analyze the stability of large scale system with intricate structure in terms of simpler system component. First we need to consider the stability of the isolated subsystem also called free subsystem. It is still obvious that the construction of Lyapunov function of the free subsystem with lower order is comparatively manageable since it is characterized by its own Lyapunov function and the characterization does not require the knowledge of others subsystems. Secondly, once the Lyapunov stability of the composite system has been established, it remains to show that the overall system is stable.

The reader is referred to [23-25] for the principal Lyapunov stability results.

In characterizing the qualitative properties of the isolated subsystem S, we use the following conventions

Definition2: Isolated subsystem S possesses Property A if there exists a continuously differentiable function $v_i : R^{n_i} \to R$ functions $\psi_{i1}, \psi_{i2} \in KR$, $\psi_{i3} \in K$ and a constant $\sigma_i \in R$ such that the inequalities

$$\psi_{i1}(\|x_i\|) \leq v_i(x_i) \leq \psi_{i2}(\|x_i\|), \tag{9a}$$

$$Dv_{i(S)}(x_i) \leq \sigma_i \psi_{i3}(\|x_i\|) \tag{9b}$$

holds for all $x_i \in R^{n_i}$

Clearly if $\sigma_i \prec 0$ the equilibrium $x_i = 0$ of S is uniformly asymptotically stable in the large. If $\sigma_i = 0$ the equilibrium is uniformly stable. If $\sigma_i \succ 0$ the equilibrium of S may be unstable.

Theorem2 [28]. The equilibrium $x_i = 0$ is uniformly asymptotically stable in the large if the following conditions are satisfied.

(i) Each isolated subsystem S possesses Property A

(ii) Given v_i and ψ_{i3} of hypothesis (i), there exist constants $a_{ij} \in R$ such that

$$\nabla v_i(x_i)^T \theta_i(x_1,\cdots,x_k) \le \left[\psi_{i3}(\|x_i\|)\right]^{1/2} \sum_{i=1}^{k} a_{ij} \left[\psi_{i3}(\|x_j\|)\right]^{1/2} \qquad (10)$$

for all $x_i \in R^{n_i}$ and where θ_i are the interactions terms.

(iii) Given σ_i of hypothesis (i) there exists an k-vectors $\alpha^T = (\alpha_1,\cdots,\alpha_k) \succ 0$ such that the test matrix $P = \lfloor p_{ij} \rfloor$ specified by

$$p_{ij} = \begin{cases} \alpha_i(\sigma_i + a_{ii} & i = j \\ (\alpha_i a_{ij} + \alpha_j a_{ji})/2 & i \ne j \end{cases} \qquad (11)$$

is negative definite.

Before proceeding further, we note that the test matrix $P = \lfloor p_{ij} \rfloor$ is negative definite if and only if

$$(-1)^m \begin{vmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \cdots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{vmatrix} \succ 0, \quad m = 1,2\cdots,k \qquad (12)$$

B. Stability of the Composite Subsystem

In this section the framework and the methodology of the work are illustrated. The sufficiency of using in our new unified approach $x = \left(x_1^T, x_2^T\right) \in R^{n_i}$ $i = 1,2$ for discussing will be evident. On the necessity side, a preliminary characterization of the system dynamic is also given to stability and performance characterizations which are directly related to contribution of Lyapunov functions. The system equation (4) is assumed to be expressible as a nonlinear interconnection of subsystems given by the following structure:

$$\dot{x}_1 = g_1(x_1) \qquad (13a)$$

$$\dot{x}_2 = g_2(x_2) + f_2(x_1, x_2) \qquad (13b)$$

One of the main impediments to the application of Lyapunov's method to composite subsystem is the lack of formal procedures to construct the required Lyapunov function for the differential equations describing the given composite sequential subsystem. This construction is an intractable problem and a crucial part of the design because of the dimensionality and the nonlinear interconnection term. For this reason, we adopt an approach which can make this construction a feasible task.

In particular, we concentrate on the case where each composite subsystem admits a simplified local subsystem by which we start the recursive design procedure; defined by the following differential equation.

$$\dot{x}_q = h_q(x_p) \qquad q = m_s + 1,\cdots, m_s + r \qquad (14)$$

where $s = j - 1$ for $j = 2,3,\cdots,m$, $x_p = x_{m_s+1},\cdots,x_{m_s+r}$

And we assume that (13) have an appropriate definite first integral $v_r(x_p)$ of order r for $m_s + r \prec m_{s+1}$.

We recall that, by first integral, we understand a differentiable function $\Gamma(X)$ defined in domain D of the state space such that when x_i's constitute a solution, $\Gamma(X)$ assumes a constant value C [18].

A necessary and sufficient condition for (14) to have first integral is given by the condition

$$\sum_{i=1}^{r} \frac{\partial h_{m_s+i}(x_p)}{\partial x_{m_s+i}} = 0 \qquad (15)$$

In what follows the constructive methodology of the first integral will be illustrated for $r = 2$ (second order). Then (14) is equivalent to the ordinary first order differential equation given by

$$\frac{d\, x_{m_s+1}}{d\, x_{m_s+2}} = \frac{h_{m_s+1}(x_p)}{h_{m_s+2}(x_p)} \qquad (16)$$

Multiplying and integrating (16) ,the result can be expressed as energy integral of (14) as follow

$$v_s(x_p) + \int_0^t -\dot{v}_s(x_p)\, dt = 0 \qquad (17)$$

where $v_s(x_p)$ is the second order first integral which we assume it is definite and constitutes a Lyapunov function of (14) and

$$\dot{v}_s = \sum_{i=1}^{r} \frac{\partial v_s}{\partial x_{m_s+i}} \dot{x}_{m_s+i} \qquad (18)$$

is the time derivative of the Lyapunov function $v_s(x_p)$ along the motions of (14).

This proposed methodology uses the resulting Lyapunov function $v_s(x_p) = V_2$ of (14) and proceeds by constructing a new Lyapunov function for the subsystem under study. Let

$$\dot{v}_c(x_e) = \sum_{i=1}^{r} \frac{\partial v_s(x_p)}{\partial x_{m_s+i}} \phi_{m_s+i}(x_e) =$$

$$\dot{v}_s(x_p) + \dot{u}_a(x_e) = u_c(x_e) \qquad (19)$$

where

$x_p = x_{m_s+1}, x_{m_s+2}$; $x_e = 1,\cdots,x_{m_s+3}$

Since the derivative (19) along the trajectories of the subsystem considered must satisfy the condition

$$\dot{v}_s(x_p) + \dot{u}_a(x_e) \le 0 \qquad (20)$$

an attempt is made to make it at least negative semi definite. This may be accomplished by grouping terms of similar state variables and choosing in obvious way a function $u_p(x_e)$ such that

$$\dot{v}_a(x_e) = \dot{v}_s(x_p) + \dot{u}_a(x_e) + u_p(x_e) \qquad (21)$$

If condition (15) is satisfied it follows that:

$$v_s[X_p(t,x_0,t_0)] = C \Rightarrow \dot{v}_s[X_p(t,x_0,t_0)] = 0 \qquad (22)$$

Therefore by augmenting the order of the subsystem by one, and by virtue of (22) equation (21) takes the following form:

$$\dot{v}_a(x_e) = u_c(x_e) + u_p(x_e) = -u_T(x_e) \qquad (23)$$

If we define

$$v_p(x_e) = \int_0^t u_p(x_e)\, dt \qquad (24)$$

The resulting Lyapunov function is found by integrating (23)

$$v(x_e) = v_c(x_e) + v_p(x_e) = V_3 \quad (25)$$

And its time derivative is given by

$$\dot{v}(x_e) = -u_T(x_e) \quad (26)$$

By reapplied iteratively the above steps, we can construct V_4 from V_3 and so on to $V_d(x_l)$ where $x_l = x_{m_x+1}, \cdots, x_{m_{s+1}}$ We proceed with the same scheme for every composite sequential subsystem to the original composite system. The methodology uses the idea of the "*top-down*" procedure to construct a sequence of Lyapunov functions started from the second order definite first integrals associated with each free subsystem. This methodology frequently enables us to circumvent difficulties which arise when the Lyapunov approach is applied to high dimensional composite subsystem with computational and analytical difficulties [13].

C. Overall Composite System Stability

Suppose that the system (13) satisfies the following assumptions.

Assumption1. The function $f_2(x_1, x_2)$ satisfies a linear growth assumption, that is, there exists two class-K functions $\gamma_1(.)$ and $\gamma_2(.)$, differentiable at $x_1 = 0$, such that

$$\| f_2(x_1; x_2 \| \le \gamma_1(\|x_1\|)\|x_2\| + \gamma_2(\|x_1\|) \quad (27)$$

Assumption 2 (Growth of the Lyapunov function $W(x_2)$)

The positive definite function $W(x_2)$ is C^2, radially unbounded, and satisfies

(i) $L_{g_2}W(x_2) \le 0$ for all x_2.

(ii) In addition, there exists constants c and b such that for $\|x_2\| \succ b$, $\left\|\frac{\partial W}{\partial x_2}\right\| \|x_2\| \le cW(x_2)$ (28)

Theorem3 If there exists a positive semi definite radially unbounded function $W(x_2)$ and a positive constants c and M such that for $\|x_2\| \succ M$ assumptions 1 and 2 are satisfied and for the equilibrium $\dot{x}_1 = 0$ the x_1-subsystem is uniformly asymptotically stable in the large, then the overall composite system is stable.

Proof: We have that for each $\tau \ge 0$

$$\left|\frac{\partial W}{\partial x_2}(\tilde{x}_2(\tau))\right| \le \left\|\frac{\partial W}{\partial x_2}\right\| (\gamma(\|x_1\|)e^{-\alpha\tau} + \gamma(\|x_1\|)e^{-\alpha\tau}\|\tilde{x}_2(\tau)\|) \quad (29)$$

Because $W(x_2)$ is radially unbounded, this implies that $\|\tilde{x}_2(\tau)\|$ and $\left\|\frac{\partial W}{\partial x_2}(\tilde{x}_2(\tau))\right\|$ are bounded on $[0, \infty)$. Therefore there exists $\gamma_1 \in K$ such that

$$\left|\frac{\partial W}{\partial x_2}(\tilde{x}_2(\tau))\right| \le \gamma_1(\|x_2, x_1\|)\, e^{-\alpha\tau} \quad (30)$$

Using (ii), we obtain the estimate

$$\dot{W} \le K_1(\|x_1(0)\|)\, e^{-\alpha t}\, \mathrm{W}$$

for $\|x_2(t)\| \succ \max\{0, M\}$. This estimate proves the bounded ness of $W(x_2)$, therefore, if we denote the estimated Lyapunov function of the overall composite system by $V_0(x_1, x_2)$ then

$$V_0(x_1, x_2) \le W(x_2(0))\, e^{\int_0^\tau K_1(\|x_1(0)\|)e^{-\alpha s}ds} \le K(\|x_1(0)\| W(x_2(0))$$

It follows that $V_0(x_1, x_2) = 0$ implies $x_1 = 0$. By construction $V_0(0, x_2) = W(x_2)$, and because the equilibrium $x_1 = 0$ of (13a) is uniformly asymptotically stable in the large and assumption 2 is satisfied , then it exists a positive semi definite radially unbounded function $V_0(x_1, x_2)$ such that

$$V_0(x_1, x_2) = 0 \Rightarrow (x_1, x_2) = (0,0).$$

We conclude that the overall composite system is stable.

V. Example

To illustrate how the results can be applied and to demonstrate the usefulness of the method of analysis advanced herein, let's consider the following composite nonlinear system given in hierarchical form by:

$$\dot{x}_{11} = -x_{11}^3 - 1.5x_{11}\left|x_{12}{}^3\right|; \quad \dot{x}_{12} = -x_{12}^5 + x_{11}^2 x_{12}^2$$

$$\dot{x}_{21} = x_{22}; \ \dot{x}_{22} = -x_{21} - x_{23}\sin x_{21};$$

$$\dot{x}_{23} = x_{22}\sin x_{21} - x_{23}x_{12}^2$$

The isolated subsystem $g_1(x_1)$ where $x_1 = (x_{11}, x_{12})$ is given by the first two equations. The required Lyapunov function of is found by decomposition. Let

$$\dot{x}_{11} = -x_{11}^3 \text{ and } \dot{x}_2 = -x_{12}^5$$

Choosing $v_1(x_{11}) = x_{11}^2$ and $v_2(x_{12}) = x_{12}^2$, we have $Dv_1(x_{11}) = -2x_{11}^4$ and $Dv_2(x_{12}) = -2x_{12}^6$. Using the notation of theorem2 we make the identification $\psi_{11}(r) = \psi_{12}(r) = r^2$, $\psi_{13}(r) = r^4$, $\psi_{21}(r) = \psi_{22}(r) = r^2$, and $\psi_{23}(r) = r^6$. The interconnecting structure is characterized by $\theta_1(x_{11}, x_{12}) = -1.5x_{11}|x_{12}|^3$ and $\theta_2(x_{11}, x_{12}) = x_{11}^2 x_{12}^2$

We have now

$$\nabla v_1(x_{11})\theta_1(x_1) = (2x_{11})(-1.5x_{11})|x_{12}|^3 = |x_{11}|^3(-3)|x_{12}|^3$$

$$= \psi_{13}(|x_{11}|)^{1/2}(-3)\psi_{23}(|x_{12}|)^{1/2}$$

$$\nabla v_2(x_{12})\theta_2(x_1) = (2x_{12})(x_{11}^2 x_{12}^2) \le |x_{12}|^3(2)|x_{11}|^2$$

$$= \psi_{23}(|x_{12}|)^{1/2}(2)\psi_{13}(|x_{11}|)^{1/2}$$

We now have $a_{11} = a_{22} = 0$, $a_{12} = -3$, $a_{21} = 2$, $\sigma_1 = \sigma_2 = -2$. Choosing $\alpha_1 = \alpha_2 = 1$, the test matrix assume the form

$$P = \begin{bmatrix} -2 & -0.5 \\ -0.5 & -2 \end{bmatrix}$$

Since P is negative definite it follows from theorem2 that the equilibrium $(x_{11}, x_{12}) = x_1 = 0$ is asymptotically stable in the large.

The composite subsystem is given by the last three equations. Using (15) we get

$\dot{x}_{21} = x_{22}$; $\dot{x}_{22} = -x_{21}$

from (17) $v_s(x_{21}, x_{22}) = 0.5(x_{21}^2 + x_{22}^2) = v_2^{(2)}$ (A)

Differentiating (A) and augmenting the subsystem order by one yield

$\dot{v}_3^{(2)} = -x_{23}^2 x_{11}^2 - \dot{x}_{23} x_{23}$, taking $u_3^{(2)} = 0.5 x_{23} \dot{x}_{23}$ yields

$V_2 = v_3^{(2} = 0.5(x_{21}^2 + x_{22}^2 + x_{23}^2) = W(x_1, x_2)$

$\dot{W}(x_1, x_2) = -x_{12}^2 x_{23}^2 \Rightarrow$ the composite subsystem is stable.

Since the x_1-subsystem is uniformly asymptotically stable in the large, and when $x_1 = 0$, the x_2-subsystem is globally stable with Lyapunov function V_2 and because the conditions of theorem3 are satisfied, the overall composite system is stable.

VI. CONCLUSION

This paper has treated in a very general setting the stability of a large scale nonlinear dynamical system. Due to the functional analysis approach, the assumptions required on the subsystems is minimal, that is the overall system is algorithmically decomposed into a hierarchy of strongly connected subsystems which may be complex or relatively of high dimension and provide computational and analytical difficulties. Simplified stability conditions and computational advantage obtained from this structural decomposition are presented. This novel approach establishes that the overall system is stable once the strongly connected subsystems are stable. However, it is found that the method proposed in this paper provides a systematic approach of Lyapunov stability for a large scale nonlinear system expressed as first order nonlinear differential equations. The method was successfully applied to fifth order differential equations.

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