Fuzzy Boundary Control of Nonlinear Distributed Parameter Systems
Through an Equivalent Punctual Control Form

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Abstract—**Relying on the idea of formulating the problem of boundary control by means of equivalent punctual control problem, we present in this paper a novel design methodology of a fuzzy partial differential equation model based boundary geometric controller for nonlinear distributed parameter systems (DPSs). With regard to the distributed dynamics of DPSs, this control problem is formulated as a set-point reference tracking problem with an infinite characteristic index defined for a punctual output variable at a yields spatial position. To deal with this control design, it is first suggested to transform the boundary control problem to an equivalent in domain punctual control form by using Laplace transform in the spatial domain. Then, considering a punctual output, a weighted value of the variable, along the spatial domain, is defined as a controlled output. Processing the controlled output by means of geometric control rules and control objectives leads to the determination of the proposed fuzzy boundary geometric controller. Temperature gradient control in a thin nonisothermal catalytic reactor is considered as an application example to show the stabilizing performance of the developed control strategy.**

Keywords— Partial differential systems, nonlinear dynamics, Takagi-Sugeno fuzzy model, boundary control, punctual control, geometric control.

I. INTRODUCTION

Nonlinear distributed parameter systems dynamical behavior is basically described by means of partial differential equations (PDEs) [2] and [1]. This class of physical systems is infinite dimensional whose control design is usually regarded as a very challenging task because of the strong variations of the systems dynamics over both time and space dimensions.

To achieve control objectives, two main control approaches can be employed, namely the early lumping approach and the late lumping approach [3]. The early lumping approach [24] and [25] applies dynamics reduction to the nonlinear distributed parameter system by using approximation methods such as finite difference method, finite element method and Galerkin method [4], [6], [7] and [5]. This leads to a lumped parameter model representation which is, obviously, finite dimensional in time, but high-order and hence rather difficult to handle systematically for control design and real-time implementation. Contrary to early lumping, the late lumping approach [8] and [9] represents an interesting design method, which involves directly the distributed parameter model in the controller design procedure without any prior approximation. This way of dealing with distributed parameter systems control problems may preserve the distributed nature of the system dynamics together with their inherent physical features. Consequently, the resulting controller is infinite dimensional of distributed nature which requires powerful mathematical tools

for comportemental analysis in closed-loop operation [8] and [12].

The control methodology presented in this paper is achieved on the basis of the late lumping approach. The distributed parameter system is approximated by a T-S fuzzy partial differential equation model [11], [15], [13], [20] and [19], based on which a fuzzy controller is derived through fuzzy interpolation of local geometric controllers that are determined by assuming directly the local linear partial differential equation models without any reduction [21] and [22]. Nevertheless, for boundary control strategy, the characteristic index is infinite [14], which prevents the application of the geometric control concept. Thus, in order to get a finite characteristic index, we propose to consider an equivalent T-S fuzzy PDE model with punctual control. The boundary control form is hence converted into an equivalent punctual control problem form by using the Laplace transform in the spatial domain [16]. Based on the local linear distributed parameter systems with a punctual control, local geometric controllers of infinite dimensionsare designed and the overall fuzzy geometric controller is obtained using fuzzy operators.

The rest of the paper is structured as follows. Section II provides a description of the nonlinear distributed parameter models and formulates the problem of fuzzy boundary geometric control. The complete design procedure of the fuzzy boundary geometric controller is detailed in Section III. The proposed control methodology is applied to a thin nonisothermal catalytic reactor model in Section V, where simulation results are given to show the stabilizing performance of the developed controller. The last Section concludes the paper.

II. FUZZY BOUNDARY CONTROL PROBLEM FORMULATION

A. Control Problem Description

Consider the nonlinear DPS described by the following model [17]:

$$\frac{\partial x(z,t)}{\partial t} = -\beta\big(x(z,t)\big)\frac{\partial x(z,t)}{\partial z} + \gamma\big(x(z,t)\big)\frac{\partial^2 x(z,t)}{\partial z^2} + f\big(x(z,t)\big) \tag{1}$$

with boundary conditions:

$$\frac{\partial x(z,t)}{\partial z}\bigg|_{z=0} = u(t), \qquad \frac{\partial x(z,t)}{\partial z}\bigg|_{z=l} = 0 \tag{2}$$

and initial condition:

$$x(z,0) = x_0(z) \tag{3}$$

where $x(z,t)$ is the state, $z \in [0,l]$ and $t \in [0,+\infty[$ are space and time variables, respectively. $u(t)$ is the manipulated

boundary control input applied at $z = 0$ and $x_0(z)$ is the initial spatial profile. $\beta(x(z,t))$, $\gamma(x(z,t))$ and $f(x(z,t))$ are sufficiently smooth nonlinear functionsin $x(z,t)$, satisfying $\beta(0) = 0, \gamma(0) = 0$ and $f(0) = 0$, respectively.

We define the punctual output variable $y_p(t)$ as follows:

$$y_p(t) = \int_0^l \delta(z - z_{ref}) x(z,t) \, dz$$
$$= x(z_{ref}, t) \tag{4}$$

where z_{ref} is a given spatial position with $z_{ref} \in [0, l]$.

Let the measured output $y(t)$, taken as the weighted mean value of the state variable $x(z,t)$ along the z-axis, be defined as:

$$y(t) = \int_0^l c(z) x(z,t) \, dz \tag{5}$$

where $c(z)$ is a smooth shaping function.

B. Equivalent In domain Punctual Control Problem Form

Formulating the boundary control problem as a punctual control one is the first step in the procedure of deriving the fuzzy boundary geometric control. This can be achieved by using a punctual actuation. The system variables are manipulated by means of Laplace transform in the domain$[0, l]$ under the following changes:

$$z = l - \eta \tag{6}$$
$$x(z,t) = X(\eta, t) \tag{7}$$
$$c(z) = C(\eta) \tag{8}$$

As a result, the nonlinear distributed parameter system (1) is rewritten as:

$$\frac{\partial X(\eta, t)}{\partial t} = -\beta(X(\eta, t)) \frac{\partial X(\eta, t)}{\partial \eta} + \gamma(X(\eta, t)) \frac{\partial^2 X(\eta, t)}{\partial \eta^2}$$
$$+ f(X(\eta, t)) \tag{9}$$

with the following boundary conditions

$$\left. \frac{\partial X(\eta, t)}{\partial \eta} \right|_{\eta=0} = u(t), \qquad \left. \frac{\partial X(\eta, t)}{\partial \eta} \right|_{\eta=l} = 0 \tag{10}$$

By applying the Laplace transform, the punctual output variable $y_p(t)$ takes the following form:

$$y_p(t) = X(\eta_{ref}, t) \tag{11}$$

and the output variable (4) can be written as:

$$y(t) = \int_0^l C(\eta) X(\eta, t) \, d\eta \tag{12}$$

The dynamical behavior of the nonlinear DPS (1)-(2) can be approximated by a T-S fuzzy partial differential equation model composed of a set of fuzzy rules whose consequents are described by local linear PDE models as follows:

Model Rule k:

IF $\psi_1(\eta, t)$ is about $G_{k1} and \dots and \psi_n(\eta, t)$ is about G_{kn},

$$\text{THEN } \frac{\partial X(\eta, t)}{\partial t} = -\beta_k \frac{\partial X(\eta, t)}{\partial \eta} + \gamma_k \frac{\partial^2 X(\eta, t)}{\partial \eta^2}$$
$$+ \theta_k X(\eta, t) \tag{13}$$

with the boundary conditions

$$\left. \frac{\partial X(\eta, t)}{\partial \eta} \right|_{\eta=0} = u(t), \qquad \left. \frac{\partial X(\eta, t)}{\partial \eta} \right|_{\eta=l} = 0 \tag{14}$$

where G_{kq}, $k \in \mathbb{S} \triangleq \{1, 2, \dots, m\}$, $q \in \{1, 2, \dots, n\}$ are fuzzy sets; $\psi_q(\eta, t)$ are the premise variables and m is the number of IF-THEN rules. β_k, γ_k and θ_k are real known matrices.

Then, the overall fuzzy partial differential equation dynamics of the nonlinear DPS (13)-(14) can be expressed as follows:

$$\frac{\partial X(\eta, t)}{\partial t} = \sum_{k=1}^{m} \mu_k(\psi(\eta, t)) \left(-\beta_k \frac{\partial X(\eta, t)}{\partial \eta} + \gamma_k \frac{\partial^2 X(\eta, t)}{\partial \eta^2} \right.$$
$$\left. + \theta_k X(\eta, t) \right) \tag{15}$$

where $\psi(\eta, t) = [\psi_1(\eta, t) \dots \psi_m(\eta, t)]^T$,
and $\mu_k(\psi(\eta, t)) = w_k(\psi(\eta, t)) / \sum_{k=1}^{m} w_k(\psi(\eta, t)), k \in \mathbb{S}$,
with $w_k(\psi(\eta, t)) = \prod_{q=1}^{n} G_{kq}(\psi_q(\eta, t))$ such that $w_k(\psi(\eta, t)) \geq 0, k \in \mathbb{S}$ and $\sum_{k=1}^{m} w_k(\psi(\eta, t)) > 0$ for all $z \in [0, l]$ and $t \geq 0$.

According to these considerations, we have $\mu_k(\psi(\eta, t)) \geq 0, k \in \mathbb{S}$, and $\sum_{k=1}^{m} \mu_k(\psi(\eta, t)) = 1$ for all $z \in [0, l]$ and $t \geq 0$.

Relying on the T-S fuzzy PDE model (13), the fuzzy boundary geometric controller is expressed linguistically by:

Control Rule k:
IF $\psi_1(\eta, t)$ is about $G_{k1} and \dots and \psi_n(\eta, t)$ is about G_{kn},

$$\text{THEN } u_k(t) = \varphi_k(X(\eta, t)) \tag{16}$$

Where $\varphi_k(X(\eta, t))$ stands for a local boundary geometric control law.

Based on the above description, the overall fuzzy boundary geometric controller can be determined by:

$$u(t) = \sum_{k=1}^{m} \mu_k(\psi(\eta, t)) \varphi_k(X(\eta, t)) \tag{17}$$

Considering Laplace transform tools, namely:

$$\mathcal{L}\left(\frac{d^2 h(z)}{dz^2}\right) = s^2 \bar{h}(s) - \left. \frac{dh(z)}{dz} \right|_{z=0} - s h(0) \tag{18}$$

$$\mathcal{L}\left(\frac{dh(z)}{dz}\right) = s \bar{h}(s) - h(0) \tag{19}$$

Where $h(0)$ is thevalue of h for strictly negative values of z.

Applying (18) and (19), the rule-consequent equation in model (13) becomes:

$$\frac{\partial \bar{X}(s, t)}{\partial t} = -\beta_k(s \bar{X}(s, t) - X(0, t))$$

$$+\gamma_k\left(s^2\bar{X}(s,t)-\left.\frac{\partial X(\eta,t)}{\partial\eta}\right|_{\eta=0}-sX(0,t)\right)$$

$$+\theta_k\bar{X}(s,t) \tag{20}$$

where $\bar{X}(s,t)$ is the Laplace transform of $X(\eta,t)$.

Considering the boundary condition (14), we get:
$$\frac{\partial\bar{X}(s,t)}{\partial t}=-\beta_k\big(s\bar{X}(s,t)-X(0,t)\big)$$
$$+\gamma_k\big(s^2\bar{X}(s,t)-u_k(t)-sX(0,t)\big)+\theta_k\bar{X}(s,t)$$

$$=\gamma_k\big(s^2\bar{X}(s,t)-sX(0,t)\big)-\beta_k\big(s\bar{X}(s,t)-X(0,t)\big)$$
$$+\theta_k\bar{X}(s,t)-\gamma_k u_k(t) \tag{21}$$

Therefore, the corresponding equivalence form of (21) is:
$$\frac{\partial\bar{X}(s,t)}{\partial t}=-\beta_k\big(s\bar{X}(s,t)-X(0,t)\big)$$
$$+\gamma_k\left(s^2\bar{X}(s,t)-\left.\frac{\partial X(\eta,t)}{\partial\eta}\right|_{\eta=0}-sX(0,t)\right)$$
$$+\theta_k\bar{X}(s,t)-\gamma_k u_k(t) \tag{22}$$

Thus, applying the inverse Laplace transform to equation (22), it results the following equivalent punctual control form with homogeneous boundary conditions:
$$\frac{\partial X(\eta,t)}{\partial t}=-\beta_k\frac{\partial X(\eta,t)}{\partial\eta}+\gamma_k\frac{\partial^2 X(\eta,t)}{\partial\eta^2}+\theta_k X(\eta,t)$$
$$-\gamma_k\delta(\eta)u_k(t) \tag{23}$$

subject to
$$\left.\frac{\partial X(\eta,t)}{\partial\eta}\right|_{\eta=0}=0,\qquad\left.\frac{\partial X(\eta,t)}{\partial\eta}\right|_{z=l}=0 \tag{24}$$

Where $\delta(\eta)$ is the Dirac delta-function.

III. DESIGN OF THE FUZZY BOUNDARY GEOMETRIC CONTROLLER

A key issue in the proposed fuzzy geometric control design methodology consists in the application of the concept of characteristic index. Considering the measured output $y(t)$, our objective is to ensure good tracking of the desired reference input $\vartheta(t)$ following a first-order-like dynamic behavior which is expressed by:
$$v\frac{dy(t)}{dt}+y(t)=\vartheta(t) \tag{25}$$

where v is a design control parameter.

More precisely, the design procedure is built upon the nonlinear partial differential equation (1). Thus, taking the first derivative of (12) yields:
$$\frac{dy(t)}{dt}=\int_0^l C(\eta)\frac{\partial X(\eta,t)}{\partial t}d\eta$$

$$=\int_0^l C(\eta)\left(-\beta_k\frac{\partial X(\eta,t)}{\partial\eta}+\gamma_k\frac{\partial^2 X(\eta,t)}{\partial\eta^2}+\theta_k X(\eta,t)\right.$$
$$\left.-\gamma_k\delta(\eta)u_k(t)\right)d\eta$$

$$=\int_0^l C(\eta)\left(-\beta_k\frac{\partial X(\eta,t)}{\partial\eta}+\gamma_k\frac{\partial^2 X(\eta,t)}{\partial\eta^2}+\theta_k X(\eta,t)\right)d\eta$$
$$-\gamma_k\underbrace{\left[\int_0^l C(\eta)\delta(\eta)\,d\eta\right]}_{J_1}u_k(t) \tag{26}$$

Evaluating the integral term J_1 in equation (26), we obtain:
$$J_1=\gamma_k C(0)u_k(t) \tag{27}$$

The resulting local geometric controller $u_k(t)$ takes then the form:
$$u_k(t)=-\frac{1}{v\gamma_k C(0)}\left[\vartheta(t)-y(t)+v\beta_k\int_0^l C(\eta)\frac{\partial X(\eta,t)}{\partial\eta}d\eta\right.$$
$$\left.-v\gamma_k\underbrace{\int_0^l C(\eta)\frac{\partial^2 X(\eta,t)}{\partial\eta^2}d\eta}_{J_2}-v\,\theta_k\int_0^l C(\eta)\,X(\eta,t)\,d\eta\right] \tag{28}$$

Integrating the term J_2 defined in equation (28), we have:
$$J_2=v\gamma_k\int_0^l C(\eta)\frac{\partial^2 X(\eta,t)}{\partial\eta^2}d\eta$$
$$=v\gamma_k\left(\left.C(\eta)\frac{\partial X(\eta,t)}{\partial\eta}\right|_{\eta=0}^{\eta=l}-\left.X(\eta,t)\frac{\partial C(\eta)}{\partial\eta}\right|_{\eta=0}^{\eta=l}\right.$$
$$\left.+\int_0^l \ddot{C}(\eta)X(\eta,t)\,d\eta\right) \tag{29}$$

Besides, by taking into account the boundary conditions (22), the integral term J_2 becomes:
$$J_2=-v\gamma_k X(l,t)\left.\frac{\partial C(\eta)}{\partial\eta}\right|_{\eta=l}+v\gamma_k X(0,t)\left.\frac{\partial C(\eta)}{\partial\eta}\right|_{\eta=0}$$
$$+v\gamma_k\int_0^l \ddot{C}(\eta)X(\eta,t)\,d\eta \tag{30}$$

Then, by assuming a smooth shaping function $C(z)$ that satisfies $C(0)\neq 0$, and substituting (30) into (28), it results the following local control law:

$$u_k(t) = -\frac{1}{v\gamma_k C(0)}\left[\vartheta(t) - y(t) + v\beta_k \int_0^l C(\eta)\frac{\partial X(\eta,t)}{\partial \eta}d\eta\right.$$
$$+ v\gamma\dot{C}(l)X(l,t) - v\gamma_k\dot{C}(0)X(0,t)$$
$$\left. - v\int_0^l \left(\gamma_k\ddot{C}(\eta) + \theta_k C(\eta)\right)X(\eta,t)d\eta\right]$$
(31)

Hence, the global fuzzy boundary geometric control can be expressed by:

$$u(t) = \sum_{k=1}^m \mu_k\left(\psi(\eta,t)\right)\left(-\frac{1}{v\gamma_k C(0)}\left[\vartheta(t) - y(t)\right.\right.$$
$$+ v\beta_k \int_0^l C(\eta)\frac{\partial X(\eta,t)}{\partial \eta}d\eta$$
$$- v\int_0^l \left(\gamma_k\ddot{C}(\eta) + \theta_k C(\eta)\right)X(\eta,t)d\eta$$
$$\left.\left. + v\gamma_k\dot{C}(l)X(l,t) - v\gamma_k\dot{C}(0)X(0,t)\right]\right)$$
(32)

Rewriting the obtained fuzzy boundary geometric controller (32) with respect to the variable changes (6)-(8) gives the final control law:
$$u(t)$$
$$= \sum_{k=1}^m \mu_k\left(\psi(z,t)\right)\left(-\frac{1}{v\gamma_k C(0)}\left[\vartheta(t) - y(t)\right.\right.$$
$$+ v\beta_k \int_0^l c(z)\frac{\partial x(z,t)}{\partial z}dz - v\int_0^l \left(\gamma_k\ddot{c}(z) + \theta_k c(z)\right)x(z,t)dz$$
$$\left.\left. + v\gamma_k\dot{c}(l)x(l,t) - v\gamma_k\dot{c}(0)x(0,t)\right]\right)$$
(33)

IV. APPLICATION EXAMPLE

The above-described control design procedure is applied in this section to a thin non isothermal catalytic reactor model with a nonlinear source for temperature control purpose. The process model is described by [18] and [23]:
$$\frac{\partial T_g(z,t)}{\partial t} = \frac{\partial^2 T_g(z,t)}{\partial z^2} - \frac{\partial T_g(z,t)}{\partial z} + \alpha_c\left(T_c(z,t) - T_g(z,t)\right)$$
(34)
$$\frac{\partial T_c(z,t)}{\partial t} = \frac{\partial^2 T_c(z,t)}{\partial z^2} + \beta_0 e^{(-\gamma T_c(z,t)/(1 + T_c(z,t)))}$$
$$-\beta_c\left(T_c(z,t) - T_g(z,t)\right)$$
(35)
with the boundary conditions
$$\left.\frac{\partial T_g(z,t)}{\partial z}\right|_{z=0} = u_g(t), \quad \left.\frac{\partial T_c(z,t)}{\partial z}\right|_{z=0} = u_c(t)$$
(36)
$$\left.\frac{\partial T_g(z,t)}{\partial z}\right|_{z=l} = \left.\frac{\partial T_c(z,t)}{\partial z}\right|_{z=l} = 0$$
(37)

and the initial conditions
$$T_g(z,0) = y_{g,0}(z); \quad T_c(z,0) = y_{c,0}(z)$$
(38)
where $T_g(z,t)$ and $T_s(z,t)$ are the temperature of the gas and the catalytic, respectively. $u_g(t)$ and $u_c(t)$ are the manipulated boundary control inputs.

Setting $\beta_0 = -0.003$, $\beta_c = 1$, $\gamma = 21.14$, $\alpha_c = 0.5$ and $l = 1$, and the initial conditions $y_{g,0}(z) = 0.2$ and $y_{c,0}(z) = 0.2$, the process model (34)-(37) is approximated by the following T-S fuzzy PDE model composed of two fuzzy rules with the boundary conditions (36)-(37):
System rule 1:
IF $T_c(z,t)$ is "about 0" THEN

$$\frac{\partial T_g(z,t)}{\partial t} = \frac{\partial^2 T_g(z,t)}{\partial z^2} - \frac{\partial T_g(z,t)}{\partial z} + A_{11}T_c(z,t) - \alpha_c T_g(z,t)$$

$$\frac{\partial T_c(z,t)}{\partial t} = \frac{\partial^2 T_c(z,t)}{\partial z^2} + A_{12}T_c(z,t) + \beta_c T_g(z,t)$$

System rule 2:
IF $T_c(z,t)$ is "not about 0" THEN

$$\frac{\partial T_g(z,t)}{\partial t} = \frac{\partial^2 T_g(z,t)}{\partial z^2} - \frac{\partial T_g(z,t)}{\partial z} + A_{21}T_c(z,t) - \alpha_c T_g(z,t)$$

$$\frac{\partial T_c(z,t)}{\partial t} = \frac{\partial^2 T_c(z,t)}{\partial z^2} + A_{22}T_c(z,t) + \beta_c T_g(z,t)$$

with $A_{11} = \alpha_c$, $A_{12} = 0.1354\beta_0 - \beta_c$, $A_{21} = \alpha_c$, and $A_{22} = -\beta_c$.

The structure of the fuzzy model and its parameters are determined by using the sector nonlinearity method [11]. The number of rules is deduced according to the nonlinearities involved in the process model (34)-(37) and the fuzzy partitions are obtained as described below.

Assuming $T_c(z,t) \in [0.1990, 0.2005]$ and $z \in [0,1]$, the nonlinear term $\varphi(T_c(z,t)) = \exp(-\gamma T_c(z,t)/(1 + T_c(z,t)))$ in (35) can be expressed as follows:
$$\varphi(T_c(z,t)) = \exp(-\gamma T_c(z,t)/(1 + T_c(z,t))) =$$
$$0.1990\, T_c(z,t)\mu_1(T_c(z,t)) + 0.1995\, T_c(z,t)\mu_2(T_c(z,t))$$
(39)

where $\mu_1(T_c(z,t))$, $\mu_2(T_c(z,t)) \in [0,1]$ and

$$\mu_1(T_c(z,t)) + \mu_2(T_c(z,t)) = 1$$
(40)

By solving equations (39) and (40), the membership functions $\mu_1(T_c(z,t))$ and $\mu_2(T_c(z,t))$ are obtained as:
$$\mu_1(T_c(z,t)) = \begin{cases} \dfrac{0.1995\, T_c(z,t) - \varphi(T_c(z,t))}{(0.1995 - 0.1990)T_c(z,t)}, & T_c(z,t) \neq 0 \\ 1, & T_c(z,t) = 0 \end{cases}$$
(41)

$$\mu_2(T_c(z,t)) = 1 - \mu_1(T_c(z,t))$$ (42)

The overall process model is written as follows:

$$\frac{\partial T_g(z,t)}{\partial t} = \frac{\partial^2 T_g(z,t)}{\partial z^2} - \frac{\partial T_g(z,t)}{\partial z}$$
$$+ \sum_{k=1}^{2} \mu_k(\psi(z,t)) \Big(A_{k1} T_c(z,t) - \alpha_c T_g(z,t) \Big) \quad (43)$$

$$\frac{\partial T_c(z,t)}{\partial t} = \frac{\partial^2 T_c(z,t)}{\partial z^2}$$
$$+ \sum_{k=1}^{2} \mu_k(\psi(z,t)) \Big(A_{k2} T_c(z,t) + \beta_c T_g(z,t) \Big) \quad (44)$$

Following the above described control design methodology and considering the boundary control input at $z = 0$, we obtain the fuzzy controller:

$$u_g(t) = \sum_{k=1}^{2} \mu_k(\psi(z,t)) \left(\frac{1}{v} \Bigg[\vartheta(t) - y_g(t) \right.$$
$$+ v \int_0^l c(z) \frac{\partial T_g(z,t)}{\partial z} dz$$
$$- v A_{k1} \int_0^l c(z) T_c(z,t) dz$$
$$- v \int_0^l \big(\ddot{c}(z) - \alpha_c c(z) \big) T_g(z,t) dz$$
$$\left. + v \, \dot{c}(l) T_g(l,t) - v \, \dot{c}(0) T_g(0,t) \Bigg] \right) \quad (45)$$

$$u_c(t) = \sum_{k=1}^{2} \mu_k(\psi(z,t)) \left(\frac{1}{v} \Bigg[\vartheta(t) - y_c(t) \right.$$
$$- v \beta_c \int_0^l c(z) T_g(z,t) dz$$
$$- v \int_0^l \big(\ddot{c}(z) + A_{k2} c(z) \big) T_c(z,t) dz$$
$$\left. + v\dot{c}(l) T_c(l,t) - v\dot{c}(0) T_c(0,t) \Bigg] \right) \quad (46)$$

Consider the following outputs:
$$y_g(t) = \int_0^l c_g(z) T_g(z,t) dz \,;\; y_c(t) = \int_0^l c_c(z) T_c(z,t) dz \quad (47)$$
with $c_g(z) = c_c(z) = z$.

Applying the fuzzy boundary geometric control laws described by (45)-(46) to the process model (34)-(38), yields the closed-loop trajectory of $\|T_g(.,t)\|_2$ and $\|T_c(.,t)\|_2$ shown in Fig. 1. The closed-loop profile of $T_g(z,t)$ and $T_c(z,t)$ are depicted in Fig. 2 and Fig. 3, respectively, which demonstrate that the fuzzy boundary geometric control can stabilize efficiently the process model (34)-(38) around the desired set

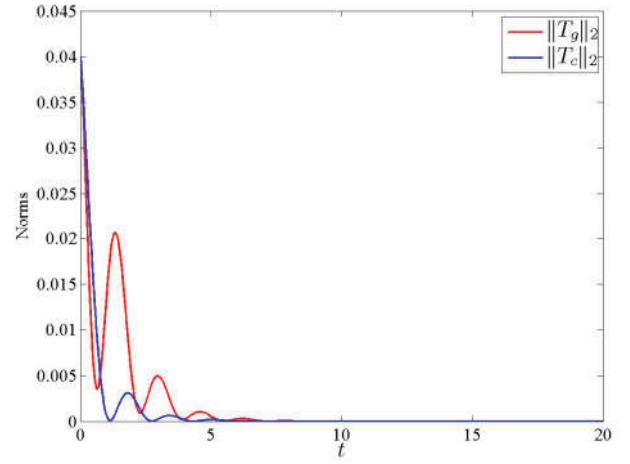point. The generatedfuzzy control inputs $u_g(t)$ and $u_c(t)$ are illustrated in Fig. 4.

Fig. 1. Closed-loop trajectories of $\left\|T_g(.,t)\right\|_2$ and $\left\|T_c(.,t)\right\|_2$ using the fuzzy boundary geometric controller.
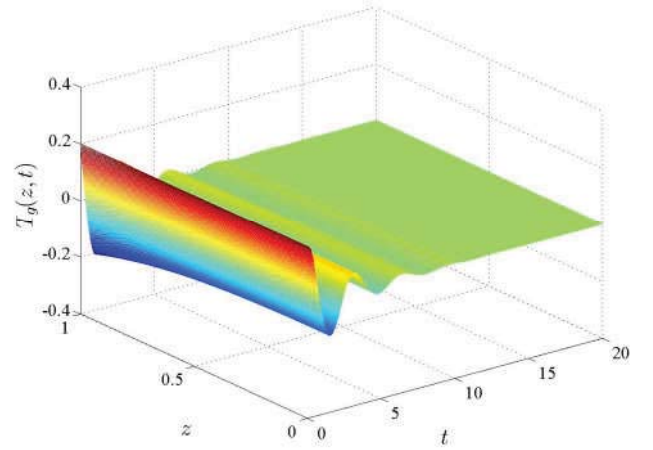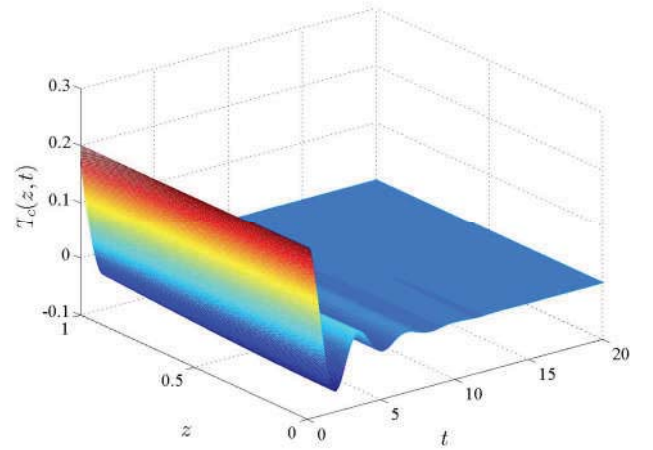

Fig. 2. Closed-loop profile of $T_g(z,t)$.

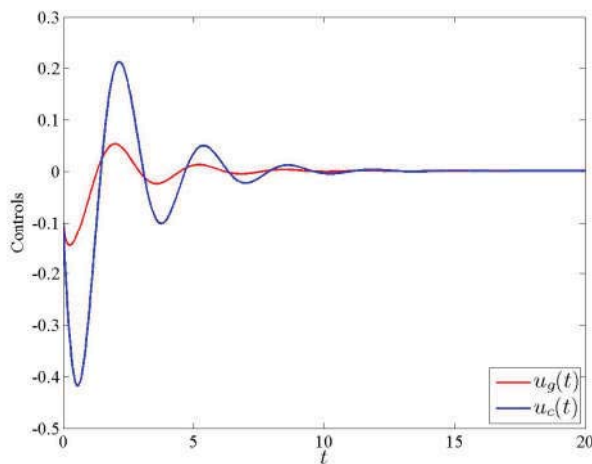Fig. 3. Closed-loop profile of $T_c(z,t)$.

Fig. 4. The manipulated fuzzy geometric control variables $u_g(t)$ and $u_c(t)$.

V. CONCLUSION

The present contribution addressed the problem of fuzzy boundary geometric control of nonlinear DPS by using the concept of characteristic index. The dynamical behavior of the nonlinear DPS is first approximated by a Takagi-Sugeno (T-S) fuzzy PDE model with boundary control which has been constructed by interpolating linear partial differential equation models using fuzzy inference operators. A problematic issue in the design process is that the resulting T-S fuzzy model has an infinite characteristic index. To overcome this constraint, a control problem transformation has been introduced. More precisely, an equivalent in domain punctual control form has been derived for the linear PDE models that describe locally the overall nonlinear T-S fuzzy PDE model. For this purpose, Laplace transformation in the space domain has been applied. Manipulating the resulting models with the aid of geometric control theory concepts and tools conducted us to the determination of a nonlinear controller with fully specified control objectives. Applying the proposed methodology to control temperature gradient in a thin nonisothermal catalytic reactor has shown interesting results that assessed clearly the stabilizing performance of the designed fuzzy boundary geometric controller. Future studies might consider further engineering applications of the proposed methodology with closed-loop stability analysis which might be checked by using the Lyapunov direct method. Semi-group theory might also be emphasized for this particular purpose. .

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