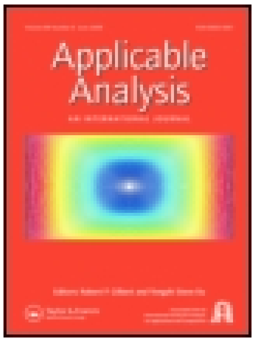

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Controllability of coupled systems for impulsive ϕ-Hilfer fractional integro-differential inclusions

A. Boudjerida[a], D. Seba[a] and G. M. N'Guérékata[b]

[a]Dynamic of Engines and Vibroacoustic Laboratory, University M'hamed Bougara of Boumerdes, Boumerdes, Algeria; [b]Department of Mathematics, Morgan State University, Baltimore, MD, USA

ABSTRACT
This paper studies the controllability of a coupled system for a class of impulsive fractional integro-differential inclusions involving ϕ-Hilfer fractional derivative and subject to coupled nonlocal integral initial conditions in the case of convex set-valued maps. Some auxiliary conditions are introduced in order to apply a fixed point theorem due to Bohnenblust–Karlin. An illustrative example is provided to exemplify our theoretical results.

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1. Introduction

Over the years, prominent mathematicians have shown very important and accurate results in fractional differential equations research field, mainly because of its effectiveness, along with fractional derivatives, as a tool of modeling different phenomena in physics, biology, chemistry, etc. Countless manuscripts have been published aiming to study the existence and uniqueness of solutions for fractional differential, integral and integro-differential equations and inclusions. For recent developments in this field, see [1–8] and the references therein. Within this context, fractional calculus has been receiving great attention in the scientific research community of mathematics, as well as other sciences, due to its rapid growth and development in both theoretical and practical domains. Consequently, the road was paved for the appearance of new and unified derivatives producing the most recent applications. Following the same line of research, Sousa and Oliveira [9] introduced the operator ϕ with Hilfer fractional derivative to get a new and general fractional derivative called 'ϕ-Hilfer fractional derivative' which contains a variety of fractional derivatives (ϕ-Riemann–Liouville and ϕ-Caputo fractional derivatives) in order to unify the enormous sum of such definitions in one fractional operator. However, so far there have been only few works that deal with it, see [10–13].

Impulse conditions drive differential equations to provide an appropriate framework for mathematical modeling. That is due to the unexpected changes in such phenomena during particular moments in their evolution process which cannot be described using regular differential equations. In this regard, impulsive fractional differential equations and inclusions have emerged as an

CONTACT G. M. N'Guérékata gaston.n'guerekata@morgan.edu

active research field in recent times, which resulted in numerous research articles related to various fractional derivatives [14–22].

Due to the fact that a lot of practical and applied problems within the fields of biology, chemistry, physics, and especially in computer network, are modeled mathematically in the form of coupled systems of fractional differential equations, several authors were devoted to examine the existence and uniqueness of the solution of such type of systems. In this framework of research works, some fruitful achievements have been obtained in relation to coupled systems of fractional differential equations [23–28], and yet results are almost rare about coupled systems of fractional differential inclusions. For example, Alsaedi et al. [29] studied the existence of a solution for coupled systems of time-fractional differential inclusions by using a new fractional derivative in the case of compact and convex valued L^1-Carathéodory multi-valued map. Jin et al. [30] solved a coupled system of hybrid fractional differential inclusions with coupled boundary conditions by using Bohnenblust–Karlin fixed point theorem. In addition, Blouhi and Ferhat [31] investigated the existence of mild solution for a coupled system of second-order impulsive semilinear stochastic differential inclusions by wiener process and Poisson jumps. On the other hand, we noted that controllability results associated with the coupled system of fractional differential inclusions are very rare and have not yet been processed. By presenting this work, we aim to fill this gap in the literature. Therefore, our results are entirely new and contribute to giving a valuable idea on this subject.

This paper is devoted to study the controllability of the following problem:

$$\begin{cases} ({}^hD^{\sigma_1,\delta_1;\phi}_{T^+}\omega_1)(t) \in \mathcal{Q}_1(t,\omega_1(t),\omega_2(t)) + \dfrac{1}{\Gamma(\sigma_1)}\displaystyle\int_T^t \phi'(\varrho)(\phi(t)-\phi(\varrho))^{\sigma_1-1} \\ \quad \times \mathcal{J}_1(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho + \mathcal{C}u_1(t), \quad t\in\mathcal{I}, t\neq t_k, \\ ({}^hD^{\sigma_2,\delta_2;\phi}_{T^+}\omega_2)(t) \in \mathcal{Q}_2(t,\omega_1(t),\omega_2(t)) + \dfrac{1}{\Gamma(\sigma_2)}\displaystyle\int_T^t \phi'(\varrho)(\phi(t)-\phi(\varrho))^{\sigma_2-1} \\ \quad \times \mathcal{J}_2(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho + \mathcal{C}u_2(t), \quad t\in\mathcal{I}, t\neq t_k \\ \Delta I^{1-r_1;\phi}_{T^+}\omega_1(t_k) = \Lambda^1_k \in \mathbb{R}, \quad \Delta I^{1-r_2;\phi}_{T^+}\omega_2(t_k) = \Lambda^2_k \in \mathbb{R}, \quad k\in\{1,2,\dots,\ell\} \end{cases} \tag{1}$$

supplemented with coupled nonlocal integral initial conditions of the form:

$$(I^{1-r_1;\phi}_{T^+}\omega_1)(T) = \int_T^b \mathcal{E}_1(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho, \quad r_1 = \sigma_1+\delta_1-\sigma_1\delta_1, \tag{2}$$

$$(I^{1-r_2;\phi}_{T^+}\omega_2)(T) = \int_T^b \mathcal{E}_2(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho, \quad r_2 = \sigma_2+\delta_2-\sigma_2\delta_2, \tag{3}$$

where $t\in\mathcal{I} := [T,b]$, $0<T<b<\infty$, $j=1,2$ and ${}^hD^{\sigma_j,\delta_j;\phi}_T$ is the ϕ-Hilfer fractional derivative of order $\sigma_j\in(0,1)$ and type $\delta_j\in[0,1]$ with T its lower limit, $I^{1-r_j;\phi}$ is the left-sided fractional integrals with respect to another function ϕ of order $1-r_j$ such that $\phi\in C^1_{\mathbb{R}}(\mathcal{I})$ is an increasing function and $\phi'(t)\neq 0$, for every $t\in\mathcal{I}$. Moreover, $\mathcal{Q}_j:\mathcal{I}\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ are multi-valued functions with convex and compact values, $\mathcal{J}_j,\mathcal{E}_j:\mathcal{I}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given functions, and $\Delta I^{1-r_j;\phi}_{T^+}\omega_j(t_k) = I^{1-r_j;\phi}_{T^+}\omega_j(t_k^+) - I^{1-r_j;\phi}_{T^+}\omega_j(t_k^-)$, whereas $I^{1-r_j;\phi}_{T^+}\omega_j(t_k^+) = \lim_{\varsigma\to 0^+} I^{1-r_j;\phi}_{T^+}\omega_j(t_k+\varsigma)$ and $I^{1-r_j;\phi}_{T^+}\omega_j(t_k^-) = \lim_{\varsigma\to 0^-} I^{1-r_j;\phi}_{T^+}\omega_j(t_k+\varsigma)$ with $T=t_0<t_1<\cdots<t_\ell<t_{\ell+1}:=b$. In additions, $\mathcal{C}$ is a bounded linear operator from $\mathcal{K}$ to $\mathbb{R}$ where $\mathcal{K}$ a Banach space, and $u_j(\cdot)$ belong to $L^2(\mathcal{I},\mathcal{K})$.

Section 2 gathers all the definitions, lemmas, remarks and theorems that are needed in Section 3. In this last, a controllability result is presented using Bohnenblust–Karlin fixed point theorem. The last section concludes the work with an illustrative example.

2. Preliminaries

Let $\mathcal{I} := [T, b], 0 < T < t_1 < \ldots < t_\ell < b < \infty$ and $C_{\mathbb{R}}(\mathcal{I})$ the space of all continuous functions on $\mathcal{I}$ endowed with the norm $\|v\|_C = \sup_{t\in\mathcal{I}} |v(t)|$.

Denote by $C_{1-r;\phi}(\mathcal{I}, \mathbb{R}) = \{v : (T, b] \to \mathbb{R}; (\phi(t) - \phi(T))^{1-r} v(t) \in C_{\mathbb{R}}(\mathcal{I}); 0 \le r \le 1\}$, the weighted space of the continuous function v on $\mathcal{I}$ and we define the weighted space of all piecewise continuous functions v on $C_{1-r_j;\phi}((t_k, t_{k+1}], \mathbb{R}), k = 1, \ldots, \ell$, by

$\mathcal{H}_j := PC_{1-r_j;\phi}(\mathcal{I}, \mathbb{R}) = \{v : (T, b] \to \mathbb{R}; v \in C_{1-r_j;\phi}((t_k, t_{k+1}], \mathbb{R}),$ which $I_{T^+}^{1-r_j;\phi} v(t_k^+), I_{T^+}^{1-r_j;\phi} v(t_k^-)$ exist and $I_{T^+}^{1-r_j;\phi} v(t_k^-) = I_{T^+}^{1-r_j;\phi} v(t_k), k = 1, \ldots, \ell\}$ for each $j = 1, 2$, normed by

$$\|v\|_{\mathcal{H}_j} = \sup\left\{|(\phi(t) - \phi(T))^{1-r_j} v(t)|, t \in \mathcal{I}\right\}.$$

Furthermore, we define $\mathcal{H} := \mathcal{H}_1 \times \mathcal{H}_2$ as the product weighted space normed by

$$\|(v, \omega)\|_{\mathcal{H}} = \|v\|_{\mathcal{H}_1} + \|\omega\|_{\mathcal{H}_2}.$$

Let $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ be given Banach space.

Definition 2.1 ([32]): The multi-valued map $\mathcal{N}$ defined from $\mathcal{F}$ into $\mathcal{P}(\mathcal{F})$ is

(1) convex (closed) valued if $\mathcal{N}(y)$ is convex (closed) for all $y \in \mathcal{F}$;
(2) bounded on bounded sets if $\mathcal{N}(\mathfrak{D}) = \bigcup_{y\in\mathfrak{D}} \mathcal{N}(y)$ is bounded in $\mathcal{F}$ for each bounded set $\mathfrak{D}$ of $\mathcal{F}$;
(3) upper semi-continuous (u.s.c.) if for every $y_0 \in \mathcal{F}$, the set $\mathcal{N}(y_0)$ is nonempty closed subset of $\mathcal{F}$, and for every open set $\mathfrak{B}$ of $\mathcal{F}$ containing $\mathcal{N}(y_0)$, there exists an open neighborhood $\mathfrak{B}_0$ of y_0 such that $\mathcal{N}(\mathfrak{B}_0) \subset \mathfrak{B}$;
(4) completely continuous if $\mathcal{N}(\mathfrak{D})$ is relatively compact for each bounded subset $\mathfrak{D} \subset \mathcal{F}$.

Furthermore, if for each $y \in \mathcal{N}$, the function $t \to d(y, \mathcal{N}(t)) = \inf\{d(y, \psi), \psi \in \mathcal{N}(t)\}$ is measurable, then $\mathcal{N}$ is measurable.

Definition 2.2: Let $\mathcal{N}$ be a multi-valued map defined from $\mathcal{F}$ to a compact Banach space $\mathcal{Z}$ with nonempty values then: if the graph of $\mathcal{N}$ is closed, thus $\mathcal{N}$ is upper semi-continuous.

Definition 2.3 ([29]): A multi-valued map $\mathcal{N} : \mathcal{I} \times \mathcal{F} \times \mathcal{F} \to \mathcal{P}(\mathcal{F})$ is called Carathéodory if

(1) $t \mapsto \mathcal{N}(t, \omega_1, \omega_2)$ is measurable for each $\omega_j \in \mathcal{F}, j = 1, 2$;
(2) $(\omega_1, \omega_2) \mapsto \mathcal{N}(t, \omega_1, \omega_2)$ is upper semi-continuous ($u.s.c$) for each $t \in \mathcal{I}$.

Now, we define the selections set of a multi-valued map $\mathcal{N}_j$ at ω_j for each $j = 1, 2$ by

$$\Theta_{\mathcal{N}_j,\omega_j} = \left\{\varphi_j \in L^1(\mathcal{I}, \mathcal{F}); \varphi_j(t) \in \mathcal{N}_j(t, \omega_1(t), \omega_2(t)), \text{ for a.e., } t \in \mathcal{I}\right\}, \tag{4}$$

which are nonempty if $\dim \mathcal{F} < \infty$.

Lemma 2.4 ([14]): *Let $\mathcal{Z}$ be a separable Banach space and $\mathcal{N} : \mathcal{I} \times \mathcal{Z} \times \mathcal{Z} \to \mathcal{P}_{c,cp}(\mathcal{Z})$ be a Carathéodory multi-valued map with $\Theta_{\mathcal{N},\omega}$ is nonempty set ($\mathcal{P}_{c,cp}(\mathcal{Z})$ denotes the family of nonempty, convex and compact subsets of $\mathcal{Z}$) and let $\Upsilon : L^1(\mathcal{I}, \mathcal{Z}) \to C_{\mathcal{Z}}(\mathcal{I})$ be a linear continuous function, then $\Upsilon \circ \Theta_{\mathcal{N}} : C_{\mathcal{Z}}(\mathcal{I}) \to \mathcal{P}_{c,cp}(C_{\mathcal{Z}}(\mathcal{I}))$, where $\omega \mapsto (\Upsilon \circ \Theta_{\mathcal{N}})(\omega) = \Upsilon(\Theta_{\mathcal{N},\omega})$ is a closed graph operator in $C_{\mathcal{Z}}(\mathcal{I}) \times C_{\mathcal{Z}}(\mathcal{I})$.*

In the remainder of this paper, we recall the necessary basic notions and properties related to fractional calculus then we give the fixed point theorem used to investigate our controllability results.

Definition 2.5 ([9]): Let $\sigma > 0$ and $\phi(t)$ be an increasing and positive monotone function on $\mathcal{I}' = (T, b]$ having a continuous derivative $\phi'(t)$ on (T, b). The ϕ-Riemann–Liouville fractional integral of order $\sigma > 0$ of a function $\mathcal{R}$ on $\mathcal{I}$ is defined by:

$$I^{\sigma;\phi}\mathcal{R}(t) = \frac{1}{\Gamma(\sigma)} \int_0^t \phi'(s)(\phi(t) - \phi(s))^{\sigma-1}\mathcal{R}(s)\,\mathrm{d}s.$$

In the following $n-1 < \sigma < n$ with $n \in \mathbb{N}$, and $\mathcal{R}, \phi \in C^n(\mathcal{I}, \mathbb{R})$ two functions such that ϕ is increasing and $\phi'(t) \neq 0, \forall t \in \mathcal{I}$.

Definition 2.6 ([9]): The ϕ-Riemann–Liouville fractional derivative of order σ of a function $\mathcal{R}$ is defined by

$$D^{\sigma,\phi}\mathcal{R}(t) = \frac{1}{\Gamma(n-\sigma)} \left(\frac{1}{\phi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \phi'(s)(\phi(t) - \phi(s))^{n-\sigma-1}\mathcal{R}(s)\,\mathrm{d}s, \quad t > 0,$$

such that $n = [\sigma] + 1$.

Definition 2.7 ([9]): The ϕ-Caputo fractional derivative of order σ of function $\mathcal{R}$ is defined by

$${}^cD^{\sigma;\phi}\mathcal{R}(t) = I^{n-\sigma;\phi}\left(\frac{1}{\phi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \mathcal{R}(t),\ t > 0.$$

Definition 2.8 ([9]): The ϕ-Hilfer fractional derivative of function $\mathcal{R}$ of order σ and type $0 \leq \delta \leq 1$ is defined by:

$$\begin{aligned}{}^hD^{\sigma,\delta;\phi}\mathcal{R}(t) =& I^{r-\sigma;\phi}D^{r;\phi}\mathcal{R}(t);\ t > 0, r = \sigma + \delta(n-\sigma)\\ =& I^{\delta(n-\sigma);\phi}\left(\frac{1}{\phi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n I^{(1-\delta)(n-\sigma);\phi}\mathcal{R}(t)\end{aligned} \tag{5}$$

Remark 2.9: In the above definitions,

(1) if $\phi(t) = t$, we find the classical fractional integral and derivative of Riemann–Liouville, classical fractional derivative of Caputo and Hilfer, respectively;
(2) if $\phi(t) = \ln(t)$, we find the fractional integral and derivative of Hadamard, the fractional derivative of Caputo–Hadamard and Hilfer–Hadamard, respectively.

Remark 2.10: (1) If $\delta \to 0$ and $0 < \sigma \leq 1$, the ϕ-Hilfer fractional derivative (5) equivalent to the ϕ-Riemann–Liouville derivative;
(2) If $\delta \to 1$ and $0 < \sigma \leq 1$, the ϕ-Hilfer fractional derivative (5) equivalent to the ϕ-Caputo derivative.For more details on ϕ-fractional derivative, see [9].

Lemma 2.11 ([9]): *Let $\sigma > 0, \delta > 0$, and $r > 0$. Then*

(1) $I_{T^+}^{\sigma;\phi} I_{T^+}^{\delta;\phi}\mathcal{R}(t) = I_{T^+}^{\sigma+\delta;\phi}\mathcal{R}(t)$;
(2) *if* $\mathcal{R}(t) = (\phi(t) - \phi(T))^{r-1}$, *then* $I_{T^+}^{\sigma;\phi}\mathcal{R}(t) = (\Gamma(r)/(\Gamma(\sigma + r)))(\phi(t) - \phi(T))^{\sigma+r-1}$.

Theorem 2.12 ([9]): *Let $\mathcal{R} \in C^1[T, b], \sigma > 0$ and $0 \leq \delta \leq 1$, then*

$${}^hD_{T^+}^{\sigma,\delta;\phi} I_{T^+}^{\sigma;\phi}\mathcal{R}(t) = \mathcal{R}(t).$$

Theorem 2.13 (Bohnenblust–Karlin [14]): *Let Λ be a nonempty subset of a Banach space $\mathcal{F}$, which is bounded, convex and closed. Assume that $\mathcal{A}: \Lambda \to \mathcal{P}(\mathcal{F}) \setminus \{0\}$ is u.s.c. with closed and convex values such that $\mathcal{A}(\Lambda) \subset \Lambda$ and $\overline{\mathcal{A}(\Lambda)}$ is compact. Then $\mathcal{A}$ has a fixed point.*

We conclude this section by defining the solution to our problem (1)–(3). Before that we will present the following auxiliary Lemma.

Lemma 2.14: *Let $0 < \sigma < 1$ and $0 \leq \delta \leq 1, r = \sigma + \delta - \sigma\delta$ and $\Psi \in C_{\mathbb{R}}(\mathcal{I})$. Then for each $t \in \mathcal{I}$ a function $w \in C_{1-r,\phi}(\mathcal{I}, \mathbb{R})$ given by*

$$w(t) = \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ I_{T^+}^{1-r;\phi} w(b) - I_{T^+}^{1-r+\sigma;\phi} \Psi(t) \mid_{t=b} \right\} + I_{T^+}^{\sigma;\phi} \Psi(t), \tag{6}$$

is the solution of the ϕ-Hilfer fractional differential equation

$${}^h D_{T^+}^{\sigma,\delta;\phi} w(t) = \Psi(t), \quad t \in \mathcal{I}.$$

Proof: The application of the operator ${}^h D_{T^+}^{\sigma,\delta;\phi}(\cdot)$ to both sides of (6) gives us

$${}^h D_{T^+}^{\sigma,\delta;\phi} w(t) = \left\{ I_{T^+}^{1-r;\phi} w(b) - I_{T^+}^{1-r+\sigma;\phi} \Psi(t) \mid_{t=b} \right\} {}^h D_{T^+}^{\sigma,\delta;\phi} \left(\frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \right) + {}^h D_{T^+}^{\sigma,\delta;\phi} I_{T^+}^{\sigma;\phi} \Psi(t),$$

As a consequence of the result ${}^h D_{T^+}^{\sigma,\delta;\phi}((\phi(t) - \phi(T))^{r-1}) = 0,\ 0 < r < 1$ and Theorem 2.12, we can conclude that

$${}^h D_{T^+}^{\sigma,\delta;\phi} w(t) = \Psi(t), \quad t \in \mathcal{I}. \qquad \blacksquare$$

Lemma 2.15: *Let $0 < \sigma < 1, 0 \leq \delta \leq 1$, and $\Psi \in C_{\mathbb{R}}(\mathcal{I})$, then the impulsive ϕ-Hilfer fractional differential equations of the form*

$$({}^h D_{T^+}^{\sigma,\delta;\phi} w)(t) = \Psi(t), \quad t \in \mathcal{I}, t \neq t_k; \tag{7}$$

$$\Delta I_{T^+}^{1-r;\phi} w(t_k) = I_{T^+}^{1-r;\phi} w(t_k^+) - I_{T^+}^{1-r;\phi} w(t_k^-) = \Lambda_k \in \mathbb{R}, \quad k \in \{1, \ldots, \ell\}; \tag{8}$$

$$(I_{T^+}^{1-r;\phi} w)(T) = \int_T^b \mathcal{E}(\varrho, w(\varrho))\, d\varrho, \quad r = \sigma + \delta - \sigma\delta, \tag{9}$$

has a solution given by

$$w(t) = \begin{cases} \dfrac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \displaystyle\int_T^b \mathcal{E}(\eta, \omega(\eta))\, d\eta \\ \quad + \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma-1} \Psi(\eta)\, d\eta, \ t \in [T, t_1]; \\ \dfrac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left(\displaystyle\int_T^b \mathcal{E}(\eta, \omega(\eta))\, d\eta + \sum_{\iota=1}^{k} \Lambda_\iota \right) \\ \quad + \dfrac{1}{\Gamma(\sigma)} \displaystyle\int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma-1} \Psi(\eta)\, d\eta, \quad t \in (t_k, t_{k+1}], k \in \{1, \ldots, \ell\}; \end{cases} \tag{10}$$

Proof: First of all, we suppose that $w \in PC_{1-r;\phi}(\mathcal{I}, \mathbb{R})$ satisfies Equations (7)–(9) and we check that w achieve (10)

If $t \in [T, t_1]$: with the application of the fractional integral operator $I_{T^+}^{\sigma;\phi}(\cdot)$ to both sides of (7) and via Lemma 2.9 [11], we get

$$w(t) = \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + I_{T^+}^{\sigma;\phi} \Psi(t). \tag{11}$$

If $t \in (t_1, t_2]$: by means of Lemma 2.14, we obtain

$$\begin{aligned} w(t) &= \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ I_{T^+}^{1-r;\phi} w(t_1^+) - I_{T^+}^{1-r+\sigma;\phi} \Psi(t) \big|_{t=t_1} \right\} + I_{T^+}^{\sigma;\phi} \Psi(t) \\ &= \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ I_{t^+}^{1-r;\phi} w(t_1^-) + \Lambda_1 - I_{T^+}^{1-r+\sigma;\phi} \Psi(t) \big|_{t=t_1} \right\} + I_{T^+}^{\sigma;\phi} \Psi(t). \end{aligned} \tag{12}$$

In view of (11), we find

$$I_{T^+}^{1-r;\phi} w(t) = \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + I_{T^+}^{1-r+\sigma;\phi} \Psi(t),$$

then

$$I_{T^+}^{1-r;\phi} w(t_1^-) - I_{T^+}^{1-r+\sigma;\phi} \Psi(t)|_{t=t_1} = \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta, \tag{13}$$

linking (12) and (13), we can see

$$w(t) = \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \Lambda_1 \right\} + I_{T^+}^{\sigma;\phi} \Psi(t), t \in (t_1, t_2]. \tag{14}$$

Similarly, if $(t_2, t_3]$, Lemma 2.14 implies that

$$\begin{aligned} w(t) &= \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ I_{T^+}^{1-r;\phi} w(t_2^+) - I_{T^+}^{1-r+\sigma;\phi} \Psi(t) \big|_{t=t_2} \right\} + I_{T^+}^{\sigma;\phi} \Psi(t) \\ &= \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ I_{T^+}^{1-r;\phi} w(t_2^-) + \Lambda_2 - I_{T^+}^{1-r+\sigma;\phi} \Psi(t) \big|_{t=t_2} \right\} + I_{T^+}^{\sigma;\phi} \Psi(t). \end{aligned} \tag{15}$$

In view of (14), we find

$$I_{T^+}^{1-r;\phi} w(t) = \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \Lambda_1 \right\} + I_{T^+}^{1-r+\sigma;\phi} \Psi(t),$$

then

$$I_{T^+}^{1-r;\phi} w(t_2^-) - I_{T^+}^{1-r+\sigma;\phi} \Psi(t)|_{t=t_2} = \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \Lambda_1, \tag{16}$$

linking (15) and (16), we can see

$$w(t) = \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \Lambda_1 + \Lambda_2 \right\} + I_{T^+}^{\sigma;\phi} \Psi(t). \tag{17}$$

pursue the above process, we get for $t \in (t_k, t_{k+1}], k \in \{1, \ldots, \ell\}$

$$w(t) = \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota \right\} + I_{T^+}^{\sigma;\phi} \Psi(t). \tag{18}$$

Reciprocally, we suppose $w \in PC_{1-r;\phi}(\mathcal{I}, \mathbb{R})$ achieves (10) and we check that is verify (7)–(9).

For $t \in [T, t_1]$, we have

$$w(t) = \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + I_{T^+}^{\sigma;\phi} \Psi(t),$$

by implement the operator ${}^{h}D_{T^+}^{\sigma,\delta;\phi}(\cdot)$ on both sides of the above equality, we obtain

$${}^{h}D_{T^+}^{\sigma,\delta;\phi} w(t) = \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta\, {}^{h}D_{T^+}^{\sigma,\delta;\phi} \left(\frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \right) + {}^{h}D_{T^+}^{\sigma,\delta;\phi} I_{T^+}^{\sigma;\phi} \Psi(t),$$

in fact that

$${}^{h}D_{T^+}^{\sigma,\delta;\phi} (\phi(t) - \phi(T))^{r-1} = 0, \quad 0 < r < 1, \tag{19}$$

and from Theorem 2.12, we can write

$${}^{h}D_{T^+}^{\sigma,\delta;\phi} w(t) = \Psi(t), \quad t \in [T, t_1].$$

Arguing as above, for $t \in (t_k, t_{k+1}], k \in \{1, \dots, \ell\}$,

$$w(t) = \frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota \right\} + I_{T^+}^{\sigma;\phi} \Psi(t),$$

by applying the operator ${}^{h}D_{T^+}^{\sigma,\delta;\phi}(\cdot)$ on both sides of the above equality, we have

$${}^{h}D_{T^+}^{\sigma,\delta;\phi} w(t) = \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota \right\} {}^{h}D_{T^+}^{\sigma,\delta;\phi} \left(\frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \right) + {}^{h}D_{T^+}^{\sigma,\delta;\phi} I_{T^+}^{\sigma;\phi} \Psi(t),$$

via (19) and Theorem 2.12, we get

$${}^{h}D_{T^+}^{\sigma,\delta;\phi} w(t) = \Psi(t), \quad t \in (t_k, t_{k+1}], k \in \{1, \dots, \ell\}.$$

New, we show that w also satisfies (8) and (9) According to (10), for $t \in (t_k, t_{k+1}]$ and $k \in \{1, \dots, \ell\}$, we have

$$\begin{aligned} I_{T^+}^{1-r;\phi} w(t) &= \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota \right\} I_{T^+}^{1-r;\phi} \left(\frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \right) + I_{T^+}^{1-r;\phi} I_{T^+}^{\sigma;\phi} \Psi(t) \\ &= \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota \right\} + I_{T^+}^{1-r+\sigma;\phi} \Psi(t), \end{aligned} \tag{20}$$

and for $t \in (t_{k-1}, t_k], k \in \{1, \dots, \ell\}$, we find

$$\begin{aligned} I_{T^+}^{1-r;\phi} w(t) &= \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k-1} \Lambda_\iota \right\} I_{T^+}^{1-r;\phi} \left(\frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \right) + I_{T^+}^{1-r;\phi} I_{T^+}^{\sigma;\phi} \Psi(t) \\ &= \left\{ \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k-1} \Lambda_\iota \right\} + I_{T^+}^{1-r+\sigma;\phi} \Psi(t), \end{aligned} \tag{21}$$

as outcome of (20) and (21), we obtain

$$\Delta I_{T^+}^{1-r;\phi} w(t_k) = I_{T^+}^{1-r;\phi} w(t_k^+) - I_{T^+}^{1-r;\phi} w(t_k^-) = \sum_{\iota=1}^{k} \Lambda_\iota - \sum_{\iota=1}^{k-1} \Lambda_\iota = \Lambda_k, \quad k \in \{1, \ldots, \ell\}.$$

On the other hand, by the application of the operator $I_{T^+}^{1-r;\phi}(\cdot)$ to both sides of Equation (11) we obtain

$$\begin{aligned} I_{T^+}^{1-r;\phi} w(t) &= \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta I_{T^+}^{1-r;\phi} \left(\frac{(\phi(t) - \phi(T))^{r-1}}{\Gamma(r)} \right) + I_{T^+}^{1-r;\phi} I_{T^+}^{\sigma;\phi} \Psi(t) \\ &= \int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta + I_{T^+}^{1-r+\sigma;\phi} \Psi(t), \end{aligned}$$

and then $I_{T^+}^{1-r;\phi} w(T) = \displaystyle\int_T^b \mathcal{E}(\eta, \omega(\eta))\, \mathrm{d}\eta.$ ■

Consequently, we have the following definition.

Definition 2.16: $(\omega_1, \omega_2) \in \mathcal{H}$ is said to be a solution for the coupled system of impulsive fractional integro-differential inclusions (1)–(3) if it satisfies the nonlocal integral conditions (2) and (3), and there exist $\varphi_1, \varphi_2 \in L^1(\mathcal{I}, \mathbb{R}) \times L^1(\mathcal{I}, \mathbb{R})$ with $\varphi_j \in \mathcal{Q}_j(t, \omega_1(t), \omega_2(t))$ for a.e., $t \in \mathcal{I}, j = 1, 2,\ k \in \{1, \ldots, \ell\}$ and

$$\begin{aligned} \omega_j(t) = &\frac{(\phi(t) - \phi(T))^{r_j - 1}}{\Gamma(r_j)} \left(\int_T^b \mathcal{E}_j(\eta, \omega_1(\eta), \omega_2(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota^j \right) \\ &+ \frac{1}{\Gamma(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma_j - 1} [\varphi_j(\eta) + \mathcal{C} u_j(\eta)]\, \mathrm{d}\eta \\ &+ \frac{1}{\Gamma^2(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma_j - 1} \int_T^\eta \phi'(\varrho)(\phi(\eta) - \phi(\varrho))^{\sigma_j - 1} \\ &\times \mathcal{J}_j(\varrho, \omega_1(\varrho), \omega_2(\varrho))\, \mathrm{d}\varrho\, \mathrm{d}\eta \quad \text{for each } t \in \mathcal{I}, \quad \text{and} \quad j = 1, 2. \end{aligned}$$

Definition 2.17: The coupled system of impulsive fractional integro-differential inclusion (1)–(3) is called controllable on $\mathcal{I}$, if for any initial state $\omega_T = (\omega_1^T, \omega_2^T) \in \mathcal{H}$ and any final state $\omega_b = (\omega_1^b, \omega_2^b) \in \mathcal{H}$, there exist two control functions $u_1, u_2 \in L^2(\mathcal{I}, \mathcal{K})$, such that the classical solution $\omega(\cdot)$ of (1)–(3) fulfills $\omega(b) = \omega_b$.

3. Assumptions and controllability results

Suppose that

(A_1) The multi-valued map $\mathcal{Q}_j : \mathcal{I} \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is Carathéodory;

(A_2) There exists a continuous function $\mu_{\mathcal{Q}_j} : \mathcal{I} \to \mathbb{R}^+, j = 1, 2$ such that

$$\|\mathcal{Q}_j(t, \omega_1(t), \omega_2(t))\|_{\mathcal{P}(\mathbb{R})} \le \mu_{\mathcal{Q}_j}(t)(1 + \|\omega_j\|_{\mathcal{H}_j}),$$

for each $t \in \mathcal{I}, \omega_j \in \mathcal{H}_j$ and $j = 1, 2$;

(A_3). The functions $\mathcal{J}_j, \mathcal{E}_j : \mathcal{I} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, j = 1, 2$ are continuous and there exists two functions $\mu_{\mathcal{J}_j} \in L^1(\mathcal{I}, \mathbb{R}^+)$ and $\mu_{\mathcal{E}_j} \in C_{\mathbb{R}^+}(\mathcal{I})$ such that for each $t \in \mathcal{I}$ and $j = 1, 2$

$$|\mathcal{J}_j(t, \omega_1(t), \omega_2(t))| \le \mu_{\mathcal{J}_j}(t) \left(\frac{|\omega_1(t)| + |\omega_2(t)|}{1 + |\omega_1(t)| + |\omega_2(t)|} \right)$$

$$|\mathcal{E}_j(t,\omega_1(t),\omega_2(t))| \le \mu_{\mathcal{E}_j}(t)\left(\frac{|\omega_1(t)|+|\omega_2(t)|}{1+|\omega_1(t)|+|\omega_2(t)|}\right)$$

(A_4). The linear operator $\Psi : L^2(\mathcal{I},\mathcal{K}) \to \mathbb{R}$ defined by

$$\Psi u_j = \frac{1}{\Gamma(\sigma_j)}\int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1}\mathcal{C}u_j(\eta)\,\mathrm{d}\eta,$$

has a bounded inverse operator $\Psi^{-1} : \mathbb{R} \to L^2(\mathcal{I},\mathcal{K})\backslash Ker(\Psi)$ and there exists a constant $\xi_\Psi > 0$ such that $\|\Psi^{-1}\| \le \xi_\Psi$.

Theorem 3.1: *If the assumptions* (A_1)–(A_4) *are satisfied, then the coupled fractional inclusions* (1)–(3) *is controllable on* $\mathcal{I}$.

Proof: We consider the operator $\mathcal{A} : \mathcal{H} \to \mathcal{P}(\mathcal{H})$ associated with the problem (1)–(3) defined by $\mathcal{A}(\omega_1,\omega_2) = (\mathcal{A}_1(\omega_1),\mathcal{A}_2(\omega_2))$ whereas the operator $\mathcal{A}_j : \mathcal{H}_j \to \mathcal{P}(\mathcal{H}_j)$ is defined as follows $A_j(\omega_j) = \{\Omega_j \in \mathcal{H}_j\}$ such that for $t \in \mathcal{I}, \varphi_j \in \Theta_{\mathcal{Q}_j,w_j}, j = 1,2,\ k \in \{1,\dots,\ell\}$

$$\begin{aligned}\Omega_j(t) = {}& \frac{(\phi(t)-\phi(T))^{r_j-1}}{\Gamma(r_j)}\left(\int_T^b \mathcal{E}_j(\eta,\omega_1(\eta),\omega_2(\eta))\,\mathrm{d}\eta + \sum_{\iota=1}^{k}\Lambda_\iota^j\right)\\ &+ \frac{1}{\Gamma(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}[\varphi_j(\eta)+\mathcal{C}u_{j,\omega_j}(\eta)]\,\mathrm{d}\eta\\ &+ \frac{1}{\Gamma^2(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}\int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1}\mathcal{J}_j(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta.\end{aligned} \tag{22}$$

Depending on assumption four, we can define the control function $u_{j,\omega_j}, j = 1,2$ as follows:

$$\begin{aligned}u_{j,\omega_j}(t) = {}& \Psi^{-1}\Big[\omega_j^b - \frac{(\phi(b)-\phi(T))^{r_j-1}}{\Gamma(r_j)}\left(\int_T^b \mathcal{E}_j(\eta,\omega_1(\eta),\omega_2(\eta))\,\mathrm{d}\eta + \sum_{\iota=1}^{k}\Lambda_\iota^j\right)\\ &- \frac{1}{\Gamma(\sigma_j)}\int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1}\varphi_j(\eta)\,\mathrm{d}\eta\\ &- \frac{1}{\Gamma^2(\sigma_j)}\int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1}\int_T^\eta \phi'(\varrho)(\phi(\eta)\\ &- \phi(\varrho))^{\sigma_j-1}\mathcal{J}_j(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta\Big](t),\end{aligned} \tag{23}$$

where $\varphi_j \in \Theta_{\mathcal{Q}_j,\omega_j}$ for $j = 1,2$.
To simplify the calculations, let us take for each $t \in \mathcal{I}, j = 1,2, k \in \{1,\dots,\ell\}$

$$\begin{aligned}|u_{j,\omega_j}(t)| \le {}& \xi_\Psi\left[|\omega_j^b| + \frac{(\phi(b)-\phi(T))^{r_j-1}}{\Gamma(r_j)}\left(\int_T^b |\mathcal{E}_j(\eta,\omega_1(\eta),\omega_2(\eta))|\,\mathrm{d}\eta + \sum_{\iota=1}^{k}|\Lambda_\iota^j|\right)\right.\\ &+ \frac{1}{\Gamma(\sigma_j)}\int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1}|\varphi_j(\eta)|\,d\eta + \frac{1}{\Gamma^2(\sigma_j)}\int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1}\\ &\left.\times \int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1}|\mathcal{J}_j(\varrho,\omega_1(\varrho),\omega_2(\varrho))|\,\mathrm{d}\varrho\,\mathrm{d}\eta\right]\end{aligned}$$

by utilizing (A_2), (A_3) and (A_4), we find

$$
\begin{aligned}
|u_{j,\omega_j}(t)| \leq \xi_\Psi \Bigg[& |\omega_j^b| + \frac{(\phi(b)-\phi(T))^{r_j-1}}{\Gamma(r_j)} \left(\int_T^b \mu_{\mathcal{E}_j}(\eta)\,\mathrm{d}\eta + \sum_{\iota=1}^{k} |\Lambda_\iota^j| \right) \\
& + \frac{1}{\Gamma(\sigma_j)} \int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1} \mu_{\mathcal{Q}_j}(\eta)(1+\|\omega_j\|_{\mathcal{H}_j})\,\mathrm{d}\eta \\
& + \frac{1}{\Gamma^2(\sigma_j)} \int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1} \int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1} \mu_{\mathcal{J}_j}(\varrho)\,\mathrm{d}\varrho\,\mathrm{d}\eta \Bigg] \\
\leq \xi_\Psi \Bigg[& |\omega_j^b| + \frac{(\phi(b)-\phi(T))^{r_j-1}}{\Gamma(r_j)} \left(\|\mu_{\mathcal{E}_j}\|_{L^1} + \sum_{\iota=1}^{k} |\Lambda_\iota^j| \right) + \frac{1}{\Gamma(\sigma_j)} \|\mu_{\mathcal{Q}_j}\|(1+\|\omega_j\|_{\mathcal{H}_j}) \\
& \times \int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1}\,\mathrm{d}\eta + \frac{1}{\Gamma^2(\sigma_j)} \|\mu_{\mathcal{J}_j}\| \int_T^b \phi'(\eta)(\phi(b)-\phi(\eta))^{\sigma_j-1} \\
& \times \int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1}\,\mathrm{d}\varrho\,\mathrm{d}\eta \Bigg] \\
\leq \xi_\Psi \Bigg[& |\omega_j^b| + \frac{(\phi(b)-\phi(T))^{r_j-1}}{\Gamma(r_j)} \left(\|\mu_{\mathcal{E}_j}\|_{L^1} + \sum_{\iota=1}^{k} |\Lambda_\iota^j| \right) + \frac{(\phi(b)-\phi(T))^{\sigma_j}}{\Gamma(\sigma_j+1)} \|\mu_{\mathcal{Q}_j}\| \\
& \times (1+\|\omega_j\|_{\mathcal{H}_j}) + \frac{(\phi(b)-\phi(T))^{2\sigma_j}}{\Gamma^2(\sigma_j+1)} \|\mu_{\mathcal{J}_j}\| \Bigg] := \zeta_{u_j}, j=1,2.
\end{aligned} \tag{24}
$$

Note that any fixed point of the operator $\mathcal{A}$ corresponds to the classical solution of problem (1)–(3). The proof is given in the following steps:

The first step: The values of $\mathcal{A}$ are convex and closed.

Let $\Omega, \hat{\Omega} \in \mathcal{A}(\omega_1,\omega_2)$ where $\Omega=(\Omega_1,\Omega_2)$, $\hat{\Omega}=(\hat{\Omega}_1,\hat{\Omega}_2)$ and $(\omega_1,\omega_2)\in\mathcal{H}$. Thus there exist $\varphi_j, \hat{\varphi}_j \in \Theta_{\mathcal{Q}_j,\omega_j}, j=1,2$ such that $k\in\{1,\ldots,\ell\}$

$$
\begin{aligned}
\Omega_j(t) = & \frac{(\phi(t)-\phi(T))^{r_j-1}}{\Gamma(r_j)} \left(\int_T^b \mathcal{E}_j(\eta,\omega_1(\eta),\omega_2(\eta))\,\mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota^j \right) \\
& + \frac{1}{\Gamma(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1} [\varphi_j(\eta) + \mathcal{C}u_{j,\omega_j}(\eta)]\,\mathrm{d}\eta \\
& + \frac{1}{\Gamma^2(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1} \int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1} \mathcal{J}_j(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\Omega}_j(t) = & \frac{(\phi(t)-\phi(T))^{r_j-1}}{\Gamma(r_j)} \left(\int_T^b \mathcal{E}_j(\eta,\omega_1(\eta),\omega_2(\eta))\,\mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota^j \right) \\
& + \frac{1}{\Gamma(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1} [\hat{\varphi}_j(\eta) + \mathcal{C}\hat{u}_{j,\omega_j}(\eta)]\,\mathrm{d}\eta \\
& + \frac{1}{\Gamma^2(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1} \int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1} \mathcal{J}_j(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta.
\end{aligned}
$$

For $\gamma \in [0,1], t \in \mathcal{I}, k \in \{1,\ldots,\ell\}$, and $j = 1,2$ we find

$$[\gamma\Omega_j + (1-\gamma)\hat{\Omega}](t) = \frac{(\phi(t)-\phi(T))^{r_j-1}}{\Gamma(r_j)}\left(\int_T^b \mathcal{E}_j(\eta,\omega_1(\eta),\omega_2(\eta))\,\mathrm{d}\eta + \sum_{\iota=1}^{k}\Lambda_\iota^j\right)$$
$$+ \frac{1}{\Gamma(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}[(\gamma\varphi_j(\eta)$$
$$+ (1-\gamma)\hat{\varphi}(t)) + (\gamma\mathcal{C}u_{j,\omega_j}(\eta) + (1-\gamma)\mathcal{C}\hat{u}_{j,\omega_j}(\eta)]\,\mathrm{d}\eta$$
$$+ \frac{1}{\Gamma^2(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}\int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1}$$
$$\times \mathcal{J}_j(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta,$$

as the values of $\mathcal{Q}_j$ are convex then $\gamma\Omega_j + (1-\gamma)\hat{\Omega}_j \in \mathcal{A}_j(\omega_j), j = 1,2$, and therefore $\gamma\Omega + (1-\gamma)\hat{\Omega} \in \mathcal{A}(\omega_1,\omega_2)$.

On the other hand, let $(\Omega_1^n,\Omega_2^n) \in \mathcal{A}(\omega_1,\omega_2)$ for all $(\omega_1,\omega_2) \in \mathcal{H}$ such that $(\Omega_1^n,\Omega_2^n) \to (\overline{\Omega}_1,\overline{\Omega}_2)$, we need to show that $(\overline{\Omega}_1,\overline{\Omega}_2) \in \mathcal{A}(\omega_1,\omega_2)$.

For $(\Omega_1^n,\Omega_2^n) \in \mathcal{A}(\omega_1,\omega_2)$, there exists $\varphi_j^n \in \Theta_{\mathcal{Q}_j,\omega_j}$ such that for $j = 1,2$ and $k \in \{1,\ldots,\ell\}$

$$\Omega_j^n(t) = \frac{(\phi(t)-\phi(T))^{r_j-1}}{\Gamma(r_j)}\left(\int_T^b \mathcal{E}_j(\eta,\omega_1(\eta),\omega_2(\eta))\,\mathrm{d}\eta + \sum_{\iota=1}^{k}\Lambda_\iota^j\right)$$
$$+ \frac{1}{\Gamma(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}[\varphi_j^n(\eta) + \mathcal{C}u_{j,\omega_j}^n(\eta)]\,\mathrm{d}\eta$$
$$+ \frac{1}{\Gamma^2(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}\int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1}\mathcal{J}_j(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta.$$

Since $\mathcal{Q}_j, j = 1,2$ has compact values and the set $\Theta_{\mathcal{Q}_j,\omega_j}$ is nonempty for each $\omega_j \in \mathcal{H}_j$ and $j = 1,2$, we may pass to a subsequence to find that φ_j^n converge to $\overline{\varphi}_j \in L^1(\mathcal{I},\mathbb{R})$. So we conclude that $\overline{\varphi}_j \in \Theta_{\mathcal{Q}_j,\omega_j}$. Then from the Lebesgue dominated convergence theorem, we deduce that for each $t \in \mathcal{I}, j = 1,2$, and $k \in \{1,\ldots,\ell\}$

$$\Omega_j^n(t) \to \overline{\Omega}_j(t) = \frac{(\phi(t)-\phi(T))^{r_j-1}}{\Gamma(r_j)}\left(\int_T^b \mathcal{E}_j(\eta,\omega_1(\eta),\omega_2(\eta))\,\mathrm{d}\eta + \sum_{\iota=1}^{k}\Lambda_\iota^j\right)$$
$$\frac{1}{\Gamma(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}[\overline{\varphi}_j(\eta) + \mathcal{C}\overline{u}_{j,\omega_j}(\eta)]\,\mathrm{d}\eta$$
$$+ \frac{1}{\Gamma^2(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}\int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1}$$
$$\times \mathcal{J}_j(\varrho,\omega_1(\varrho),\omega_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta.$$

Then $(\overline{\Omega_1},\overline{\Omega_2}) \in \mathcal{A}(\omega_1,\omega_2)$.

The second step: $\mathcal{A}(\Sigma_\kappa) \subset \Sigma_\kappa$, where κ is a positive constant and the bounded set $\Sigma_\kappa \subset \mathcal{H}$ is given by $\Sigma_\kappa = \{(\omega_1,\omega_2) \in \mathcal{H}, \|(\omega_1,\omega_2)\|_{\mathcal{H}} = \|\omega_1\|_{\mathcal{H}_1} + \|\omega_2\|_{\mathcal{H}_2} \le \kappa\}$. Σ_κ is convex and closed in $\mathcal{H}$.

Let $(\Omega_1,\Omega_2) \in \mathcal{A}(\omega_1,\omega_2)$ for each $(\omega_1,\omega_2) \in \Sigma_\kappa$, which means the existence of $\varphi_j \in \Theta_{\mathcal{Q}_j,\omega_j}$ such that $\Omega_j(t)$ achieves (22) for $j = 1,2$. Thus, from $(A_2) - (A_4)$ together with (24), for $k \in \{1,\ldots,\ell\}$ we get

$$
\begin{aligned}
|(\phi(t)-\phi(T))^{1-r_1}\Omega_1(t)| \leq{}& \frac{1}{\Gamma(r_1)}\left(\int_T^b |\mathcal{E}_1(\eta,\omega_1(\eta),\omega_2(\eta))|\,\mathrm{d}\eta + \sum_{\iota=1}^{k}|\Lambda_\iota^1|\right)\\
&+\frac{(\phi(t)-\phi(T))^{1-r_1}}{\Gamma(\sigma_1)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_1-1}|\varphi_1(\eta)|\,\mathrm{d}\eta\\
&+\frac{(\phi(t)-\phi(T))^{1-r_1}}{\Gamma(\sigma_1)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_1-1}\|\mathcal{C}\||u_{1,\omega_1}(\eta)|\,\mathrm{d}\eta\\
&+\frac{(\phi(t)-\phi(T))^{1-r_1}}{\Gamma^2(\sigma_1)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_1-1}\\
&\times\int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_1-1}|\mathcal{J}_1(\varrho,\omega_1(\varrho),\omega_2(\varrho))|\,\mathrm{d}\varrho\,\mathrm{d}\eta\\
\leq{}& \frac{1}{\Gamma(r_1)}\left(\int_T^b \mu_{\mathcal{E}_1}(\eta)\,d\eta + \sum_{\iota=1}^{k}|\Lambda_\iota^1|\right)+\frac{(\phi(t)-\phi(T))^{1-r_1}}{\Gamma(\sigma_1)}\\
&\times\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_1-1}\mu_{\mathcal{Q}_1}(\eta)(1+\|\omega_1\|_{\mathcal{H}_1})\,\mathrm{d}\eta\\
&+\frac{(\phi(t)-\phi(T))^{1-r_1}}{\Gamma(\sigma_1)}\|\mathcal{C}\|\zeta_{u_1}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_1-1}\,\mathrm{d}\eta\\
&+\frac{(\phi(t)-\phi(T))^{1-r_1}}{\Gamma^2(\sigma_1)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_1-1}\\
&\times\int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_1-1}\mu_{\mathcal{J}_1}(\varrho)\,\mathrm{d}\varrho\,\mathrm{d}\eta\\
\leq{}& \frac{1}{\Gamma(r_1)}\left(\|\mu_{\mathcal{E}_1}\|_{L^1}+\sum_{\iota=1}^{k}|\Lambda_\iota^1|\right)+\frac{1}{\Gamma(\sigma_1+1)}\|\mu_{\mathcal{Q}_1}\|(1+\|\omega_1\|_{\mathcal{H}_1})\\
&\times(\phi(t)-\phi(T))^{1-r_1+\sigma_1}+\frac{1}{\Gamma(\sigma_1+1)}\|\mathcal{C}\|\zeta_{u_1}(\phi(t)-\phi(T))^{1-r_1+\sigma_1}\\
&+\frac{1}{\Gamma^2(\sigma_1+1)}\|\mu_{\mathcal{J}_1}\|(\phi(t)-\phi(T))^{1-r_1+2\sigma_1}\\
\leq{}& \frac{1}{\Gamma(r_1)}\left(\|\mu_{\mathcal{E}_1}\|_{L^1}+\sum_{\iota=1}^{k}|\Lambda_\iota^1|\right)+\frac{(\phi(b)-\phi(T))^{1-r_1+\sigma_1}}{\Gamma(\sigma_1+1)}\\
&\times\left[\|\mu_{\mathcal{Q}_1}\|(1+\kappa)+\|\mathcal{C}\|\zeta_{u_1}+\frac{(\phi(b)-\phi(T))^{\sigma_1}}{\Gamma(\sigma_1+1)}\|\mu_{\mathcal{J}_1}\|\right]:=\kappa_1. \quad (25)
\end{aligned}
$$

Following the same way, as before we find

$$
\begin{aligned}
|(\phi(t)-\phi(T))^{1-r_2}\Omega_2(t)| \leq{}& \frac{1}{\Gamma(r_2)}\left(\|\mu_{\mathcal{E}_2}\|_{L^1}+\sum_{\iota=1}^{k}|\Lambda_\iota^2|\right)+\frac{(\phi(b)-\phi(\eta))^{1-r_2+\sigma_2}}{\Gamma(\sigma_2+1)}\\
&\times\left[\|\mu_{\mathcal{Q}_2}\|(1+\kappa)+\|\mathcal{C}\|\zeta_{u_2}+\frac{(\phi(b)-\phi(T))^{\sigma_2}}{\Gamma(\sigma_2+1)}\|\mu_{\mathcal{J}_2}\|\right]:=\kappa_2. \quad (26)
\end{aligned}
$$

As a outcome of (25) and (26) for each $t \in \mathcal{I}$, we conclude the existence of a constant $\kappa > 0$ such that $\|\Omega\|_{\mathcal{H}} = \|\Omega_1\|_{\mathcal{H}_1} + \|\Omega_2\|_{\mathcal{H}_2} \leq \kappa_1 + \kappa_2 := \kappa$. Namely, $\mathcal{A}(\Sigma_\kappa) \subset \Sigma_\kappa$.

The third step: $\mathcal{A}(\Sigma_\kappa)$ is equicontinuous in $\mathcal{H}$. Let $(\omega_1,\omega_2)\in\Sigma_\kappa$ and $(\Omega_1,\Omega_2)\in\mathcal{A}(\omega_1,\omega_2)$. For this, there exist $\varphi_1,\varphi_2\in\Theta_{\mathcal{Q}_j,\omega_j}$ such that $\Omega_j(t)$ satisfies (22), $j=1,2$. Take $t_1,t_2\in J$, $t_1<t_2$, and put $\mathfrak{P}:=|(\phi(t_2)-\phi(T))^{1-r_1}\Omega_1(t_2)-(\phi(t_1)-\phi(T))^{1-r_1}\Omega_1(t_1)|$, then

$$
\begin{aligned}
\mathfrak{P}\leq{}& \frac{(\phi(t_2)-\phi(T))^{1-r_1}}{\Gamma(\sigma_1)}\int_{t_1}^{t_2}\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}|\varphi_1(\eta)+\mathcal{C}u_{1,\omega_1}(\eta)|\,\mathrm{d}\eta\\
&+\frac{1}{\Gamma(\sigma_1)}\int_T^{t_1}|(\phi(t_2)-\phi(T))^{1-r_1}\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}-(\phi(t_1)-\phi(T))^{1-r_1}\\
&\times\phi'(\eta)(\phi(t_1)-\phi(\eta))^{\sigma_1-1}||\varphi_1(\eta)+\mathcal{C}u_{1,\omega_1}(\eta)|\,\mathrm{d}\eta\\
&+\frac{(\phi(t_2)-\phi(T))^{1-r_1}}{\Gamma^2(\sigma_1)}\int_{t_1}^{t_2}\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}\\
&\times\int_T^{\eta}\phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_1-1}|\mathcal{J}_1(\varrho,\omega_1(\varrho),\omega_2(\varrho))|\,\mathrm{d}\varrho\,\mathrm{d}\eta\\
&+\frac{1}{\Gamma^2(\sigma_1)}\int_T^{t_1}|(\phi(t_2)-\phi(T))^{1-r_1}\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}-(\phi(t_1)-\phi(T))^{1-r_1}\\
&\times\phi'(\eta)(\phi(t_1)-\phi(\eta))^{\sigma_1-1}|\int_T^{\eta}\phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_1-1}|\mathcal{J}_1(\varrho,\omega_1(\varrho),\omega_2(\varrho))|\,\mathrm{d}\varrho\,\mathrm{d}\eta\\
\leq{}&\frac{(\phi(t_2)-\phi(T))^{1-r_1}}{\Gamma(\sigma_1)}\int_{t_1}^{t_2}\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}[\mu_{\mathcal{Q}_1}(\eta)(1+\|\omega_1\|_{\mathcal{H}_1})+\|\mathcal{C}\|\zeta_{u_1}]\,\mathrm{d}\eta\\
&+\frac{1}{\Gamma(\sigma_1)}\int_T^{t_1}[(\phi(t_1)-\phi(T))^{1-r_1}\phi'(\eta)(\phi(t_1)-\phi(\eta))^{\sigma_1-1}-(\phi(t_2)-\phi(T))^{1-r_1}\\
&\times\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}][\mu_{\mathcal{Q}_1}(\eta)(1+\|\omega_1\|_{\mathcal{H}_1})+\|\mathcal{C}\|\zeta_{u_1}]\,\mathrm{d}\eta\\
&+\frac{(\phi(t_2)-\phi(T))^{1-r_1}}{\Gamma^2(\sigma_1)}\int_{t_1}^{t_2}\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}\int_T^{\eta}\phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_1-1}\mu_{\mathcal{J}_1}(\varrho)\,\mathrm{d}\varrho\,\mathrm{d}\eta\\
&+\frac{1}{\Gamma^2(\sigma_1)}\int_T^{t_1}[(\phi(t_1)-\phi(T))^{1-r_1}\phi'(\eta)(\phi(t_1)-\phi(\eta))^{\sigma_1-1}-(\phi(t_2)-\phi(T))^{1-r_1}\\
&\times\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}]\int_T^{\eta}\phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_1-1}\mu_{\mathcal{J}_1}(\varrho)\,\mathrm{d}\varrho\,\mathrm{d}\eta\\
\leq{}&\frac{(\phi(t_2)-\phi(T))^{1-r_1}}{\Gamma(\sigma_1)}\int_{t_1}^{t_2}\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}[\mu_{\mathcal{Q}_1}(\eta)(1_+\|\omega_1\|_{\mathcal{H}_1})+\|\mathcal{C}\|\zeta_{u_1}]\,\mathrm{d}\eta\\
&+\frac{1}{\Gamma(\sigma_1+1)}[\|\mu_{\mathcal{Q}_1}\|(1+\|\omega_1\|_{\mathcal{H}1})+\|\mathcal{C}\|\zeta_{u_1}][(\phi(t_1)-\phi(T))^{1-r_1+\sigma_1}-(\phi(t_2)-\phi(T))^{1-r_1+\sigma_1}]\\
&+\frac{(\phi(t_2)-\phi(T))^{1-r_1}}{\Gamma^2(\sigma_1)}\int_{t_1}^{t_2}\phi'(\eta)(\phi(t_2)-\phi(\eta))^{\sigma_1-1}\int_T^{\eta}\phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_1-1}\mu_{\mathcal{J}_1}(\varrho)\,\mathrm{d}\varrho\,\mathrm{d}\eta\\
&+\frac{1}{\Gamma^2(\sigma_1+1)}\|\mu_{\mathcal{J}_1}\|(\phi(b)-\phi(T))^{\sigma_1}[(\phi(t_1)-\phi(T))^{1-r_1+\sigma_1}-(\phi(t_2)-\phi(T))^{1-r_1+\sigma_1}].
\end{aligned}
$$

The second term of the previous inequality shows that $|(\phi(t_2)-\phi(T))^{1-r_1}\Omega_1(t_2)-(\phi(t_1)-\phi(T))^{1-r_1}\Omega_1(t_1)|\to 0$, as $|t_2-t_1|\to 0$. By following the same process, we find $|(\phi(t_2)-\phi(T))^{1-r_2}\Omega_2(t_2)-(\phi(t_1)-\phi(T))^{1-r_2}\Omega_2(t_1)|\to 0$, as $|t_2-t_1|\to 0$. This results prove the equicontinuity of $\mathcal{A}(\Sigma_\kappa)$. Therefore, we conclude from step two and three with Arzela–Ascoli theorem, the compactness of the operator $\mathcal{A}$.

The fourth step: In order to proof that $\mathcal{A}$ is u.s.c we have to show that its graph is closed.

Let $(\Omega_1^n, \Omega_2^n) \in \mathcal{A}(\omega_1^n, \omega_2^n)$ such that $(\Omega_1^n, \Omega_2^n) \to (\overline{\Omega}_1, (\overline{\Omega}_2)$ and $(\omega_1^n, \omega_2^n) \to (\overline{\omega}_1, \overline{\omega}_2)$ in $\mathcal{H}$, we ensure that $(\overline{\Omega_1}, \overline{\Omega_2}) \in \mathcal{A}(\overline{\omega_1}, \overline{\omega_2})$.

From $(\Omega_1^n, \Omega_2^n) \in \mathcal{A}(\omega_1^n, \omega_2^n)$ follows the existence of $\varphi_j^n \in \Theta_{\mathcal{Q}_j, \omega_j^n}$ such that for each $j = 1, 2, k \in \{1, \dots, \ell\}$,

$$\begin{aligned}\Omega_j^n(t) = {} & \frac{(\phi(t) - \phi(T))^{r_j - 1}}{\Gamma(r_j)} \left(\int_T^b \mathcal{E}_j(\eta, \omega_1^n(\eta), \omega_2^n(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota^j \right) \\ & + \frac{1}{\Gamma(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma_j - 1} [\varphi_j^n(\eta) + \mathcal{C} u_{j,\omega_j}^n(\eta)]\, \mathrm{d}\eta \\ & + \frac{1}{\Gamma^2(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma_j - 1} \int_T^\eta \phi'(\varrho)(\phi(\eta) - \phi(\varrho))^{\sigma_j - 1} \\ & \times \mathcal{J}_j(\varrho, \omega_1^n(\varrho), \omega_2^n(\varrho))\, \mathrm{d}\varrho\, \mathrm{d}\eta,\end{aligned}$$

and there exist $\overline{\varphi}_j \in \Theta_{\mathcal{Q}_j, \overline{\omega}_j}$ with $j = 1, 2$ such that

$$\begin{aligned}\overline{\Omega}_j(t) = {} & \frac{(\phi(t) - \phi(T))^{r_j - 1}}{\Gamma(r_j)} \left(\int_T^b \mathcal{E}_j(\eta, \overline{\omega}_1(\eta), \overline{\omega}_2(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota^j \right) \\ & + \frac{1}{\Gamma(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma_j - 1} [\overline{\varphi}_j(\eta) + \mathcal{C} \overline{u}_{j,\omega_j}(\eta)]\, \mathrm{d}\eta \\ & + \frac{1}{\Gamma^2(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma_j - 1} \int_T^\eta \phi'(\varrho)(\phi(\eta) - \phi(\varrho))^{\sigma_j - 1} \\ & \times \mathcal{J}_j(\varrho, \overline{\omega}_1(\varrho), \overline{\omega}_2(\varrho))\, \mathrm{d}\varrho\, \mathrm{d}\eta,\end{aligned}$$

To reach the desired result, we define the following function $\Upsilon : L^1(\mathcal{I}, \mathbb{R}) \to C_{\mathbb{R}}(\mathcal{I})$ as

$$\begin{aligned}\Upsilon(\varphi)(t) = {} & \frac{1}{\Gamma(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma_j - 1} \Big[\varphi(\eta) - \mathcal{C}\Psi^{-1} \\ & \times \left(\frac{1}{\Gamma(\sigma_j)} \int_T^b \phi'(\theta)(\phi(b) - \phi(\theta))^{\sigma_j - 1} \varphi(\theta)\, \mathrm{d}\theta \right) \Big]\, \mathrm{d}\eta\end{aligned}$$

Moreover, through the above definition of Υ we note that

$$\begin{aligned}& \Omega_j^n(t) - \frac{(\phi(t) - \phi(T))^{r_j - 1}}{\Gamma(r_j)} \left(\int_T^b \mathcal{E}_j(\eta, \omega_1^n(\eta), \omega_2^n(\eta))\, \mathrm{d}\eta + \sum_{\iota=1}^{k} \Lambda_\iota^j \right) \\ & - \frac{1}{\Gamma(\sigma_j)} \int_T^t \phi'(\eta)(\phi(t) - \phi(\eta))^{\sigma_j - 1} \mathcal{C}\Psi^{-1} \Bigg[\omega_j^b - \frac{(\phi(b) - \phi(T))^{r_j - 1}}{\Gamma(r_j)} \\ & \times \left(\int_T^b \mathcal{E}_j(\theta, \omega_1^n(\theta), \omega_2^n(\theta))\, d\theta + \sum_{\iota=1}^{k} \Lambda_\iota^j \right) - \frac{1}{\Gamma^2(\sigma_j)} \int_T^b \phi'(\theta)(\phi(b) - \phi(\theta))^{\sigma_j - 1} \\ & \times \int_T^\theta \phi'(\varrho)(\phi(\theta) - \phi(\varrho))^{\sigma_j - 1} \mathcal{J}_j(\varrho, \omega_1^n(\varrho), \omega_2^n(\varrho))\, \mathrm{d}\varrho\, \mathrm{d}\theta \Bigg]\, \mathrm{d}\eta\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\Gamma^2(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}\int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1}\\
&\times \mathcal{J}_j(\varrho,\omega_1^n(\varrho),\omega_2^n(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta \in \Upsilon(\Theta_{\mathcal{Q}_j,\omega_j^n}),\quad j=1,2,\ k\in\{1,\ldots,\ell\}.
\end{aligned}
$$

In addition, by linking the continuity of $\mathcal{J}_j$ and $\mathcal{E}_j$ for each $j=1,2$, $C_{\mathcal{H}}(\mathcal{D})\subset L^p(\mathcal{D},\mathcal{H})(1<p<\infty)$, and Lebesgue dominated convergence theorem, we acquire the uniform convergence of the previous relationship to

$$
\begin{aligned}
&\overline{\Omega}_j(t)-\frac{(\phi(t)-\phi(T))^{r_j-1}}{\Gamma(r_j)}\left(\int_T^b \mathcal{E}_j(\eta,\overline{\omega}_1(\eta),\overline{\omega}_2(\eta))\,\mathrm{d}\eta+\sum_{\iota=1}^{k}\Lambda_\iota^j\right)\\
&-\frac{1}{\Gamma(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}\mathcal{C}\Psi^{-1}\left[\omega_j^b-\frac{(\phi(b)-\phi(T))^{r_j-1}}{\Gamma(r_j)}\right.\\
&\times\left(\int_T^b \mathcal{E}_j(\theta,\overline{\omega}_1(\theta),\overline{\omega}_2(\theta))\,d\theta+\sum_{\iota=1}^{k}\Lambda_\iota^j\right)-\frac{1}{\Gamma^2(\sigma_j)}\int_T^b \phi'(\theta)(\phi(b)-\phi(\theta))^{\sigma_j-1}\\
&\left.\times\int_T^\theta \phi'(\varrho)(\phi(\theta)-\phi(\varrho))^{\sigma_j-1}\mathcal{J}_j(\varrho,\overline{\omega}_1(\varrho),\overline{\omega}_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\theta\right]\mathrm{d}\eta\\
&-\frac{1}{\Gamma^2(\sigma_j)}\int_T^t \phi'(\eta)(\phi(t)-\phi(\eta))^{\sigma_j-1}\int_T^\eta \phi'(\varrho)(\phi(\eta)-\phi(\varrho))^{\sigma_j-1}\\
&\times \mathcal{J}_j(\varrho,\overline{\omega}_1(\varrho),\overline{\omega}_2(\varrho))\,\mathrm{d}\varrho\,\mathrm{d}\eta,\ \text{as } n\to+\infty\quad \forall j=1,2,\ k\in\{1,\ldots,\ell\}. \qquad (27)
\end{aligned}
$$

According to Lemma 2.4, $\Upsilon\circ\Theta_{\mathcal{Q}_j,\omega_j}$ is a closed graph operator and since $\omega_j^n\to\overline{\omega}_j$, for $n\in\mathbb{N}$ and $j=1,2$, so the formula (27) belong to $\Upsilon(\Theta_{\mathcal{Q}_j,\overline{\omega}_j})$, for $j=1,2$. Thus, $(\overline{\Omega}_1,\overline{\Omega}_2)\in\mathcal{A}(\overline{\omega}_1,\overline{\omega}_2)$. i.e. the graph of $\mathcal{A}$ is closed. Subsequently, we deduce that $\mathcal{A}$ is u.s.c.

Consequently, all the conditions of Theorem 2.13 are realized and therefore the operator $\mathcal{A}$ has a fixed point, which represents the solution of the fractional coupled system (1)–(3). Moreover, through (23) It is clear that any solution of (1)–(3) achieves $\omega(b)=\omega_b$, this result means the controllability of (1)–(3). ■

4. Applications

In this section, we provide an example to illustrate the applicability of the theoretical techniques presented in this paper.

We consider the following coupled system of impulsive fractional integro-differential inclusion with nonlocal integral condition by taking $\phi(t)=t$:

$$
\begin{cases}
({}^hD_{1^+}^{1/3,1/4;t}\nu_1)(t)\in\left[\dfrac{(e^t+\sin t)(t+t^{3/2}|\nu_1(t)|)e^{|\cos\nu_2(t)|}}{\frac{3}{2}+t},\left(\dfrac{|\nu_2(t)|^2-1}{|\nu_2(t)|^2}+\sqrt{t^2+4t}\right)t^{1/2}|\nu_1(t)|\right]\\
+\dfrac{1}{\Gamma(\frac{1}{3})}\displaystyle\int_1^t (t-\varrho)^{-2/3}\dfrac{\ln(\varrho^2+2)(\nu_1(\varrho)+\nu_2(\varrho)\sin\varrho)}{(\varrho^2+3)(1+\nu_1(\varrho)+\nu_2(\varrho))}\,\mathrm{d}\varrho+\mathcal{C}u_1(t),\quad t\in\mathcal{I}=[1,5]-\{2,3,4\},\\
({}^hD_{1^+}^{2/3,\frac{3}{4};t}\nu_2)(t)\in\left[\dfrac{(e^t+\cos t)(t+t^{7/4}|\nu_2(t)|)e^{|\sin\nu_1(t)|}}{\frac{5}{3}+t},\left(\dfrac{|\nu_1(t)|^3-1}{|\nu_1(t)|^3}+\sqrt{t}(4+t)\right)t^{3/4}|\nu_2(t)|\right]\\
+\dfrac{1}{\Gamma(\frac{2}{3})}\displaystyle\int_1^t (t-\varrho)^{-1/3}\dfrac{2e^{1/3\varrho}(\nu_1(\varrho)+\nu_2(\varrho)\cos\varrho)}{(e^\varrho+12)(1+\nu_1(\varrho)+\nu_2(\varrho))}\,\mathrm{d}\varrho+\mathcal{C}u_2(t),\quad t\in\mathcal{I}=[1,5]-\{2,3,4\},\\
\Delta I_{1^+}^{1-1/2;t}\nu_1(t_k)=\Lambda_k^1\in\mathbb{R},\quad \Delta I_{1^+}^{1-3/4;\phi}\nu_2(t_k)=\Lambda_k^2\in\mathbb{R},\quad t_k=k+1,k=\{1,2,3\},
\end{cases}
\qquad (28)
$$

with the coupled nonlocal integral conditions:

$$(I_{1^+}^{1-1/2;t}v_1)(1) = \int_1^5 \frac{e^{-2\varrho}(v_1(\varrho)+v_2(\varrho))}{(\varrho+8)^2(1+v_1(\varrho)+v_2(\varrho))}\,\mathrm{d}\varrho, \tag{29}$$

$$(I_{1^+}^{1-3/4;t}v_2)(1) = \int_1^5 \frac{\varrho^{1/2}e^{\varrho}\ln(1+v_1(\varrho)+v_2(\varrho))}{(t+2)^2(1+v_1(\varrho)+v_2(\varrho))}\,\mathrm{d}\varrho, \tag{30}$$

where $\sigma_1 = \frac{1}{3}, \delta_1 = \frac{1}{4}, \sigma_2 = \frac{2}{3}$ and $\delta_1 = \frac{3}{4}$, here for each j = 1,2, $\mathcal{Q}_j$, $\mathcal{J}_j$ and $\mathcal{E}_j$ are given by

$$\begin{cases} \mathcal{Q}_1(t,v_1,v_2) = \left[\dfrac{(e^t+\sin t)(t+t^{3/2}|v_1|)e^{|\cos v_2|}}{\frac{3}{2}+t}, \left(\dfrac{|v_2|^2-1}{|v_2|^2}+\sqrt{t^2+4t}\right)t^{1/2}|v_1|\right] \\ \mathcal{J}_1(t,v_1,v_2) = \dfrac{(\ln t^2+2)(v_1+v_2\sin t)}{(t^2+3)(1+v_1+v_2)} \\ \mathcal{E}_1(t,v_1,v_2) = \dfrac{e^{-2t}(v_1+v_2)}{(t+8)^2(1+v_1+v_2)}, \end{cases}$$

and

$$\begin{cases} \mathcal{Q}_2(t,v_1,v_2) = \left[\dfrac{(e^t+\cos t)(t+t^{7/4}|v_2|)e^{|\sin v_1|}}{\frac{5}{3}+t}, \left(\dfrac{|v_1|^3-1}{|v_1|^3}+\sqrt{t}(4+t)\right)t^{3/4}|v_2|\right] \\ \mathcal{J}_2(t,v_1,v_2) = \dfrac{2e^{1/3t}(v_1+v_2\cos t)}{(e^t+12)(1+v_1+v_2)} \\ \mathcal{E}_2(t,v_1,v_2) = \dfrac{t^{1/2}e^t\ln(1+v_1+v_2)}{(t+2)^2(1+v_1+v_2)}. \end{cases}$$

Clearly, $\mathcal{Q}_j$ satisfies (A_1), and $\mathcal{J}_j, \mathcal{E}_j$ are continuous for $j = 1, 2$. In addition

$$\|\mathcal{Q}_1(t,v_1,v_2)\| \le e(e^t+1)(1+\|v_1\|_{\mathcal{H}_1}), \quad \|\mathcal{Q}_2(t,v_1,v_2)\| \le e(e^t+1)(1+\|v_2\|_{\mathcal{H}_2}),$$

$$|\mathcal{J}_1(t,v_1,v_2)| \le \frac{|\ln(t^2+2)|(|v_1|+|v_2|)}{(t^2+3)(1+|v_1|+|v_2|)}, \quad |\mathcal{J}_2(t,v_1,v_2)| \le \frac{2e^{1/3t}(|v_1|+|v_2|)}{(e^t+12)(1+|v_1|+|v_2|)},$$

$$|\mathcal{E}_1(t,v_1,v_2)| \le \frac{e^{-2t}(|v_1|+|v_2|)}{(t+8)^2(1+|v_1|+|v_2|)}, \quad |\mathcal{E}_2(t,v_1,v_2)| \le \frac{t^{1/2}e^t(|v_1|+|v_2|)}{(t+2)^2(1+|v_1|+|v_2|)}.$$

Whereas

$$\mu_{\mathcal{Q}_j} = e(e^t+1), \quad \mu_{\mathcal{J}_1} = \frac{|\ln(t^2+2)|}{t^2+3}, \quad \mu_{\mathcal{J}_2} = \frac{2e^{1/3t}}{e^t+12},$$

$$\mu_{\mathcal{E}_1} = \frac{e^{-2t}}{(t+8)^2}, \quad \text{and} \quad \mu_{\mathcal{E}_2} = \frac{t^{1/2}e^t}{(t+2)^2}.$$

Thus, the assumptions (A_2) and (A_3) are satisfied. Now, we suppose that the linear operator ψ : $L^2(\mathcal{I},\mathcal{K}) \to \mathbb{R}$ defined by

$$\psi u_1(t) = \frac{1}{\Gamma(\frac{1}{3})}\int_1^5 (t-\eta)^{-2/3}Cu_1(\eta)\,\mathrm{d}\eta,$$

$$\psi u_2(t) = \frac{1}{\Gamma(\frac{2}{3})}\int_1^5 (t-\eta)^{-1/3}Cu_2(\eta)\,\mathrm{d}\eta,$$

has a bounded invertible operator and satisfies the assumption (A_4).

From above, we note that all the condition of Theorem 2.13 are met, so the coupled system (28)–(30) is controllable on $\mathcal{I} = [1, 5]$.

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