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Bernstein Polynomials Based-Solution For Linear Fractional Differential Equations

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Abstract. In this paper, a numerical approximation for the solution of linear fractional differential equations, based on Galerkin method and Bernstein polynomials, is proposed. A system of linear equations is obtained and the coefficients of Bernstein polynomials, whose linear combination is used to approximate the solution, are determined. Matrix formulation is used throughout the whole procedure. The accuracy of the proposed technique has been evaluated via different degrees of Bernstein polynomials.
Keywords: Fractional differential equations, Bernstein polynomials, Galerkin method.
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INTRODUCTION

Fractional differential equations are encountered in model problems in fluid flow, viscoelasticity, finance, engineering, and other areas of applications[1].
In this paper, we consider the linear fractional differential equation in the form

$$l(y(x)) = g(x), 0 \leq x \leq 1 \tag{1}$$

subject to the following conditions

$$y^{(i)}(\beta) = \sigma_i, i \leq \overline{k-1}, \beta = \{0, 1\}, \tag{2}$$

where l: is a linear fractional differential operator.
The solution of (1) is approximated by a weighted sum of Bernstein polynomials as

$$\tilde{y}(x) = \sum_{l=0}^{n} c_l B_{l,n}(x), \tag{3}$$

here:
$\tilde{y}(x)$: The approximate solution of the fractional differential equation given in equation (1).
$B_{(i,n)}(x)$: i-th Bernstein polynomial of degree n.
$C = [C_0, C_1, , C_n]^T$: the coefficients of Bernstein polynomials.

PROPERTIES OF BERNSTEIN POLYNOMIALS

Bernstein polynomials of the n-th degree are defined on the interval $[0, 1]$ as in [2]

$$B_{l,n}(x) = nl\,(1-x)^{n-l}\,x^l. \tag{4}$$

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There are $(n+1)$ n-th degree polynomials and for convention, we set $B_{l,n}(x) = 0$, for $l < 0$ or $l > n$. A recursive definition also can be used to generate the Bernstein polynomials over $[0, 1]$

$$B_{l,n}(x) = (1-x)B_{l,n-1}(x) + xB_{l-1,n-1}(x). \tag{5}$$

It can be easily shown that any given polynomial of degree n can be expressed as a linear combination of the Bernstein basis[3].
The expression in equation (4) can be transformed , using the binomial expansion of $(1-x)^{n-l}$ [4] to

$$B_{l,n}(x) = \sum_{j=l}^{n} (-1)^{j-l} \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} n-j \\ j-l \end{pmatrix} x^j. \tag{6}$$

The derivatives of the n-th degree Bernstein polynomials are a linear combination of Bernstein polynomials of degree $(n-1)$. They are given by

$$\frac{dB_{l,n}(x)}{dx} = n(B_{l-1,n-1}(x) - B_{l,n-1}(x)). \tag{7}$$

In what follows, we consider $\alpha \in \mathbb{R}$, $\alpha \notin \mathbb{N}$ and $\alpha > 0$. So, the α-fractional derivative of Bernstein Polynomials in Caputo-sense has the form

$$D_x^{\alpha} B_{l,n}(x) = \sum_{j=l}^{n} (-1)^{j-l} \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} n-j \\ j-l \end{pmatrix} D_x^{\alpha} x^j = \sum_{j=\lceil \alpha \rceil}^{n} (-1)^{j-l} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} n-j \\ j-l \end{pmatrix} \frac{\Gamma(j+1)}{\Gamma(j+1-\alpha)} x^{j-\alpha}. \tag{8}$$

DESCRIPTION OF THE METHOD

The solution of linear fractional differential equations can be approximated by a linear combination of Bernstein polynomials. The coefficients are determined by Galerkin method[5]. The latter is based on taking the inner product of the fraction differential equation, in which the approximate solution is substituted, and the basis that is used in approximating the solution. A linear algebraic system of equation is obtained in the form of

$$AX = B, \tag{9}$$

where $X = \begin{bmatrix} C_0 & C_1 & \dots & C_n \end{bmatrix}^T$ represents the vector of the unknown coefficients.
A is a $(n+1)$ by $(n+1)$ matrix given by

$$A = \begin{bmatrix} B_{l,n}^{(i)}(\beta), \\ \int_0^1 l(B_{l,n}(x))B_{m,n}(x)dx. \end{bmatrix} \tag{10}$$

B is a $(n+1)$ vector given by

$$B = \begin{bmatrix} \sigma_i, \\ \int_0^1 g(x)B_{m,n}(x)dx \end{bmatrix} \tag{11}$$

for $l = \overline{0,n}$, $j = \overline{0,k-1}$, , $m = \overline{k,n}$

ILLUSTRATIVE EXAMPLE

Consider the following linear boundary value problem

$$4(x+1)D^{5/2}y(x) + 4D^{3/2}y(x) + \frac{y(x)}{\sqrt{x+1}} = \sqrt{x} + \sqrt{\pi}\; y(0) = \sqrt{\pi}, y'(0) = \frac{\sqrt{\pi}}{2}, y(1) = \sqrt{2\pi}. \tag{12}$$

The exact solution is $y(x) = \sqrt{(x+1)}$. The method previously described is applied. The results of the approximation are shown in Table 1, in which a comparison of the L_∞ and L_2 errors has been made between the present technique and the methods in [6] and [7]. The method of [6] requires the solution of large systems of algebraic equations to get high level of accuracy, and this is obviously time consuming. The work done in [7] is based on constructing Bernstein operational matrix of fractional derivatives and using collocation method to construct algebraic system of equations.

TABLE 1. Results of the presented example

Methods	number of polynomials terms	L_∞ *error*	L_2 *error*
Method in [6]	J=5	3.5×10^{-3}	1.2×10^{-3}
	J=7	3.5×10^{-4}	1.2×10^{-4}
	J=8	3.5×10^{-5}	1.2×10^{-5}
Method in [7]	n=3	1.5×10^{-3}	8.8×10^{-4}
	n=6	1.6×10^{-5}	6.1×10^{-6}
	n=12	1.4×10^{-6}	9.7×10^{-7}
	n=15	6.7×10^{-7}	4.6×10^{-7}
Proposed method	n=3	1.3×10^{-3}	1.3×10^{-3}
	n=6	2.3×10^{-5}	1.1×10^{-5}
	n=12	4.2×10^{-7}	1.6×10^{-7}
	n=15	6.3×10^{-8}	4.3×10^{-8}

CONCLUSION

In this paper, a numerical solution for linear fractional differential equations using Galerkin method and Bernstein polynomials has been presented. The method showed its effectiveness through its simple implementation, convergence and the high accuracy it gave.

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